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Kyoto University
ON FINITE GALOIS COVERINGS OF
COMPACT COMPLEX MANIFOLDS

Makoto Namba

1. Introduction. A finite branched covering of an
n-dimensional connected compact complex manifold \( M \) is,
by definition, an irreducible normal complex space \( X \)
together with a surjective proper finite holomorphic
mapping \( \pi : X \to M \). A point \( p \in X \) is called an unramified
point if \( \pi \) is locally biholomorphic around \( p \). Otherwise,
p is called a ramified point. The set of all ramified
points forms a hypersurface \( R_\pi \) of \( X \), called the ramification
locus of \( \pi \). The image \( B_\pi = \pi(R_\pi) \) of \( R_\pi \) is called the
branch locus of \( \pi \), which is a hypersurface of \( M \).

A morphism (resp. isomorphism) of \( \pi : X \to M \) to
\( \pi' : X' \to M \) is a holomorphic (resp. biholomorphic) mapping
\( \phi : X \to X' \) such that \( \pi' \phi = \pi \). An automorphism of \( \pi \) is
called a covering transformation of $\pi$. The set of all covering transformations of $\pi$ forms a group $G_\pi$ under composition, called the covering transformation group of $\pi$. The covering $\pi : X \to M$ is called a Galois covering if $G_\pi$ acts transitively on every fiber.

Let $\pi : X \to M$ be a finite Galois covering and $B_\pi = D_1 \cup \ldots \cup D_s$ be the irreducible decomposition of the branch locus $B_\pi$. For each non-singular point $q$ of $B_\pi$, every point $p \in \pi^{-1}(q)$ is a non-singular point of both $X$ and $\pi^{-1}(B_\pi)$. Moreover, there are coordinate systems $(w_1, \ldots, w_n)$ and $(z_1, \ldots, z_n)$ around $q$ and $p$, respectively, such that $q = (0, \ldots, 0)$, $p = (0, \ldots, 0)$, $B_\pi = \{w_1 = 0\}$, $\pi^{-1}(B_\pi) = \{z_1 = 0\}$ and

$$\pi : (z_1, \ldots, z_n) \rightarrow (w_1, \ldots, w_n) = (z_1^e, z_2, \ldots, z_n),$$

locally. The positive integer $e = e_j \ (\geq 2)$ is constant as a function of $q$ on $D_j - \text{Sing} B_\pi$ for every $D_j$ and is called the ramification index of $\pi$ around $D_j$. (Sing
is the singular locus of \( B_\pi \). In this case, \( \pi \) is said to \textbf{branch along the positive divisor} \( D = e_1 D_1 + \ldots + e_S D_S \) on \( M \).

The purpose of this note is to give a sufficient condition for the existence of a finite Galois covering \( \pi : X \to M \) of \( M \) with non-singular \( X \) which branches along a given positive divisor \( D = e_1 D_1 + \ldots + e_S D_S \) (\( e_j \geq 2 \)) on \( M \). The existence of such coverings for some special cases were proved in the interesting papers Hirzebruch [3], Höfer [4] and Kato [6]. In this note, we make use of Selberg's theorem on the monodromy representation of a Fuchsian differential equation.

2. Kato's criterion. Let \( B \) be a hypersurface of \( M \) and \( B = D_1 \cup \ldots \cup D_S \) be its irreducible decomposition.
We fix a point \( q_0 \in M - B \) once for all. Let \( \gamma_j \) be a loop in \( M - B \) starting and terminating at \( q_0 \) encircling a point of \( D_j \in \text{Sing} \, B \) in the positive sense as in Figure 1. We identify \( \gamma_j \) with its homotopy class in the fundamental group \( \pi_1(M - B, q_0) \). Let \( e_1, \ldots, e_s \) be integers greater than one. We denote by

\[
J = \langle \gamma_1^{e_1}, \ldots, \gamma_s^{e_s} \rangle^{\pi_1}
\]

the smallest normal subgroup of \( \pi_1(M - B, q_0) \) which contains \( \gamma_1^{e_1}, \ldots, \gamma_s^{e_s} \).

For a subgroup \( K \) of \( \pi_1(M - B, q_0) \) with \( J \subset K \), we consider the following condition:

**Condition A.** If \( \gamma_j^d \) belongs to \( K \), then \( d \equiv 0 \pmod{e_j} \) for every \( j \) \((1 \leq j \leq s)\).

Then we have

**Theorem 1 (Namba [8]).** There exists a finite Galois covering \( \pi : X \to M \) which branches along \( D = e_1D_1 + \ldots + e_sD_s \) if and only if there exists a normal subgroup \( K \) of
\[ \pi_1(M - B, q_0) \] of finite index which contains \( J \) and satisfies Condition A. The correspondence

\[ \pi \rightarrow K = K(\pi) = \pi_*(\pi_1(X - \pi^{-1}(B))) \]

between (isomorphism classes of) such \( \pi \)'s and such \( K \)'s is one-to-one. \( G_\pi \) is isomorphic to \( \pi_1(M - B, q_0)/K \).

Henceforth, we suppose that \( B \) is simple normally crossing.

For any point \( q \in B \), we take a local coordinate system \((w_1, \ldots, w_n)\) around \( q \) such that \( q = (0, \ldots, 0) \) and \( B = \{w_1 \ldots w_t = 0\} \) locally. We may suppose \( D_j = \{w_j = 0\} \) locally for \( 1 \leq j \leq t \).

![Figure 2](image-url)
Let \( \hat{\gamma}_j \) be a loop in \( M - B \) starting and terminating at \( q_o \) encircling a point of \( D_j - \text{Sing } B \) near \( q \) in the positive sense as in Figure 2. \( \hat{\gamma}_j \) is then conjugate to \( \gamma_j \) in \( \pi_1(M - B, q_o) \). Note that \( \hat{\gamma}_1, \ldots, \hat{\gamma}_t \) are mutually commutative.

For a subgroup \( K \) of \( \pi_1(M - B, q_o) \) with \( J \) \( K \), we consider the following condition:

**Condition A'.** If \( \hat{\gamma}_1^{d_1} \ldots \hat{\gamma}_t^{d_t} \) belongs to \( K \), then \( d_1 \equiv 0 \pmod{e_1}, \ldots, d_t \equiv 0 \pmod{e_t} \) for every point \( q \in B \).

Then, as a special case of Kato [5], we have

**Theorem 2 (Kato).** In Theorem 1, if \( K = K(\pi) \) satisfies Condition A', then \( X \) is non-singular.

3. Fuchsian differential equations. Let \( B = D_1 \cup \ldots \cup D_s \) be as above simple normally crossing. Let \( \Omega \) be an \((r \times r)\)-matrix-valued meromorphic 1-form on \( M \) such that \( d\bar{\Omega} + \Omega \wedge \Omega = 0 \) and such that \( \Omega \) is holomorphic on \( M - B \).
For an unknown $r$-vector-valued function $Y$, the differential equation

$$dY = Y\Omega$$

(of order $r$) is called a Fuchsian differential equation with regular singularity along $B$, if, for every point $q \in B$, $\Omega$ can be locally written as

$$\Omega = A_1(w)\frac{dw_1}{w_1} + \ldots + A_t(w)\frac{dw_t}{w_t} + A_{t+1}(w)dw_{t+1} + \ldots + A_n(w)dw_n,$$

around $q$, under the notations in §2, where $A_j(w)$ are $(r \times r)$-matrix-valued holomorphic functions around $q$. In this case,

$$\text{Res}_{D_j} \Omega = A_j(q) \quad (1 \leq j \leq t)$$

is a constant matrix on $D_j$, called the residue of $\Omega$ at $D_j$.

**Theorem 3** (Gérard [2], Yoshida-Takano [10]). A fundamental matrix solution $F(w)$ around $q \in B$ of the Fuchsian differential equation (1) with regular singularity along $B$ can be written as

$$F(w) = w_1^{C_1} \ldots w_t^{C_t} w_1^{N_1} \ldots w_t^{N_t} G(w),$$

where $C_1, \ldots, C_t$ are mutually commutative constant matrices,
$N_1, \ldots, N_t$ are diagonal matrices whose components are non-negative integers, and $G(w)$ is an $(r \times r)$-matrix-valued holomorphic function around $q$ with $\det G(w)$ nowhere vanishing. Moreover, if none of the differences of the eigenvalues of $\text{Res}_{D_j} \Omega$ are non-zero integers for $1 \leq j \leq t$, then $C_j$ and $N_j$ can be so chosen that $N_j = 0$ $(1 \leq j \leq t)$ and $C_j$ is conjugate to $\text{Res}_{D_j} \Omega$ $(1 \leq j \leq t)$.

4. Existence of finite Galois coverings. Let $B = D_1 U \ldots U D_s$ be as above simple normally crossing. Suppose

(i) $\pi_1(M - B, q_0)$ is finitely generated. (This condition is satisfied if $M$ is projective as Prof. M. Oka informed us.)

Suppose that there is a Fuchsian differential equation (1) with regular singularity along $B$ such that (ii) the order $r$ of (1) satisfies $r \geq n$, (iii) every $\text{Res}_{D_j} \Omega$ $(1 \leq j \leq s)$ is diagonalizable and (iv) every eigenvalue of $\text{Res}_{D_j} \Omega$ is a rational number.

We write the eigenvalue as

$$\frac{a_{j1}}{e_j}, \ldots, \frac{a_{jr}}{e_j},$$

(2)
where $e_j \geq 2$ and $a_{j\nu}$ are integers such that

$$(a_{j1}, \ldots, a_{jr}, e_j) = 1,$$

where $(\ast, \ldots, \ast)$ denotes the GCD of the components.

Suppose moreover (v) if $a_{j\nu} \equiv a_{j\mu} \pmod{e_j}$, then $a_{j\nu} = a_{j\mu}$.

For a point $q \in \text{Sing } B$, $\text{Res}_{D_j} \Omega$ $(1 \leq j \leq t)$ under the notations in §2 are mutually commutative. Hence they can be simultaneously diagonalizable:

$$P(\text{Res}_{D_j} \Omega)P^{-1} = \begin{pmatrix}
a_{j1}/e_j & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & a_{jr}/e_j & 0
\end{pmatrix}$$

for $1 \leq j \leq t$, where $P$ is a non-singular matrix.

Let

$$\Delta_1, \ldots, \Delta_N \quad (N = \binom{r}{t}).$$

be the $(t \times t)$-minors of the $(t \times r)$-matrix $(a_{j\nu})$. Put

$$f_j = e_j < e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_t > / < e_1, \ldots, e_t >$$

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for \( 1 \leq j \leq t \), where \(<*, ..., *>\) denotes the LCM of the components.

**Theorem 4.** Under the above notations and assumptions (i) - (v), suppose moreover (vi) \((\Delta_1, ..., \Delta_N, f_j) = 1\)
\(1 \leq j \leq t\) for every point \(q \in \text{Sing } B\). Then there exists a finite Galois covering \(\pi : X \to M\) with non-singular \(X\) which branches along \(D = e_1D_1 + ... + e_sD_s\).

For the proof of Theorem 4, we make use of the following theorem of Selberg [9] (see also Borel [1]).

**Theorem 5 (Selberg).** Let \(\Gamma\) be a finitely generated subgroup of \(\text{GL}(r, \mathbb{C})\). Then there exists a normal subgroup \(\Gamma_0\) of \(\Gamma\) of finite index and of torsion free. If \(\Gamma \neq \{1\}\), then \(\Gamma_0\) can be so chosen that \(\Gamma_0 \neq \Gamma\).

**Proof of Theorem 4.** Let \(R : \pi_1(M - B, q_0) \to \text{GL}(r, \mathbb{C})\) be the monodromy representation of the equation (1) and put
\(\Gamma = R(\pi_1(M - B, q_0))\). Let \(\Gamma_0\) be a normal subgroup of \(\Gamma\) of finite index and of torsion free in Theorem 5. Put \(K = R^{-1}(\Gamma_0)\).
Then \( K \) is a normal subgroup of \( \pi_1(M - B, q_0) \) of finite index.

We show that \( K \) satisfies the conditions of Theorem 2.

We first show that \( K \) contains \( J \). By Theorem 3 and by the assumption (v),

\[
R(\gamma_j^{e_j}) = R(\gamma_j)^{e_j} = (\exp 2\pi i \text{Res}_{D_j} \Omega)^{e_j}
\]

\[
= \exp 2\pi i \text{Res}_{D_j} \Omega \\
= \exp 2\pi i \begin{pmatrix}
    a_{jl} & 0 \\
    0 & \ddots & \ddots \\
    0 & \ddots & a_{jr}
\end{pmatrix} = 1,
\]

where \( \sim \) means the conjugacy relation. Hence

\[
\gamma_j^{e_j} \in \ker(R) \subseteq K,
\]

so

\( J \subseteq K \).

Next, suppose \( \gamma_j^d \in K \). Then \( R(\gamma_j)^d \in \Gamma_0 \). Note that

\[
(R(\gamma_j)^d)^{e_j} = (R(\gamma_j)^{e_j})^d = 1.
\]

Since \( \Gamma_0 \) is torsion free, we have \( R(\gamma_j)^d = 1 \). This means that

//
$d_{aj_1}/e_j, \ldots, d_{aj_r}/e_j$

are integers. Hence $d \equiv 0 \pmod{e_j}$. This holds for $1 \leq j \leq s$.

Finally, for a point $q \in \text{Sing } B$, suppose $\hat{\gamma}_1^{d_1} \ldots \hat{\gamma}_t^{d_t} \in K$, under the notations in §2. Then $R(\hat{\gamma}_1)^{d_1} \ldots R(\hat{\gamma}_t)^{d_t} \in \Gamma_0$. Note that

$$(R(\hat{\gamma}_1)^{d_1} \ldots R(\hat{\gamma}_t)^{d_t})^{e_1} \ldots e_t = 1.$$ 

Since $\Gamma_0$ is torsion free, we have $R(\hat{\gamma}_1)^{d_1} \ldots R(\hat{\gamma}_t)^{d_t} = 1$.

This means that

$$d_1a_{l_1}/e_1 + \ldots + d_ta_{t_1}/e_t$$

.................................

$$d_1a_{l_r}/e_1 + \ldots + d_ta_{t_r}/e_t$$

are integers. Now the assumption (vi) implies easily that

$$d_1 \equiv 0 \pmod{e_1}, \ldots, d_t \equiv 0 \pmod{e_t}.$$ 

Q.E.D.
5. Application to Appell's $F_1$. Let $(Z_0 : Z_1 : Z_2)$ be a homogeneous coordinate system on the complex projective plane $\mathbb{P}^2$ and let $(x, y) = (Z_1/Z_0, Z_2/Z_0)$ be the affine coordinate system. Appell's hypergeometric differential equation $F_1$ can be written as

$$d(f, x^{3/3} f, y^{3/3} f) = (f, x^{3/3} f, y^{3/3} f) \Omega_o,$$

where

$$\Omega_o = A\frac{dx}{x} + B\frac{dy}{y} + C\frac{dx}{x-1} + D\frac{dy}{y-1} + E\frac{d(x-y)}{x-y},$$

with

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1-\gamma+\beta' & -\beta' \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\beta & 1-\gamma+\beta \\ 1 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & -\alpha\beta & 0 \\ 0 & \gamma-\alpha-\beta-1 & 0 \\ 0 & -\beta & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & -\alpha\beta' \\ 0 & 0 & -\beta' \\ 0 & 0 & \gamma-\alpha-\beta'-1 \end{pmatrix}$$
\[
E = \begin{pmatrix}
0 & 0 & 0 \\
0 & -\beta' & \beta' \\
0 & \beta & -\beta
\end{pmatrix},
\]

where \(\alpha, \beta, \beta',\) and \(\gamma\) are constants, (see Kimura [7]).

\(\Omega_0\) is a \((3 \times 3)\)-matrix-valued meromorphic 1-form on \(\mathbb{F}^2\) which is holomorphic on \(\mathbb{F}^2 - B_0\), where

\[
B_0 = D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \cup D_6,
\]

\[
D_1 = \{x = 0\}, \quad D_2 = \{y = 0\},
\]

\[
D_3 = \{x = 1\}, \quad D_4 = \{y = 1\},
\]

\[
D_5 = \{x = y\}, \quad D_6 = \text{the line at infinity}.
\]

\[\text{Figure 3}\]
Consider the blowing up $\rho : M = \mathbb{P}^2 \rightarrow \mathbb{P}^2$ at the four points
$p_0 = (1 : 0 : 0), p_1 = (0 : 1 : 0), p_2 = (0 : 0 : 1), p_3 = (1 : 1 : 1)$

![Figure 4](image)

Then the differential equation

$$dY = Y\Omega$$ \hspace{1cm} (3)

on $\mathbb{P}^2$, where $\Omega = \rho^*\Omega_0$, is Fuchsian with regular singularity along

$$B = D_1 \cup \ldots \cup D_{10},$$

where $D_7, D_8, D_9, D_{10}$ are exceptional curves as in Figure 4.
(By abuse of notation, we write the strict transform of $D_j$ as $D_j$ again for $1 \leq j \leq 6$). Note that $B$ is simple normally crossing.

We apply Theorem 4 to the equation (3). Suppose that $\alpha, \beta, \beta'$ and $\gamma$ are rational numbers and suppose

\[
\begin{align*}
1 - \gamma + \beta' &= a_1/e_1, \text{ where } e_1 \geq 2 \text{ and } (a_1, e_1) = 1, \\
1 - \gamma + \beta &= a_2/e_2, \text{ where } e_2 \geq 2 \text{ and } (a_2, e_2) = 1, \\
\gamma - \alpha - \beta - 1 &= a_3/e_3, \text{ where } e_3 \geq 2 \text{ and } (a_3, e_3) = 1, \\
\gamma - \alpha - \beta' - 1 &= a_4/e_4, \text{ where } e_4 \geq 2 \text{ and } (a_4, e_4) = 1, \\
-\beta - \beta' &= a_5/e_5, \text{ where } e_5 \geq 2 \text{ and } (a_5, e_5) = 1, \\
1 - \gamma &= a_7/e_7, \text{ where } e_7 \geq 2 \text{ and } (a_7, e_7) = 1, \\
\gamma - 1 - \alpha - \beta - \beta' &= a_{10}/e_{10}, \text{ where } e_{10} \geq 2 \text{ and } (a_{10}, e_{10}) = 1, \\
\beta + \beta' - \alpha &= a_6/e_6, \text{ where } e_6 \geq 2 \text{ and } (a_6, b_6, e_6) = 1, \\
\beta - \alpha &= a_8/e_8, \text{ where } e_8 \geq 2 \text{ and } (a_8, b_8, e_8) = 1, \\
\beta' - \alpha &= a_9/e_9, \text{ where } e_9 \geq 2 \text{ and } (a_9, b_9, e_9) = 1.
\end{align*}
\]

Then, after some simple calculations, we have by

Theorem 4:
Theorem 6. Under the above notations and assumptions, suppose moreover that \( a_j \equiv 0 \pmod{e_j} \) for \( j = 6, 8, 9 \), and \((a_6, e_6, a_8, e_8) = 1\) and \((a_6, e_6, a_9, e_9) = 1\). Then there exists a finite Galois covering \( \pi: X \to \mathbb{F}^2 \) with non-singular \( X \) which branches along \( D = e_1D_1 + \ldots + e_{10}D_{10} \).

There are a lot of examples in which the conditions of Theorem 6 are satisfied. We give three such examples.
(The numbers attached to the curves in the figures below mean the ramification indices along the curves.)

Example 1. \( \alpha = 3 - 1/a, \beta = \beta' = 1 - 1/4a \) and \( \gamma = 3 - 3/4a \), where \( a \geq 2 \) and \( (a, 3) = 1 \).

![Figure 5](image-url)
Example 2. $\alpha = 3 - \frac{1}{3}a$, $\beta = \beta' = 1 - \frac{1}{12}a$ and $\gamma = 3 - \frac{1}{4}a$, where $a$ is a positive integer.

Figure 6

Example 3. $\alpha = 3 - \frac{4}{3}a$, $\beta = \beta' = 1 - \frac{1}{3}a$ and $\gamma = 3 - \frac{1}{a}$, where $a \geq 3$ and $(a, 2) = 1$.

Figure 7
References


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