

Branched Coverings of Complex Manifolds

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Introduction. The theory of branched coverings is one of good examples of amalgamation of different branches of mathematics : topology, complex analysis and algebraic geometry. See, for example, Zariski [31], Fox [6], Kato [17], Hirzebruch [11], Höfer [13], Ishida [15], Fukui [7], Gaffney-Lazarsfeld [8], etc..

It need not to be mentioned that the theory of (Galois) branched coverings is a geometric counterpart of the Galois theory of function fields.

In this article, we present a theory of Galois and abelian branched coverings of complex manifolds, emphasizing existence theorems and examples mainly along the line of Namba [22]. In the last section, we discuss the equivalence problem of Kummer coverings after Kato [18].

## Chapter 1. Galois Coverings.

1. Definition of Branched Coverings. First of all, we give a definition of branched coverings of complex manifolds. Since we treat infinite coverings as well as finite coverings, we define branched coverings as follows:

Definition 1.1. Let  $M$  be an  $n$ -dimensional connected complex manifold. A branched covering of  $M$  is an irreducible normal complex space  $X$  together with a surjective holomorphic mapping  $\pi : X \rightarrow M$  satisfying the following 4 conditions:

- i) Every fiber of  $\pi$  is discrete.
- ii)  $R_\pi = \{p \in X \mid \pi^* : \mathcal{O}_{M, \pi(p)} \rightarrow \mathcal{O}_{X, p} \text{ is not isomorphic}\}$  and  $B_\pi = \pi(R_\pi)$  are hypersurfaces (i.e., pure codimension 1) of  $X$  and  $M$ , respectively, called the ramification locus and the branch locus of  $\pi$ , respectively. (Here,  $\mathcal{O}_{X, p}$  is the local ring of germs of holomorphic functions around  $p$ .)
- iii)  $\pi : X - \pi^{-1}(B_\pi) \rightarrow M - B_\pi$  is a topological (i.e., unbranched) covering.
- iv) For every point  $q \in B_\pi$ , there is an open neighborhood  $W$  of  $q$  in  $M$  such that, for every connected component  $U$  of

$\pi^{-1}(W)$ ,  $\pi^{-1}(q) \cap U$  consists of one point and  $\pi|_U : U \rightarrow W$  is a surjective proper mapping (hence a finite mapping).

If  $R_\pi$  is empty, then  $\pi : X \rightarrow M$  should be called an unbranched covering. But we call such a covering also a branched covering by abuse of language. A branched covering is said to be finite if every fiber is a finite set. The mapping degree of  $\pi : X - \pi^{-1}(B_\pi) \rightarrow M - B_\pi$  is called the degree of  $\pi$ . Using the purity of branch loci (see Fischer [4]), we have easily

Proposition 1. 2. An irreducible normal complex space  $X$  together with a surjective finite proper holomorphic mapping  $\pi : X \rightarrow M$  is a finite branched covering, and vice versa.

Let  $\pi : X \rightarrow M$  and  $\pi' : X' \rightarrow M$  be branched coverings of  $M$ . A morphism of  $\pi$  to  $\pi'$  is, by definition, a surjective holomorphic mapping  $\phi : X \rightarrow X'$  such that  $\pi' \circ \phi = \pi$ . Thus we have the category of branched coverings of  $M$ .  $\phi$  is an isomorphism if  $\phi : X \rightarrow X'$  is biholomorphic. In this case, we say that  $\pi$  and  $\pi'$  are isomorphic. In particular, if  $X = X'$  and  $\pi = \pi'$ , then an isomorphism is called a covering transformation of  $\pi$ . The set  $G_\pi$  of all covering transformations of  $\pi$  forms a group under compositions, called the covering transformation group.  $G_\pi$  acts on every fiber of  $\pi$ . A branched covering  $\pi : X \rightarrow M$  is called a Galois covering if  $G_\pi$  acts transitively on every fiber.  $\pi : X \rightarrow M$  is called an abelian (resp. a cyclic) covering if  $\pi$  is a Galois covering and  $G_\pi$  is an abelian (resp. a cyclic) group.

We denote by  $\text{Sing } B_\pi$  the singular locus of the branch

locus  $B_\pi$ . It can be shown that, for every point  $q \in B_\pi - \text{Sing}B_\pi$ , every point  $p \in \pi^{-1}(q)$  is a non-singular point of both  $X$  and  $\pi^{-1}(B_\pi)$ . Moreover, for any sufficiently small open neighborhood  $W$  of  $q$  with a coordinate system  $(w_1, \dots, w_n)$  such that  $q = (0, \dots, 0)$  and  $B_\pi \cap W = \{w_n = 0\}$ , there is an open neighborhood  $U$  of  $p$  with a coordinate system  $(z_1, \dots, z_n)$  such that  $U$  is a connected component of  $\pi^{-1}(W)$ ,  $p = (0, \dots, 0)$  and  $\pi$  is locally given by

$$\pi|_U : (z_1, \dots, z_n) \longrightarrow (w_1, \dots, w_n) = (z_1, \dots, z_{n-1}, z_n^e),$$

where  $e$  is a positive integer, (see Roan [25] and Namba [22]). For an irreducible component  $C$  of  $\pi^{-1}(B_\pi)$ , the integer  $e$  is constant for points of  $C - \pi^{-1}(\text{Sing}B)$ , and is called the ramification index of  $\pi$  along  $C$ . (For convenience, the ramification index of  $\pi$  along an irreducible hypersurface of  $X$  which is not contained in  $\pi^{-1}(B_\pi)$  is defined to be 1.) If  $\pi$  is a Galois covering, then, for any irreducible component  $D_1$  of  $B_\pi$ , the ramification index  $e$  of  $\pi$  along irreducible components of  $\pi^{-1}(D_1)$  is constant. In this case,  $e$  is called the ramification index of  $\pi$  along  $D_1$ .

Let a hypersurface  $B$  of  $M$  be given. Suppose for simplicity that  $B$  has a finite number of irreducible components  $D_1, \dots, D_s$ :

$$B = D_1 \cup \dots \cup D_s.$$

Let  $e_1, \dots, e_s$  be positive integers greater than one, and

$$D = e_1 D_1 + \dots + e_s D_s$$

be a positive divisor on  $M$ .

Definition 1.3. A branched covering  $\pi : X \longrightarrow M$  is

said to branch along  $D$  (resp. at most along  $D$ ) if (i)  $B_\pi = B$  (resp.  $B_\pi \subset B$ ) and (ii) for every  $j$  ( $1 \leq j \leq s$ ) and for every irreducible component  $C$  of  $\pi^{-1}(D_j)$ , the ramification index of  $\pi$  along  $C$  is  $e_j$  (resp. divides  $e_j$ ).

For branched coverings  $\pi : X \rightarrow M$  and  $\pi' : X' \rightarrow M$  of  $M$ , we denote  $\pi \geq \pi'$  or  $\pi' \leq \pi$  if there is a morphism of  $\pi$  to  $\pi'$ . If  $\pi \geq \pi'$  and  $\pi$  branches at most along  $D$ , then  $\pi'$  branches at most along  $D$ . If  $\pi$  is a Galois covering,  $\pi \geq \pi'$  and  $\pi \leq \pi'$ , then  $\pi$  and  $\pi'$  are isomorphic.

Definition 1.4. A Galois covering  $\pi : X \rightarrow M$  is called a D-universal covering if (i)  $\pi$  branches along  $D$  and (ii) for any covering  $\pi' : X' \rightarrow M$  which branches at most along  $D$ , the relation  $\pi \geq \pi'$  holds.

By the above remark, a D-universal covering is unique up to isomorphisms, if it exists. We denote it by

$$\tilde{\pi} : \tilde{M}(D) \rightarrow M.$$

We now propose the following two problems:

Problem 1. When does a D-universal covering exist?

Problem 2. When does a finite Galois covering which branches along  $D$  exist?

As for a compact Riemann surface  $M$ , the problems were answered completely by Bundgaard-Nielsen [1] and Fox [5]:

Theorem 1. 5. Let  $M$  be a compact Riemann surface of genus  $g$ ,  $p_1, \dots, p_s$  be points of  $M$ ,  $e_1, \dots, e_s$  be positive integers greater than 1, and  $D = e_1 p_1 + \dots + e_s p_s$  be a positive divisor on  $M$ . Then the following three conditions are equivalent:

(i) There does not exist a  $D$ -universal covering of  $M$ .

(ii) There does not exist a finite Galois covering  $\pi : X \rightarrow M$  which branches along  $D$ .

(iii) Either (iii-1)  $g = 0$  and  $s = 1$  or (iii-2)  $g = 0$ ,  $s = 2$  and  $e_1 \neq e_2$ .

Example 1. 6. If  $M$  is a compact Riemann surface and  $\tilde{\pi} : \tilde{M}(D) \rightarrow M$  exists, then  $\tilde{\pi}$  is an infinite covering, unless  $M = \tilde{M}(D) = \mathbb{P}^1$ , the complex projective line, and  $\tilde{\pi}$  is isomorphic to one of the following rational functions, (see Klein [20], Hochstadt [12]):

$$(1) \quad w = z^m \quad (m = 1, 2, \dots),$$

$$D = m(\infty) + m(0), \quad \tilde{G} \simeq C_m \quad (m\text{-th cyclic group}).$$

$$(2) \quad w = -(z^m - 1)^2 / 4z^m,$$

$$D = m(\infty) + 2(0) + 2(1), \quad \tilde{G} \simeq D_m \quad (m\text{-th dihedral group}).$$

$$(3) \quad w = (z^4 + 2\sqrt{3}z^2 - 1)^3 / (z^4 - 2\sqrt{3}z^2 - 1)^3,$$

$$D = 3(\infty) + 3(0) + 2(1), \quad \tilde{G} \simeq A_4.$$

$$(4) \quad w = (z^8 + 14z^4 + 1)^3 / 108z^4(z^4 - 1)^4,$$

$$D = 4(\infty) + 3(0) + 2(1), \quad \tilde{G} \simeq S_4.$$

$$(5) \quad w = \frac{(z^{20} - 228z^{15} + 494z^{10} + 228z^5 + 1)^3}{-1728z^5(z^{10} + 11z^5 - 1)^5}$$

$$D = 5(\infty) + 3(0) + 2(1), \quad \tilde{G} \simeq A_5.$$

(Here  $(\alpha)$  is the point divisor of  $\alpha \in \mathbb{P}^1$ ,  $\tilde{G} = G_{\tilde{\pi}}$  and  $A_n$  (resp.  $S_n$ ) is the alternating (resp. symmetric) group of  $n$  letters.)

2. D-universal coverings. In this section, we give answers to the problems at the end of §1, using language of fundamental groups.

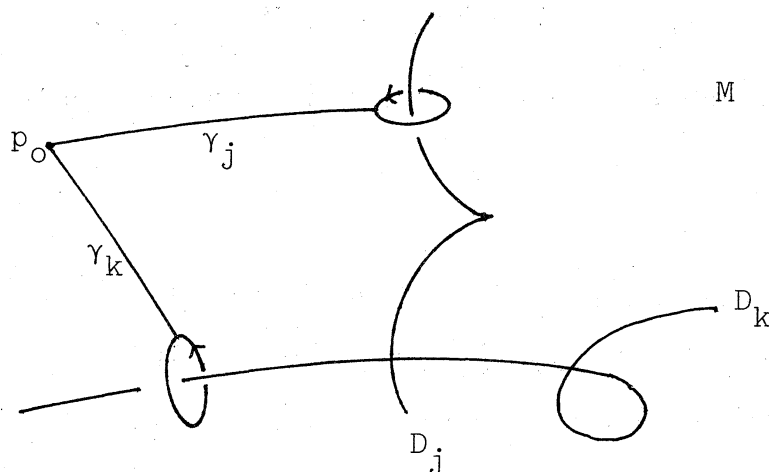


Figure 1

Take a point  $p_0 \in M - B$  and fix it once for all. Let  $\gamma_j$  be a loop in  $M - B$  starting and terminating at  $p_0$ , encircling a point  $p \in D_j - \text{Sing}B$  in the positive sense as in Figure 1.

$\gamma_j$  is called a normal loop of  $D_j$ . We identify  $\gamma_j$  with its homotopy class in  $\pi_1(M - B, p_0)$ . Let

$$J = \langle \gamma_1^{e_1}, \dots, \gamma_s^{e_s} \rangle_{\pi_1}$$

be the smallest normal subgroup of  $\pi_1(M - B, p_0)$  which contains  $\gamma_1^{e_1}, \dots, \gamma_s^{e_s}$ .

Definition 2. 1. A subgroup  $K$  of  $\pi_1(M - B, p_0)$  with  $J \subset K$  is said to be D-faithful if the following condition is satisfied: If  $\gamma_j^d$  belongs to  $K$ , then  $d \equiv 0 \pmod{e_j}$  for every  $j$  ( $1 \leq j \leq s$ ).

For every point  $p \in \text{Sing} B$ , take a sufficiently small ball  $W$  (with respect to a metric on  $M$ ) with the center  $p$  such that

$$\pi_1(W - B) \simeq \pi_{1, \text{loc}, p}(M - B), \text{ (the local fundamental group at } p\text{).}$$

Let

$$i_* : \pi_1(W - B) \longrightarrow \pi_1(M - B, p_0)$$

be the homomorphism induced by the inclusion mapping  $i : W - B \hookrightarrow M - B$ .

Definition 2. 2. A subgroup  $K$  of  $\pi_1(M - B, p_0)$  with  $J \subset K$  is said to be locally cofinite if  $i_*^{-1}(K)$  is a subgroup of  $\pi_1(W - B)$  of finite index for every point  $p \in \text{Sing} B$ .

Theorem 2. 3. For any covering  $\pi : X \longrightarrow M$  which branches at most along  $D$ ,  $K = \pi_*(\pi_1(X - \pi^{-1}(B)))$  contains  $J$  and is locally cofinite. Conversely, for any locally cofinite subgroup  $K (\supset J)$  of  $\pi_1(M - B, p_0)$ , there exists a unique (up to isomorphisms) covering  $\pi : X \longrightarrow M$  which branches at most along  $D$  such that  $\pi_*(\pi_1(X - \pi^{-1}(B))) = K$ . In this case,  $\pi$  branches along  $D$  if and only if  $K$  is  $D$ -faithful.

For the proof of the converse, we construct a topological covering  $\pi' : X' \longrightarrow M - B$  such that  $K = \pi_*^1(\pi_1(X'))$ , and then we extend  $\pi'$  to

$$\pi : X \longrightarrow M$$

using a theorem in Grauert-Remmert [9], (see also Grothendieck-Raynaud [10], p.340). Topologically, this is so called a Fox completion, (see Fox[6]). See Namba [22] for detail. By



Theorem 2.3,

Theorem 2.4. There exists a finite Galois covering  $\pi : X \rightarrow M$  which branches along  $D$  if and only if there exists a normal subgroup  $K$  of  $\pi_1(M - B, p_0)$  of finite index which contains  $J$  and is  $D$ -faithful. The correspondence  $\pi \rightarrow K = \pi_*(\pi_1(X - \pi^{-1}(B)))$  between (isomorphism classes of) such  $\pi$ 's and such  $K$ 's is one-to-one. In this case,  $G_\pi$  is isomorphic to  $\pi_1(M - B, p_0)/K$ .

In fact, for such a normal subgroup  $K$ , we have

$$\frac{\pi_1(W-B)}{i_*^{-1}(K)} \sim \frac{i_*(\pi_1(W-B))}{K \cap i_*(\pi_1(W-B))} \sim \frac{K \cdot i_*(\pi_1(W-B))}{K} \subset \frac{\pi_1(M-B, p_0)}{K}$$

under the above notation. Hence  $K$  is necessarily locally cofinite.

Now, put

$$\tilde{K} = \bigcap K,$$

where the intersection  $\bigcap$  runs over all subgroups  $K$  of  $\pi_1(M - B, p_0)$  which contain  $J$  and are locally cofinite.  $\tilde{K}$  is then a normal subgroup of  $\pi_1(M - B, p_0)$  which contains  $J$ .

Theorem 2.5. A  $D$ -universal covering  $\tilde{\pi} : \tilde{M}(D) \rightarrow M$  exists if and only if  $\tilde{K}$  is locally cofinite and  $D$ -faithful. In this case,  $\tilde{K} = \tilde{\pi}_*(\pi_1(\tilde{M}(D) - \tilde{\pi}^{-1}(B)))$  and  $G_{\tilde{\pi}} \sim \pi_1(M - B, p_0)/\tilde{K}$ . Moreover,  $\tilde{M}(D)$  is simply connected.

It is easy to see that  $\tilde{M}(D)$  is simply connected. In fact, if  $\mu : \tilde{X} \rightarrow \tilde{M}(D)$  is a (topological) universal covering

of  $\tilde{M}(D)$ , then  $\tilde{\pi} \cdot \mu : \tilde{X} \rightarrow M$  is a covering which branches along  $D$  such that  $\tilde{\pi} \cdot \mu \geq \tilde{\pi}$ . By the  $D$ -universality of  $\tilde{\pi}$ , we have  $\tilde{\pi} \cdot \mu \leq \tilde{\pi}$ . Hence  $\mu$  is an isomorphism.

Theorem 2. 6. Let  $\pi : X \rightarrow M$  be a Galois covering which branches along  $D$ . Suppose that  $X$  is non-singular and simply connected. Then  $\pi$  is  $D$ -universal. In this case,  $\tilde{K} = J$  and  $G_\pi = \pi_1(M - B, p_0)/J$ .

In this theorem, the condition of the non-singularity of  $X$  can not be dropped, as the following example shows:

Example 2. 7. Put  $M = \mathbb{C}^2$  and let  $(u, v)$  be the coordinate system on  $\mathbb{C}^2$ . Put  $D_1 = \{u = 0\}$ ,  $D_2 = \{v = 0\}$  and  $D = 2D_1 + 2D_2$ . Put  $X = \{(u, v, w) \in \mathbb{C}^3 \mid w^2 = uv\}$  and

$$\pi : (u, v, w) \in X \mapsto (u, v) \in \mathbb{C}^2.$$

Then  $\pi$  is a cyclic covering of degree 2 which branches along  $D$ .  $X$  is simply connected, for  $X$  is a cone. But  $\pi$  is not  $D$ -universal. In fact, putting  $Y = \mathbb{C}^2$  and

$$\mu : (x, y) \in Y \mapsto (u, v, w) = (x^2, y^2, xy) \in X,$$

the composition  $\pi \cdot \mu : Y \rightarrow \mathbb{C}^2$  is a covering of degree 4 which branches along  $D$  and  $\pi \cdot \mu \geq \pi$ . (By Theorem 2. 6,  $\pi \cdot \mu$  is  $D$ -universal.)

For the rest of this section, we suppose that  $B$  is simple normally crossing.

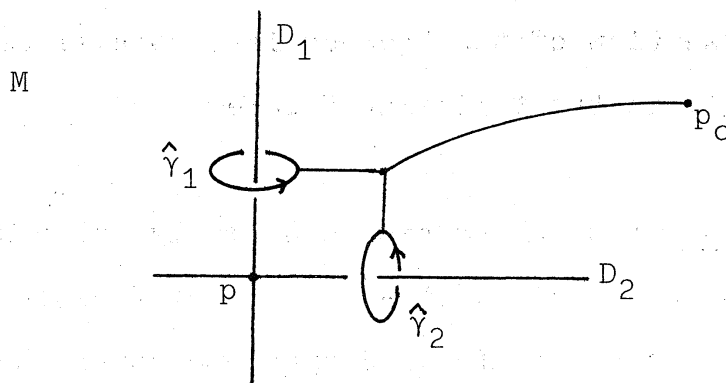


Figure 2

For any point  $p \in \text{Sing}B$ , let  $(w_1, \dots, w_n)$  be a local coordinate system around  $p$  such that  $p = (0, \dots, 0)$  and

$$B = \{(w_1, \dots, w_n) \mid w_k = \dots = w_n = 0\}.$$

locally. Let

$$\{w_j = 0\} = D_j \quad (k \leq j \leq n),$$

locally, say. Let  $\hat{\gamma}_j$  be a loop in  $M - B$  starting and terminating at  $p_0$ , encircling a point of  $D_j - \text{Sing}B$  near  $p$  in the positive sense as in Figure 2. Then  $\hat{\gamma}_j$  is conjugate to  $\gamma_j$  in  $\pi_1(M - B, p_0)$ .  $\hat{\gamma}_k, \dots, \hat{\gamma}_n$  are mutually commutative. For a sufficiently small ball  $W$  with the center  $p$ , we have

$$\pi_1(W - B) = (\hat{\gamma}_k)^{\mathbb{Z}} \cdots (\hat{\gamma}_n)^{\mathbb{Z}}$$

and

$$(\hat{\gamma}_k^{e_k})^{\mathbb{Z}} \cdots (\hat{\gamma}_n^{e_n})^{\mathbb{Z}} \subset i_*^{-1}(J) \subset \pi_1(W - B).$$

Hence  $J$  is locally cofinite, so that  $\tilde{K} = J$ . Thus

Theorem 2.8. If  $B$  is simple normally crossing, then a  $D$ -universal covering  $\tilde{\pi} : \tilde{M}(D) \rightarrow M$  exists if and only if  $J$  is  $D$ -faithful. In this case,  $J = \tilde{K}$  and  $G_{\tilde{\pi}} \cong \pi_1(M - B, p_0)/J$ . Moreover, if (under the above notation),  $(\hat{\gamma}_k^{e_k})^{\mathbb{Z}} \cdots (\hat{\gamma}_n^{e_n})^{\mathbb{Z}} = i_*^{-1}(J)$  for every point  $p \in \text{Sing}B$ , then  $\tilde{M}(D)$  is non-singular.

The last assertion of the theorem is a special case of Kato [17], as well as the following theorem.

Theorem 2. 9. Let  $B$  be simple normally crossing. Let  $K$  be a normal subgroup of  $\pi_1(M - B, p_0)$  of finite index which contains  $J$  and is  $D$ -faithful. Suppose moreover that, for any point  $p \in \text{Sing}B$ ,  $K$  satisfies the following condition :

(under the above notation)

$$\text{if } \hat{\gamma}_k^{d_k} \cdots \hat{\gamma}_n^{d_n} \in K, \text{ then } d_k \equiv 0 \pmod{e_k}, \cdots, d_n \equiv 0 \pmod{e_n}.$$

Then the irreducible normal complex space  $X$  is non-singular, where  $\pi : X \rightarrow M$  is the finite Galois covering which branches along  $D$  and corresponds to  $K$  under Theorem 2. 4.

3. Examples. It is not easy in general to apply the results of §2 to concrete examples. (Even the calculation of  $\pi_1(M - B, p_0)$  is not easy.) In this section, we discuss two examples.

Case 1. Put  $M = \mathbb{C}^2$ ,  $B = D_1 = \{(x, y) \in \mathbb{C}^2 \mid x^3 = y^2\}$ ,  $\ell :$  a positive integer greater than 1, and  $D = \ell D_1$ .

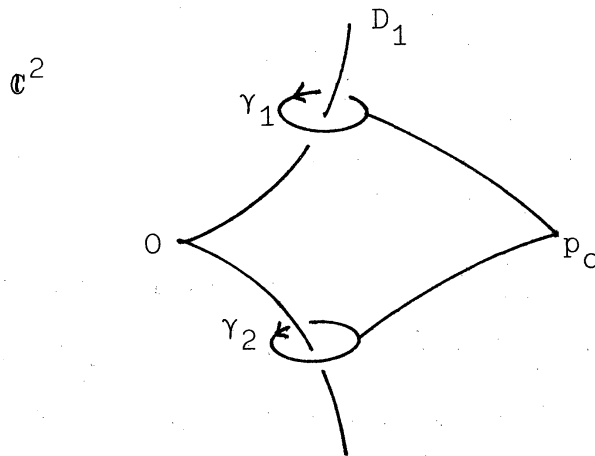


Figure 3

As is well known,  $\pi_1(\mathbb{C}^2 - B, p_0)$  is isomorphic to 3rd braid group  $B_3$ ; taking the loops  $\gamma_1$  and  $\gamma_2$  as in Figure 3, we have

$$\pi_1(\mathbb{C}^2 - B, p_0) = \langle \gamma_1, \gamma_2 \mid \gamma_1 \gamma_2 \gamma_1 = \gamma_2 \gamma_1 \gamma_2 \rangle.$$

Here the right hand side means the group generated by  $\gamma_1$  and  $\gamma_2$  with the generating relation  $\gamma_1 \gamma_2 \gamma_1 = \gamma_2 \gamma_1 \gamma_2$ . Since  $\gamma_2 = (\gamma_2 \gamma_1)^{-1} \gamma_1 (\gamma_2 \gamma_1)$ ,  $\gamma_2$  is conjugate to  $\gamma_1$ . Let  $J$  be the smallest normal subgroup of  $\pi_1(\mathbb{C}^2 - B, p_0)$  containing  $\gamma_1^\ell$  (and so  $\gamma_2^\ell$ ). Then

$$\pi_1(\mathbb{C}^2 - B, p_0)/J \cong G_\ell = \langle a, b \mid a^\ell = b^\ell = 1, aba = bab \rangle$$

by the correspondence:  $\gamma_1 \mapsto a, \gamma_2 \mapsto b$ . We identify these groups through the isomorphism.

The cyclic covering

$$\pi_\ell: X_\ell \longrightarrow M = \mathbb{C}^2,$$

corresponding to the kernel of the homomorphism

$$f_\ell: G_\ell \longrightarrow \mathbb{Z}/\ell\mathbb{Z}, (f_\ell(a) = f_\ell(b) = 1)$$

is given by

$$\begin{aligned} \pi_\ell: X_\ell = \{(x, y, z) \in \mathbb{C}^3 \mid z^\ell = x^3 - y^2\} &\longrightarrow M = \mathbb{C}^2 \\ (x, y, z) &\longmapsto (x, y) \end{aligned}$$

and branches along  $D = \ell D_1$ . But  $\pi_\ell$  is not  $D$ -universal.

The following argument on the structure of  $G_\ell$  was informed by Mr. Mizutani. See also Coxeter [2]. First of all,

Lemma 3.1.  $c = (aba)^2$  is an element of the center  $Z(G_\ell)$  of  $G_\ell$ .

Next, consider the Schwarz' triangular group

$$G(2, 3, \ell) = \langle S, T \mid S^2 = T^3 = (ST)^\ell = 1 \rangle,$$

and the homomorphism,

$$g : G_\ell \longrightarrow G(2, 3, \ell)$$

defined by  $g(a) = ST$  and  $g(b) = TS$ .

Proposition 3. 2. The following sequence is exact:

$$1 \longrightarrow \langle c \rangle \longrightarrow G_\ell \xrightarrow{g} G(2, 3, \ell) \longrightarrow 1$$

From this proposition, we have the following table:

$\ell$	ord(c)	$G(2, 3, \ell)$	$G_\ell$	ord $G_\ell$
2	1	$S_3$	$S_3$	6
3	2	$A_4$	$SL(2, \mathbb{Z}/3\mathbb{Z})$	24
4	4	$S_4$	$G_4/Z(G_4) \simeq S_4,$ $Z(G_4) \simeq \mathbb{Z}/4\mathbb{Z}$	96
5	10	$A_5$	$SL(2, \mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z})$	600
6	$\infty$	infinite group	infinite solvable group	$\infty$
$\geq 7$	$\infty$	infinite group	infinite unsolvable group	$\infty$

If  $\ell$  satisfies  $2 \leq \ell \leq 5$ , then (under the notations of

§2)  $\tilde{K} = J$  and there exists a D-universal covering

$$\tilde{\pi} : \tilde{M}(D) \longrightarrow M = \mathbb{C}^2.$$

In this case,  $\tilde{\pi}$  is a finite Galois covering such that  $G_{\tilde{\pi}} \simeq G_\ell$ .

Moreover, we have  $\tilde{M}(D) = \mathbb{C}^2$  and  $\tilde{\pi}$  is the composition

$$\tilde{M}(D) = \mathbb{C}^2 \xrightarrow{\mu} X_\ell \xrightarrow{\pi_\ell} M = \mathbb{C}^2,$$

where  $\mu$  is the projection

$$\mu : \tilde{M}(D) = \mathbb{C}^2 \longrightarrow X_\ell = \mathbb{C}^2/H,$$

where  $H$  is a finite subgroup of  $GL(2, \mathbb{C})$ . The origin of  $X_\ell$  in this case is called the Klein singularity, (see Pinkham [24]).

If  $\ell = 6$ , then we have

Proposition 3. 3. The kernel of  $f_6 : G_6 \longrightarrow \mathbb{Z}/6\mathbb{Z}$

( $f_6(a) = f_6(b) = 1$ ) is given by  $\langle a^{-1}b, ab^{-1} \rangle$  and is isomorphic

$$\text{to } N = \left\{ \begin{pmatrix} 1 & i & j \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \mid i, j, k \in \mathbb{Z} \right\}.$$

The isomorphism is given by

$$a^{-1}b \longmapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad ab^{-1} \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

We identify  $\ker(f_6)$  with  $N$  through the isomorphism. For any positive odd integer  $r$ ,

$$N(r) = \left\{ \begin{pmatrix} 1 & i & j \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \in N \mid i \equiv j \equiv k \equiv 0 \pmod{r} \right\}$$

is a normal subgroup of  $G_6$  of index  $6r^3$  and is  $D$ -faithful.

Hence

Proposition 3. 4. For any positive odd integer  $r$ , there is a Galois covering  $v_r : Y_r \longrightarrow M = \mathbb{C}^2$  of degree  $6r^3$  branching along  $6D_1$ .  $v_r \leq v_{r'}$  if and only if  $r|r'$ .

Since

$$\bigcap_{r:\text{odd}} N(r) = \{1\},$$

We have

Proposition 3.5. If  $D = 6D_1$ , then there does not exist a  $D$ -universal covering of  $M = \mathbb{C}^2$ .

Case 2. Put  $M = \mathbb{P}^2$  (the complex projective plane),  $B = D_1 \cup D_2$ ,  $D_1$  = the closure in  $\mathbb{P}^2$  of the affine curve  $\{(x, y) \in \mathbb{C}^2 \mid x^3 - y^2 = 0\}$ ,  $D_2 = L_\infty$  (the line at infinity),  $\ell, m$  : positive integers greater than 1, and  $D = \ell D_1 + m D_2$ .

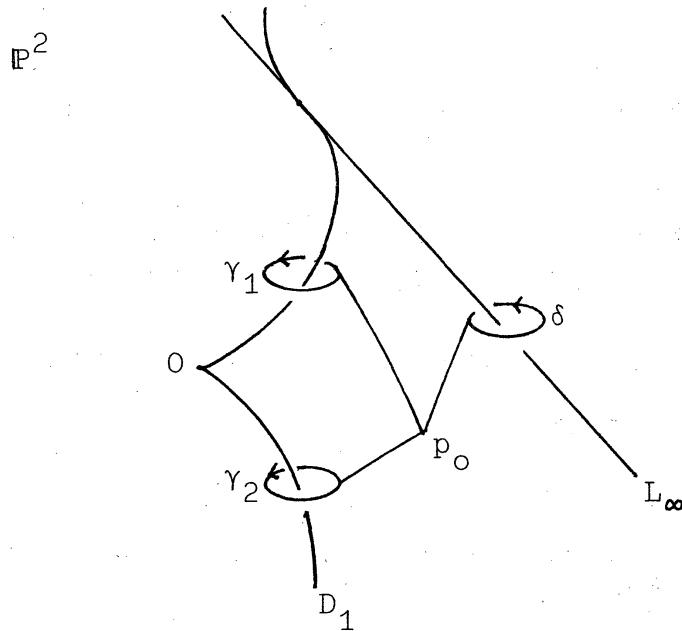


Figure 4

Taking the loops  $\gamma_1, \gamma_2$  and  $\delta$  as in Figure 4, we have

$$\pi_1(M - B, p_0) = \langle \gamma_1, \gamma_2, \delta \mid \gamma_1 \gamma_2 \gamma_1 = \gamma_2 \gamma_1 \gamma_2 = \delta^{-1} \rangle.$$

Let  $J$  be the smallest normal subgroup of  $\pi_1(M - B, p_0)$  which contains  $\gamma_1^\ell, \gamma_2^\ell$  and  $\delta^m$ . Then

$$\pi_1(\mathbb{P}^2 - B, p_0) / J \cong G_{\ell, m},$$



where

$$G_{\ell, m} = \langle \alpha, \beta, \delta \mid \alpha^\ell = \beta^\ell = \delta^m = 1, \alpha\beta\alpha = \beta\alpha\beta = \delta^{-1} \rangle,$$

$$(\gamma_1 \longmapsto \alpha, \gamma_2 \longmapsto \beta, \delta \longmapsto \delta).$$

Let  $G_\ell$  be the group in Case 1. There is a surjective homomorphism

$$h : G_\ell \longrightarrow G_{\ell, m}$$

defined by  $h(a) = \alpha$  and  $h(b) = \beta$ .

For simplicity, we assume that  $m$  is a positive even integer.

Then the following sequence is exact:

$$1 \longrightarrow \langle c^{m/2} \rangle \longrightarrow G_\ell \xrightarrow{h} G_{\ell, m} \longrightarrow 1.$$

In particular, if the pair  $(\ell, m)$  is one of the following table:

$\ell$	2	3	4	5
$m$	2	4	8	20

then  $G_{\ell, m} \cong G_\ell$  and there exists a  $D$ -universal covering

$$\tilde{\pi} : \tilde{M}(D) \longrightarrow M = \mathbb{P}^2 \quad (D = \ell D_1 + m D_2).$$

In this case,  $\tilde{\pi}$  is a finite Galois covering such that  $G_{\tilde{\pi}} \cong G_{\ell, m} \cong G_\ell$ .

If  $\ell = 6$ , then we have by Proposition 3: 3,

Proposition 3. 6. For any positive integer  $m$  such that  $m \equiv 2 \pmod{4}$ , there is a Galois covering  $\phi_m : Z_m \longrightarrow \mathbb{P}^2$  of degree  $6(m/2)^3$  branching along  $D = 6D_1 + mL_\infty$ .  $\phi_m \leq \phi_{m'}$  if and only if  $m|m'$ .

On the other hand, since the sequence

$$1 \longrightarrow \langle c \rangle / \langle c^{m/2} \rangle \longrightarrow G_{\ell, m} \longrightarrow G(2, 3, \ell) \longrightarrow 1$$

is exact, we have in particular (putting  $m = 2$ ),

$$G_{\ell, 2} \simeq G(2, 3, \ell).$$

Putting  $\ell = 2$ , we identify  $G_{6, 2}$  with  $G(2, 3, 6)$  through the isomorphism. It is well known that  $G(2, 3, 6)$  has the normal subgroup  $L$  such that

$$G(2, 3, 6)/L \simeq \mathbb{Z}/6\mathbb{Z},$$

$$L \simeq \mathbb{Z} \oplus \mathbb{Z} \quad (\text{the direct sum}).$$

Identifying  $L$  with  $\mathbb{Z} + \mathbb{Z}$  through the isomorphism, consider, for any positive integer  $q$ , the normal subgroup

$$L_q = \{(j, k) \in \mathbb{Z} \oplus \mathbb{Z} \mid j \equiv k \equiv 0 \pmod{q}\}$$

of index  $6q^2$  of  $G(2, 3, 6)$ . Since

$$\bigcap_q L_q = \{1\},$$

we have

Proposition 3. 7. If  $D = 6D_1 + 2L_\infty$ , then there does not exist a  $D$ -universal covering of  $\mathbb{P}^2$ .

By another method (see Namba [23]), we can show

Proposition 3. 8. For any positive integer  $k$ , there exists a finite Galois covering  $\pi : X \longrightarrow \mathbb{P}^2$  branching along  $D = 6kD_1 + 2kL_\infty$ .

4. Existence of Finite Galois Coverings. As for Problem 2 in §1, it is desirable to give (sufficient) conditions for the existence without using language of fundamental groups. Theorem 1.5 is such a theorem. In this section, we give such theorems.

Let  $L_1, \dots, L_s$  be distinct lines on  $\mathbb{P}^2$  and put  $B = L_1 \cup \dots \cup L_s$ . Put

$$\Delta = \{p \in B \mid m_p(B) \geq 3\},$$

where  $m_p(B)$  is the multiplicity at  $p$  of the curve  $B$ .  $\Delta$  is a finite point set.

Theorem 4. 1. Suppose that  $L_j \cap \Delta$  is non-empty for every  $j$  ( $1 \leq j \leq s$ ). Then, for any positive integers  $e_1, \dots, e_s$  greater than 1, there exists a finite Galois covering  $\pi : X \rightarrow \mathbb{P}^2$  branching along  $D = e_1 L_1 + \dots + e_s L_s$ .

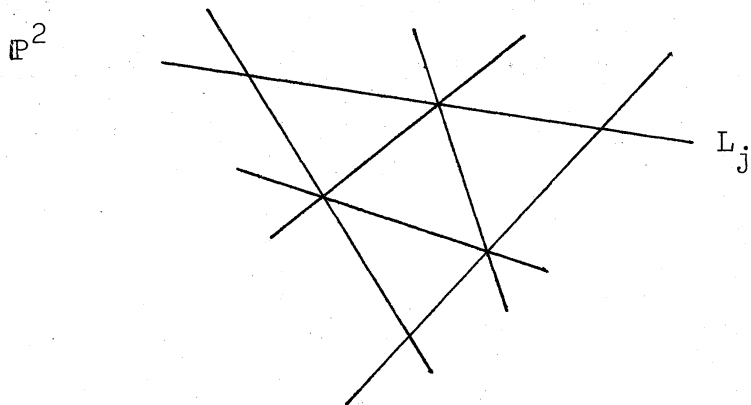


Figure 5

See Kato [16] for the proof of the theorem. We generalize the theorem as follows:

Theorem 4. 2. Let  $M$  be an  $n$  ( $\geq 2$ ) dimensional projective

manifold and  $D_1, \dots, D_s$  be distinct irreducible hypersurfaces of  $M$ . Suppose that there are fixed component free linear pencils  $\Lambda_1, \dots, \Lambda_t$  on  $M$  such that (i) every  $D_j$  is a member of some  $\Lambda_k$  and (ii) every  $\Lambda_k$  contains at least three  $D_j$ 's as its members. Then, for any positive integers  $e_1, \dots, e_s$  greater than 1, there exists a finite Galois covering  $\pi: X \rightarrow M$  branching along  $D = e_1 D_1 + \dots + e_s D_s$ .

Note that Theorem 4. 1 follows from Theorem 4. 2, putting  $M = \mathbb{P}^2$ ,  $D_j = L_j$  ( $1 \leq j \leq s$ ) and  $\Lambda_k =$  the linear pencil given by the projection with the center point  $p_k \in \Delta$ . See Namba [22] for the proofs of Theorem 4. 2 and the following theorem:

Theorem 4. 3. Let  $C_1, \dots, C_s$  be distinct irreducible conics on  $\mathbb{P}^2$  such that, for any  $C_j$ , there is a  $C_k$  which is tangent to  $C_j$  at two distinct points. Then, for any positive integers  $e_1, \dots, e_s$  greater than 1, there exists a finite Galois covering  $\pi: X \rightarrow \mathbb{P}^2$  branching along  $D = e_1 D_1 + \dots + e_s D_s$ .

$\mathbb{P}^2$

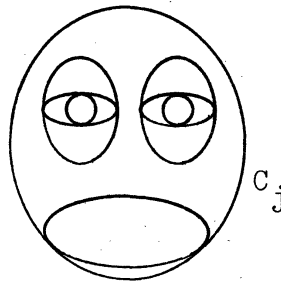


Figure 6

Chapter 2. Abelian Coverings.

5. Abelian D-universal coverings. Let  $M$  be an  $n$ -dimensional connected complex manifold. Let  $B = D_1 \cup \dots \cup D_s$ ,  $D = e_1 D_1 + \dots + e_s D_s$ ,  $p_0 \in M - B$ ,  $\gamma_j$  ( $1 \leq j \leq s$ ) and  $J = \langle \gamma_1^{e_1}, \dots, \gamma_s^{e_s} \rangle^{\pi_1}$  be as in §1 and §2. Put

$$\hat{J} = J \cdot [\pi_1(M-B, p_0), \pi_1(M-B, p_0)],$$

where  $[G, G]$  is the commutator subgroup of  $G$ . Then we can easily prove the following lemma.

Lemma 5. 1.  $\pi_1(M - B, p_0) / \hat{J} \cong H_1(M - B; \mathbb{Z}) / (\mathbb{Z}(e_1 \gamma_1) + \dots + \mathbb{Z}(e_s \gamma_s))$ .

Here  $H_1(M - B; \mathbb{Z})$  is the first homology group of  $M - B$  and  $\mathbb{Z}(e_1 \gamma_1) + \dots + \mathbb{Z}(e_s \gamma_s)$  is the subgroup of  $H_1(M - B; \mathbb{Z})$  generated by  $e_1 \gamma_1, \dots, e_s \gamma_s$ , which are regarded as elements of  $H_1(M - B; \mathbb{Z})$ .

Moreover, we can prove:

Proposition 5. 2.  $\hat{J}$  is a normal subgroup of  $\pi_1(M - B, p_0)$  which contains  $J$  and is locally cofinite.

The covering  $\pi_0 : X_0 \rightarrow M$  which branches at most along  $D$ , corresponding to  $\hat{J}$  by Theorem 2. 1 is an abelian covering. Moreover, for any abelian covering  $\pi : X \rightarrow M$  which branches at most along  $D$ , the relation  $\pi_0 \geq \pi$  holds.

Definition 5. 3. An abelian covering  $\tilde{\pi}_{ab} : \tilde{M}_{ab}(D) \rightarrow M$  is called an abelian D-universal covering if (i)  $\tilde{\pi}_{ab}$  branches

along  $D$  and (ii) for any abelian covering  $\pi : X \rightarrow M$  which branches at most along  $D$ , the relation  $\tilde{\pi}_{ab} \geq \pi$  holds.

By the above consideration, if an abelian  $D$ -universal covering  $\tilde{\pi}_{ab} : \tilde{M}_{ab}(D) \rightarrow M$  exists, then it must be isomorphic to  $\pi_0 : X_0 \rightarrow M$ . Conversely, if  $\pi_0 : X_0 \rightarrow M$  branches along  $D$ , then it is an abelian  $D$ -universal covering. Thus

Theorem 5. 4. There exists an abelian  $D$ -universal covering  $\tilde{\pi}_{ab} : \tilde{M}_{ab}(D) \rightarrow M$  if and only if the following condition is satisfied: if  $d\gamma_j \in \mathbb{Z}(e_1\gamma_1) + \cdots + \mathbb{Z}(e_s\gamma_s)$ , then  $d \equiv 0 \pmod{e_j}$  for every  $1 \leq j \leq s$ . In this case, the covering transformation group of  $\tilde{\pi}_{ab}$  is isomorphic to  $\tilde{G}_{ab} = H_1(M - B; \mathbb{Z}) / (\mathbb{Z}(e_1\gamma_1) + \cdots + \mathbb{Z}(e_s\gamma_s))$ .

For example, let  $M = \mathbb{C}^2$ ,  $B = D_1$  and  $D = \ell D_1$  be as in Case 1 of §3. Then we have  $H_1(\mathbb{C}^2 - B; \mathbb{Z}) = \mathbb{Z}\gamma_1$  and the condition in Theorem 5. 4 is clearly satisfied. In this case,  $\tilde{\pi}_{ab} : \tilde{\mathbb{C}}_{ab}^2(D) \rightarrow \mathbb{C}^2$  is nothing but the cyclic covering

$$\begin{aligned} \pi_\ell : X_\ell = \{(x, y, z) \in \mathbb{C}^3 \mid z^\ell = x^3 - y^2\} &\rightarrow \mathbb{C}^2 \\ (x, y, z) &\longmapsto (x, y) \end{aligned}$$

considered in Case 1 of §3.

Let  $M = \mathbb{P}^2$ ,  $B = D_1 \cup L_\infty$  and  $D = \ell D_1 + mL_\infty$  be as in Case 2 of §3. Then we have

$$H_1(\mathbb{P}^2 - B; \mathbb{Z}) = (\mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2) / \mathbb{Z}(3\gamma_1 + \gamma_2).$$

Hence the condition in Theorem 5. 4 is equivalent in this case to the condition:  $\ell / (3, \ell) = m$ , where  $(3, \ell)$  is the GCD of 3 and  $\ell$ . If this is the case,  $\tilde{\pi}_{ab} : \tilde{\mathbb{P}}_{ab}^2(D) \rightarrow \mathbb{P}^2$  is a finite

covering.

In general, if  $M = \mathbb{P}^n$ ,  $D_j$  is an irreducible hypersurface of degree  $d_j$  ( $1 \leq j \leq s$ ) and  $B = D_1 \cup \dots \cup D_s$ , then we have

Thus  $H_1(\mathbb{P}^n - B; \mathbb{Z}) = (\mathbb{Z}\gamma_1 + \dots + \mathbb{Z}\gamma_s) / (\mathbb{Z}(d_1\gamma_1 + \dots + d_s\gamma_s))$ .

Theorem 5. 5. Let  $D_j$  be distinct irreducible hypersurfaces of degree  $d_j$  ( $1 \leq j \leq s$ ) of the complex projective space  $\mathbb{P}^n$ .

Put  $D = e_1 D_1 + \dots + e_s D_s$ . Then there exists an abelian  $D$ -

universal covering  $\tilde{\pi}_{ab} : \tilde{\mathbb{P}}_{ab}^n(D) \rightarrow \mathbb{P}^n$  if and only if

$e_j / (d_j, e_j)$  divides

$$\langle e_1 / (d_1, e_1), \dots, e_{j-1} / (d_{j-1}, e_{j-1}), e_{j+1} / (d_{j+1}, e_{j+1}), \dots, e_s / (d_s, e_s) \rangle$$

for every  $j$  ( $1 \leq j \leq s$ ), where  $(\dots)$  and  $\langle \dots \rangle$  denote the GCD and LCM of the components, respectively. In this case,

$\tilde{\pi}_{ab}$  is a finite covering.

As for a compact Riemann surface  $M$ , Theorem 5. 4 can be rewritten as

Theorem 5. 6. Let  $p_j$  ( $1 \leq j \leq s$ ) be distinct points on a compact Riemann surface  $M$  of genus  $g$ . Put  $D = e_1 p_1 + \dots + e_s p_s$ . Then there exists an abelian  $D$ -universal covering

$\tilde{\pi}_{ab} : \tilde{M}_{ab}(D) \rightarrow M$  if and only if  $e_j$  divides

$$\langle e_1, \dots, e_{j-1}, e_{j+1}, \dots, e_s \rangle$$

for every  $j$  ( $1 \leq j \leq s$ ). In this case,  $\tilde{\pi}_{ab}$  is an infinite covering if  $g \geq 1$ .

Finally, as for finite abelian coverings of a complex manifold  $M$ , we have

Theorem 5. 7. Let  $M$  be a connected complex manifold,  $B = D_1 \cup \dots \cup D_s$ ,  $D = e_1 D_1 + \dots + e_s D_s$  and  $\gamma_j$  ( $1 \leq j \leq s$ ) be as before. Then there exists a one-to-one correspondence  $\pi \rightarrow K = K(\pi)$  between isomorphism classes of finite abelian coverings  $\pi : X \rightarrow M$  which branches at most along  $D$ , and subgroups  $K$  of finite index of

$$\tilde{G}_{ab} = H_1(M - B; \mathbb{Z}) / (\mathbb{Z}(e_1 \gamma_1) + \dots + \mathbb{Z}(e_s \gamma_s)).$$

The correspondence satisfies (1)  $G_\pi \cong \tilde{G}_{ab} / K(\pi)$ , (2)  $\pi_1 \leq \pi_2$  if and only if  $K(\pi_1) \supset K(\pi_2)$  and (3)  $\pi$  branches along  $D$  if and only if  $K(\pi)$  satisfies the following condition: if  $d\gamma_j \in K(\pi)$ , then  $d \equiv 0 \pmod{e_j}$  for  $1 \leq j \leq s$ .

6. Finite Abelian Coverings of Projective Manifolds. In this section, we suppose that  $M$  is a projective manifold. We discuss finite abelian coverings of  $M$ . Here are two typical examples of abelian coverings.

Example 6. 1. Let  $\hat{\pi} : L \rightarrow M$  be a holomorphic line bundle on  $M$  and  $\xi = \{\xi_\alpha\}$  be a holomorphic section of  $L^{\otimes e}$  (the  $e$ -times tensor product of  $L$  for a positive integer  $e$  greater than 1), where  $\xi_\alpha$  is a holomorphic function on an open set  $U_\alpha$  on which  $L$  is trivial. Suppose that the zero divisor  $(\xi)$  of  $\xi$  has no multiple component:

$$(\xi) = D_1 + \dots + D_s,$$

where  $D_j$  are distinct prime divisors. Put

$$D = e(\xi) = eD_1 + \dots + eD_s.$$



Put

$$X = \bigcup_{\alpha} \{(p, z_{\alpha}) \in U_{\alpha} \times \mathbb{C} \mid z_{\alpha}^e = \xi_{\alpha}(p)\}.$$

Then  $X$  can be considered as an irreducible normal hypersurface of the bundle space  $L$ . Put

$$\pi = \hat{\pi}|_X : X \longrightarrow M.$$

Then  $\pi$  is a cyclic covering which branches along  $D$ .

Example 6. 2. Let  $L$  be a holomorphic line bundle on  $M$  and  $\xi_1, \dots, \xi_s$  be holomorphic sections of  $L$ . Suppose that  $D_1 = (\xi_1), \dots, D_s = (\xi_s)$  are distinct prime divisors such that  $D_1 \cap \dots \cap D_s = \emptyset$ . For a positive integer  $e$  greater than 1, put

$$B = D_1 \cup \dots \cup D_s,$$

$$D = eD_1 + \dots + eD_s.$$

Consider the Kummer extension

$$F = \mathbb{C}(M)((\xi_1/\xi_s)^{1/e}, \dots, (\xi_{s-1}/\xi_s)^{1/e})$$

of the field  $\mathbb{C}(M)$  of meromorphic functions on  $M$ . Let

$$\pi : X \longrightarrow M$$

be the F-normalization of  $M$ , (see Iitaka [14]). Then  $\pi$  is a finite abelian covering of  $M$  which branches along  $D$  such that  $G_{\pi} \simeq (\mathbb{Z}/e\mathbb{Z})^{s-1}$ . The covering  $\pi : X \longrightarrow M$  is called a Kummer covering. In this case, we can prove that, if  $B$  is simple normally crossing, then  $X$  is non-singular.

Now, let  $B = D_1 \cup \dots \cup D_s$  and  $D = e_1 D_1 + \dots + e_s D_s$  be as in §1. We rewrite Theorem 5. 7 using language of rational divisors. A rational D-divisor is a rational divisor  $\hat{E}$  on  $M$

of the following type:

$$\hat{E} = (a_1/e_1)D_1 + \cdots + (a_s/e_s)D_s + E,$$

where  $a_j$  ( $1 \leq j \leq s$ ) are integers and  $E$  is an integral divisor. Rational D-divisors form an additive group  $\text{Div}^{\mathbb{Q}}(M, D)$ . Let  $\text{Div}_0^{\mathbb{Q}}(M, D)$  be the subgroup of  $\text{Div}^{\mathbb{Q}}(M, D)$  consisting of all  $\hat{E}$  such that

$$\begin{aligned} c_{\mathbb{Q}}(\hat{E}) &= (a_1/e_1)c_{\mathbb{Q}}([D_1]) + \cdots + (a_s/e_s)c_{\mathbb{Q}}([D_s]) + c_{\mathbb{Q}}(E) \\ &= 0 \in H^2(M; \mathbb{Q}), \end{aligned}$$

where  $[D_j]$  is the line bundle determined by  $D_j$  and  $c_{\mathbb{Q}}: \text{Pic}(M) \rightarrow H^2(M; \mathbb{Q})$  is the homomorphism of rational Chern class.

Two rational D-divisors  $\hat{E}$  and  $\hat{E}'$  are said to be linearly equivalent,  $\hat{E} \sim \hat{E}'$ , if  $\hat{E} - \hat{E}'$  is an integral and principal divisor. Consider the additive group

$$\text{Pic}_0^{\mathbb{Q}}(M, D) = \text{Div}_0^{\mathbb{Q}}(M, D)/\sim.$$

Theorem 6. 3. There exists a one-to-one correspondence  $\pi \rightarrow S = S(\pi)$  between isomorphism classes of finite abelian coverings  $\pi: X \rightarrow M$  which branches at most along  $D$ , and subgroups  $S$  of finite index of  $\text{Pic}_0^{\mathbb{Q}}(M, D)$ . The correspondence satisfies (1)  $G_{\pi} \cong S(\pi)$  and (2)  $\pi_1 \leq \pi_2$  if and only if  $S(\pi_1) \subset S(\pi_2)$ .

Theorem 6. 4. There exists a finite abelian covering  $\pi: X \rightarrow M$  which branches along  $D$  if and only if there is a subgroup  $S$  of finite index of  $\text{Pic}_0^{\mathbb{Q}}(M, D)$  with the following condition: for every  $j$  ( $1 \leq j \leq s$ ), there is an element  $\hat{E}(j)/\sim \in S$  such that  $(a_j, e_j) = 1$ , where

$\hat{E}(j) = (a_1/e_1)D_1 + \cdots + (a_j/e_j)D_j + \cdots + (a_s/e_s)D_s + E,$   
 ( $E$  : an integral divisor).

For the proofs of the above theorems, we make use of the theory of harmonic integrals by de Rham-Kodaira [3].

For example, the cyclic covering  $\pi : X \rightarrow M$  in Example 6.1 corresponds to

$$S = \{(a/e)(D_1 + \cdots + D_s) - aE \mid 0 \leq a \leq e - 1\} / \sim,$$

where  $E$  is an integral divisor on  $M$  such that  $[E] = L$ .

The Kummer covering  $\pi : X \rightarrow M$  in Example 6.2 corresponds to

$$S = S_{12} + S_{23} + \cdots + S_{n-1,n} + S_{n,1},$$

where

$$S_{12} = \{(a/e)D_1 - (a/e)D_2 \mid 0 \leq a \leq e - 1\} / \sim, \text{ etc..}$$

As applications of Theorem 6.4,

Theorem 6.5. Let  $D_1, \cdots, D_s$  ( $s \geq 2$ ) be linearly equivalent distinct prime divisors on a projective manifold  $M$ . Suppose that, for every  $j$  ( $1 \leq j \leq s$ ),  $e_j$  divides

$$\langle e_1, \cdots, e_{j-1}, e_{j+1}, \cdots, e_s \rangle.$$

Then there exists a finite abelian covering  $\pi : X \rightarrow M$  which branches along  $D = e_1D_1 + \cdots + e_sD_s$ .

Theorem 6.6. Let  $p_1, \cdots, p_s$  be distinct points of a compact Riemann surface  $M$ . Put  $D = e_1p_1 + \cdots + e_sp_s$ , ( $e_j \geq 2$ ). Then there exists a finite abelian covering  $\pi : X \rightarrow M$  which branches along  $D$  if and only if, for every  $j$  ( $1 \leq j \leq s$ ),  $e_j$  divides

$$\langle e_1, \dots, e_{j-1}, e_{j+1}, \dots, e_s \rangle .$$

Finally, we move  $D$  and consider various  $\text{Pic}_0^{\mathbb{Q}}(M, D)$ 's. Let  $\text{Div}^{\mathbb{Q}}(M)$  be the additive group of all rational divisors on  $M$ , and  $\text{Div}_0^{\mathbb{Q}}(M)$  be the subgroup of  $\text{Div}^{\mathbb{Q}}(M)$  consisting of all rational divisors whose rational Chern classes vanish. Two rational divisors  $\hat{E}$  and  $\hat{E}'$  are said to be linearly equivalent,  $\hat{E} \sim \hat{E}'$ , if  $\hat{E} - \hat{E}'$  is integral and principal. Consider the additive group

$$\text{Pic}_0^{\mathbb{Q}}(M) = \text{Div}_0^{\mathbb{Q}}(M)/\sim.$$

Let  $\mathbb{C}(M)$  be the field of meromorphic functions on  $M$ . Note that isomorphism classes of finite Galois (resp. abelian) branched coverings  $\pi : X \rightarrow M$  and (isomorphism classes of) finite Galois (resp. abelian) extensions  $F/\mathbb{C}(M)$  of  $\mathbb{C}(M)$  are in one-to-one correspondence under

$$\begin{aligned} \pi &\longrightarrow F = \mathbb{C}(X), \\ F &\longrightarrow F\text{-normalization of } M. \end{aligned}$$

Then, by Theorem 6. 3, we have

Theorem 6. 7. For a projective manifold  $M$ , there exists a one-to-one correspondence  $F \rightarrow S = S(F)$  between the set of all (isomorphism classes of) finite abelian extensions  $F/\mathbb{C}(M)$  and the set of all finite subgroups  $S$  of  $\text{Pic}_0^{\mathbb{Q}}(M)$ . The correspondence satisfies (1)  $S(F) \simeq \text{Gal}(F/\mathbb{C}(M))$  and (2)  $F_1 \subset F_2$  if and only if  $S(F_1) \subset S(F_2)$ .

Note that the class field theory for fields of algebraic functions (of one variable) asserts the dual version of this

theorem, using the generalized Jacobian variety, (see Serre [27]).

The content of this section can be generalized to finite Galois coverings of a projective manifold, using language of unitary flat generalized vector bundles, along the line of Weil [30]. See Namba [22] for detail.

7. Equivalence Problem and Automorphism Groups of Kummer Coverings. Let  $\pi : X \longrightarrow M$  be a Galois covering of  $M$  branching along  $D = e_1 D_1 + \dots + e_s D_s$  with the covering transformation group  $G = G_\pi$ . In this case, we also write in this section

$$\pi : (G, X) \longrightarrow (M : D).$$

For a second Galois covering  $\pi' : (G', X') \longrightarrow (M' : D')$ , a biholomorphism  $h : X \longrightarrow X'$  is referred to as an equivalence, written  $h : \pi \approx \pi'$ , if there is a biholomorphism  $\bar{h} : M \longrightarrow M'$  making a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ \pi \downarrow & & \downarrow \pi' \\ M & \xrightarrow{\bar{h}} & M' \end{array}$$

For a biholomorphism  $h : X \longrightarrow X'$ , if we put

$$G^{',h} = \{h^{-1}g'h \mid g' \in G'\},$$

Then we have

$$h : \pi \approx \pi' \iff G = G^{',h}.$$

Let  $E(\pi)$  be the subgroup of  $\text{Aut}(X)$  consisting of equivalences of  $\pi$  onto itself. We have an obvious short exact sequence:

$$\{1\} \longrightarrow G \longrightarrow E(\pi) \longrightarrow \text{Aut}(M, D),$$

where  $\text{Aut}(M, D) = \{f \in \text{Aut}(M) \mid f^* D = D\}$ .

Definition 7.1. A Galois covering  $\pi : X \longrightarrow M$  is said to be rigid, if  $E(\pi) = \text{Aut}(X)$ .

Equivalence Problem. Are which kinds of Galois coverings rigid?

This problem in the case of cyclic branched coverings of  $\mathbb{P}^1$  was proposed by H. Shiga in Wakabayashi's problem session, Wakabayashi [29].

The second named author, Namba [21] showed that cyclic branched coverings of  $\mathbb{P}^1$  are rigid under some conditions. Moreover, by making use of a theorem of Matsumura-Monsky, he proved that an  $m$ -fold cyclic covering  $\pi : X \longrightarrow \mathbb{P}^n$  branching along a non-singular hypersurface of degree  $m$  in  $\mathbb{P}^n$  is rigid, provided that

- (i)  $m \geq 4$ , if  $n = 1$ ,
- (ii)  $m \geq 3$ , if  $n \geq 2$ , and
- (iii)  $(m, n) \neq (4, 2)$ .

T. Kato [19] improved the results of Namba in the case of cyclic branched coverings of  $\mathbb{P}^1$ .

Let  $L = L_1 + \cdots + L_s$  be a reduced divisor of  $\mathbb{P}^n$  consisting of  $s$  distinct hyperplanes  $L_1, \cdots, L_s$ , which will be referred to as a hyperplane configuration of  $\mathbb{P}^n$ .

A Kummer covering

$$\pi : (G, X) \longrightarrow (\mathbb{P}^n : mL)$$

of  $\mathbb{P}^n$  branching along  $mL$  is nothing but a branched covering

obtained as the Fox completion of a covering spread

$X_0 \longrightarrow \mathbb{P}^n - L \subset \mathbb{P}^n$  associated with a  $\mathbb{Z}/m\mathbb{Z}$ -Hurewicz homomorphism

$$\pi_1(\mathbb{P}^n - L, *) \longrightarrow H_1(\mathbb{P}^n - L; \mathbb{Z}) \longrightarrow H_1(\mathbb{P}^n - L; \mathbb{Z}/m\mathbb{Z}).$$

Thus

$$G \simeq H_1(\mathbb{P}^n - L; \mathbb{Z}/m\mathbb{Z}) = (\mathbb{Z}/m\mathbb{Z})^{s-1}$$

and  $G$  is generated by covering transformations  $g_1, \dots, g_s$  corresponding to the normal loops  $\gamma_1, \dots, \gamma_s$  of  $L_1, \dots, L_s$ , respectively.

We are interested in the case where  $n = 2$ .

Let  $q$  be an  $r$ -ple point of  $L$ ;  $q = L_{i_1} \cap \dots \cap L_{i_r}$ .

Let

$$\phi : B_q(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$$

be the blowing up of  $\mathbb{P}^2$  at  $q$ . Then  $\phi^{-1}(q) = E$  is a non-singular rational curve and we have a reduced divisor

$$p_1 + \dots + p_r$$

on  $E$ , where

$$p_k = \overline{(\phi^* L_k - E)} \cap E$$

for  $k = 1, \dots, r$ .

Definition 7. 2. If a Kummer covering of  $E$  branching along  $m(p_1 + \dots + p_r)$  is rigid, then  $(\mathbb{P}^2 : mL)$  is said to be rigid at  $q$ . We shall say that  $(\mathbb{P}^2 : mL)$  is locally rigid, if for each  $r$ -ple ( $r \geq 4$ ) point  $q$  of  $L$ ,  $(\mathbb{P}^2 : mL)$  is rigid at  $q$ .

In M. Kato [18], the first named author proved essentially

Theorem 7. 3. Let  $\pi : (G, X) \longrightarrow (\mathbb{P}^2 : mL)$  and  $\pi' : (G', X') \longrightarrow (\mathbb{P}^2 : mL')$  be kummer coverings of  $\mathbb{P}^2$  such that  $L$  and  $L'$  are line configurations of  $\mathbb{P}^2$ . Suppose that

- (1)  $m \geq 6$ ,
- (2) each  $L_j$  contains at least three singular points of  $L$  and
- (3)  $(\mathbb{P}^2 : mL)$  is locally rigid.

If a biholomorphism  $h : X \longrightarrow X'$  exists, then  $h : \pi \approx \pi'$ . In particular,  $\pi : X \longrightarrow \mathbb{P}^2$  is rigid.

Since the Kummer covering  $\pi : (G, X) \longrightarrow (\mathbb{P}^n : mL)$  is an abelian  $mL$ -universal covering, it follows that a natural homomorphism

$$E(\pi) \longrightarrow \text{Aut}(\mathbb{P}^n, L) (= \text{Aut}(\mathbb{P}^n, mL), (m > 0))$$

is surjective. Thus we have

Corollary 7. 4. Under the assumption of Theorem 7. 3, we have a short exact sequence:

$$\{1\} \longrightarrow G(\simeq (\mathbb{Z}/m\mathbb{Z})^{s-1}) \longrightarrow \text{Aut}(X) \longrightarrow \text{Aut}(\mathbb{P}^2, L) \longrightarrow \{1\}.$$

The following results about rigidity of a Kummer covering  $\pi : X \longrightarrow \mathbb{P}^1$  branching along  $m(p_1 + \cdots + p_s)$  are known:

- Theorem 7. 5. (1) if  $\chi(X) \geq 0$ , i.e., either  $s = 2$  or  $s = 3$  and  $m \leq 3$ , then  $\pi$  is not rigid.
- (2) if  $s = 3$  and  $m \geq 4$ , then  $\pi$  is rigid (see Namba [21]).
- (3) if  $s \geq 4$  and  $m \geq 5(s - 1)$ , then  $\pi$  is rigid (see M. Kato [18]).



Theorem 7. 6. (Sakurai-Suzuki [26]). Suppose that  $\chi(X) < 0$ ,  $s \geq 4$  and that for any subset  $P'$  of  $\{p_1, \dots, p_s\}$  with  $\#P' \geq 4$ ,  $\text{Aut}(P^1, p') = \{1\}$ . Then  $\pi$  is rigid.

Remark 7. 7. Recently, Sakurai is improving the result above extensively. He has announced in February, 1987, that  $\pi$  is rigid, if  $\chi(X) < 0$  and  $m \geq 11$ . It is plausible that  $\pi$  is rigid, if  $\chi(X) < 0$ , i. e.,  $\text{Aut}(X)$  is finite.

The proof of Theorem 7. 3 is based on the following facts:

- (I) If  $X$  is a surface of general type, then  $\text{Aut}(X)$  is finite.
- (II) The covering transformation group  $G$  is generated by 'complex reflections'  $g_1, \dots, g_s$  of the surface  $X$ .
- (III) If a finite unitary reflection group of  $\mathbb{C}^2$  contains a unitary reflection of order  $\geq 6$ , then it is abelian, refer to Shephard-Todd [28].

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