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**Kyoto University**
On the mixed Hodge structures in the normal crossing case
Yuji Shimizu

§1. Introduction and the main result

1.0 The purpose of this note is to remark that the existence of natural mixed Hodge structures (abbreviated as MHS) on certain cohomology groups follows from the recent study of variation of MHS by Kashiwara [K1] (What he studied was named as the infinitesimal mixed Hodge module) as well as the purity theorem of the intersection cohomology in the normal crossing case by Kashiwara-Kawai [KK1] and Cattani-Kaplan-Schmid [CKS]. So this note may be regarded as an appendix to Kashiwara [K1].

1.1 To be precise, let $X$ be a compact Kähler manifold, $Y$ a hypersurface of $X$ with normal crossings, and $H$ a polarizable variation of Hodge structure of weight $w$ on $X - Y =: U$. Then our result is the following:

**Theorem 1.2** There exist MHS's on $H^i(U,H)$ and $H_c^i(U,H)$ in a functorial way.

Of course, this generalizes the classical case $H = Q_U$ by Deligne [D,II] and the one-dimensional case by Zucker [Z,§13,14]. Special case was treated in Shimizu [Sh].

We can generalize Theorem 1.2 to the case allowing $H$ to be an admissible variation of MHS (cf.4.6).
Remark 1.3 Theorem 1.2 can be proved in the frame work of mixed Hodge modules (cf. [Sa2]) at least when $X$ is projective. This follows from the following facts: (i) $j_* H$, $j! H$ are weakly mixed Hodge modules [Sa2], where $j$ denotes the inclusion $j: U \hookrightarrow X$ and $H$ is as above. ($j_*, j!$ are taken in the sense of filtered $\mathcal{O}$-modules with $\mathbb{Q}$-structure.) (ii) For a weakly mixed Hodge module $H$ on a projective manifold $X$, $R\Gamma(X, M)$ is a cohomological mixed Hodge complex $[D, \mathbb{M}(8.1)]$.

Stronger fact holds indeed: (i') $j_* H$ and $j! H$ in (i) are mixed Hodge modules, and we can show directly that at least in the algebraic category, $R\Gamma(U, H)$ and $R\Gamma_c(U, H)$ are complexes of MHS. These facts are remarked in [Sa2] and Kashiwara's theory [K1] is vital for their proof.

1.4 The outline of the proof is as follows.

We will make use of the formalism of cohomological mixed Hodge complexes by Deligne $[D, \mathbb{M}]$ to put MHS on the above cohomologies. In our setup, these cohomologies are the hypercohomologies of perverse sheaves $Rj_* H$, $j! H$ ($j$ denotes the inclusion $U \hookrightarrow X$). Then we can use the explicit description of perverse sheaves in the normal crossing case by Galligo-Granger-Maisonobe [GGM] and its counterpart in the mixed Hodge theory is more or less Kashiwara's theory [K1]. Using these, we can give the weight filtration on the above perverse sheaves. The procedure for giving the Hodge filtrations using the canonical extension of $\mathcal{O}_U \otimes H$ is well known. Finally results in [K1] and the purity theorem in the normal crossing case [KK1] or [CKS] imply that $Rj_* H$, $j! H$ are graded-polarizable cohomological mixed Hodge complexes.
We remark that our method is "ad hoc" compared to the recent theory of mixed Hodge modules (cf. Remark 1.3).

1.5 The construction of this paper is as follows. In §2,3, we recall the necessary facts on the description of perverse sheaves and infinitesimal mixed Hodge modules. Finally in §4, we construct filtrations, i.e., the data necessary for a cohomological mixed Hodge complex.

Acknowledgement: The author would like to express his sincere gratitude to Professor M. Kashiwara for inspiring conversations. The essential difficulty in this note is solved in his study [K1].
§2. Infinitesimal mixed Hodge modules

2.0 Let $H$ be a polarized variation of HS of weight $w$ on $\Delta^{*n}$, having unipotent monodromies $T_j$ $(1 \leq j \leq n)$. According to Schmid [Sc], $\exp(-\sqrt{-1}t_j N_j) \cdot F(t) \in D$ has a limit $F$ in $D^\sim$ as $t_j$ tends to infinity, where $D$ denotes an appropriate period domain, $D^\sim$ its compact dual and $t$ varies in the universal covering of $\Delta^{*n}$. Moreover, $\exp(\sqrt{-1}t_j N_j) \cdot F$ approximates $F(t)$ well with respect to an invariant metric on $D^\sim$. It is why one should study nilpotent orbits (cf. [CK]).

2.1 Recall that a nilpotent orbit of weight $w$ $(H,F,S;N_1,\ldots,N_n)$ or $(H,F;N_1,\ldots,N_n)$ consists of a $R$-vector space $H_R$, a decreasing filtration $F$ of $H := H_R \otimes \mathbb{C}$, a $(-1)^w$-symmetric bilinear form on $H_R$, and a set of mutually commuting nilpotent endomorphisms $(N_1,\ldots,N_n)$ of $H_R$. These data should satisfy the following two conditions.

(i) $N_j F^p \subseteq F^{p-1}$ for all $p,j$.

(ii) There is a constant $C > 0$ such that $(H,\exp(\sqrt{-1}t_j N_j) \cdot F, S)$ is a polarized HS of weight $w$ for $t_j > C$.

2.2 Kashiwara introduced the notion of infinitesimal mixed Hodge module (IMHM for short) as an object arising from an admissible variation of MHS [SZ, §3] as in (2.0). We recall its definition.

Definition 2.3 1) A pre-IMHM $(H;W,F,(S_k);N_1,\ldots,N_n)$ consists of a $R$-vector space $H_R$, an increasing filtration $W$ on $H_R$, a decreasing filtration $F$ on $H := H_R \otimes \mathbb{C}$, a bilinear form $S_k$ on $Gr_k^W H_R$ for each $k$, and mutually commuting nilpotent endomorphisms $(N_1,\ldots,N_n)$. 

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These data should satisfy the following conditions (i), (ii).

(i) \( N_j P^p \subseteq P^{p-1}, N_j W_k \subseteq W_k \) for all \( p, j, k \).
(ii) \( \langle \text{Gr}_k^W, P(\text{Gr}_k^W), S_k ; N_1, \ldots, N_n \rangle \) is a nilpotent orbit of weight \( k \) for all \( k \).

We don't often mention the bilinear forms \( (S_k) \) explicitly.

2) A pre-IMHM \( (H, W, F; N_1, \ldots, N_n) \) is called an IMHM if there exists a filtration \( M(J) \) for each \( J \subseteq I = \{1, \ldots, n\} \) such that

(iii) \( N_j M_k(J) \subseteq M_{k-2}(J) \) for \( j \in J \).
(iv) \( M(J) \) is the relative monodromy filtration of \( \sum_{j \in J} N_j \) with respect to \( W \) (For the definition, see (2.4) below).

It is shown in [K1] that IMHM has many nice properties concerning the relative monodromy filtration. We recall some of them, which will be used later.

2.4 Here, let \( H \) denote an object in an abelian category, \( W \) its increasing filtration and \( N \) a nilpotent endomorphism of \( H \) such that \( N W_k \subseteq W_k \). The relative monodromy filtration \( M \) of \( N \) with respect to \( W \) is, by definition, the unique (increasing) filtration (if it exists) satisfying the conditions:

(i) \( N M_k \subseteq M_{k-2} \), (ii) \( N^\ell : \text{Gr}_k^M \rightarrow \text{Gr}_{k-\ell}^M \) for all \( \ell \geq 1, k \).

We often denote it by \( M(N, W) \). Several properties follow from
its existence (cf. [SZ], [K1]).

Assuming the existence of $M = M(N, W)$, we recall two filtrations $N^*_W$ and $N^!_W$ related to the perverse sheaves $Rj^*_H$, $j^!_H$ (cf. [K1, (3.4)])). We put

\[(N^*_W)_k := NW_{k+1} + M_k \cap W_k = NW_{k+1} + M_k \cap W_{k+1}, \]
\[(N^!_W)_k := W_{k-1} + M_k \cap N^{-1}W_{k-1} = W_{k-1} + M_k \cap N^{-1}W_{k-2}. \]

**Lemma 2.6** ([K1, (3.4.2), (3.4.3)]) The following hold:

2. $Gr^W_{k+1} \leftrightarrow Gr^N_{k+1} \ (\text{resp. } Gr^W_k \leftrightarrow Gr^N_k \ (\text{For the meaning of } \leftrightarrow, \text{ see (3.4).})).$
3. $Gr^N_{k+1} = \text{Im}(N: Gr^W_k \to Gr^W_{k+1}) \oplus \text{Coker}(N: W_k \to Gr^M_k \to W_k) \ (\text{resp. } Gr^N_k = \text{Coim}(N: Gr^W_{k-1} \to Gr^W_k) \oplus \text{Ker}(N: Gr^M_k (H/W_{k-1}) \to Gr^M_k (H/W_{k-1})).$}

**Proposition 2.7** ([K1, (5.5.1), (5.5.5)])

Let $(H, W, F; N_1, \cdots, N_n)$ be an IMHM. Then, for any $i, j \in I := \{1, \cdots, n\}$, the following hold.

1. $N_i^*(N_j^* W) = N_j^* (N_i^* W)$.

Thus we can define the following: For $J = \{j_1, \cdots, j_{\ell} \} \subset I,$

\[(N^*_W) := N_{j_1}^* \cdots N_{j_{\ell}}^* W, \ N^!_W := N_{j_1}^! \cdots N_{j_{\ell}}^! W \ (N_J := \prod_{j \in J} N_j). \]
Then, putting $M(J,W) = \bigoplus_{j \in J} \Sigma N_j, W$, we have

$$M(J_1, N_{j_2}, W) = N_{j_2} M(J_1, W).$$

2.9 We recall an important property of the relative monodromy filtration; There is a canonical decomposition [Kl,(3.2.9)]:

$$\text{Gr}_k^M = \bigoplus_{\ell} \text{Gr}_k^M \text{Gr}_\ell^W.$$

Therefore, $\text{Gr}_k^M$ is polarizable if so is $\text{Gr}_\ell^W$ (cf.[Sc,(6.16)]). In particular, for an IMHM, $\text{Gr}_k^{N_{J*} W}$ or $\text{Gr}_k^{N_{J} W}$ is polarizable by (2.6).
§3. IMMM's and n-cubes

3.0 There is a description of perverse sheaves in the normal crossing case due to Galligo-Granger-Maisonobe [GGM], which we recall now.

In this section, we use the following notation:

(3.1) \( X = \Delta^n \quad z = (z_1, \ldots, z_n) \), \( Y = (z_1 \cdots z_n = 0) \), \( \Delta = \{ x \in \mathbb{C} ; |x| < 1 \} \),

\( Y_j = \cap_{j \in J} \langle z_j = 0 \rangle \), \( Y_j^* = Y_j - \cap_{j \in J} Y_j \) for \( J \subseteq I = \{ 1, \ldots, n \} \).

\( k \); a field.

Let \( \mathcal{P}(n) \) denote the category of perverse sheaves \( F \) such that \( F|_{Y_j^*} \) is a locally constant sheaf of \( k \)-modules. We refer the reader to [BBBD] about perverse sheaves. Then the objects in \( \mathcal{P}(n) \) can be described in the following way.

Proposition 3.2 ([GGM]) \( \mathcal{P}(n) \) is equivalent to the category \( \mathcal{G}(n) \) consisting of the data \( \mathcal{H} = (H_{\alpha} : f_{\alpha \beta}, \varepsilon_{\beta \alpha}) \) satisfying the following conditions (the morphisms in \( \mathcal{G}(n) \) are obviously defined):

(0) \( H_{\alpha} \) is a \( k \)-vector space \( (\alpha \subseteq I) \), and \( f_{\alpha \beta} : H_{\beta} \rightarrow H_{\alpha} \), \( \varepsilon_{\beta \alpha} : H_{\alpha} \rightarrow H_{\beta} \) are \( k \)-homomorphisms \( (\beta \subset \alpha \subset I) \).

(1) \( f_{\alpha \beta} \cdot f_{\beta \gamma} = f_{\alpha \gamma} \), \( \varepsilon_{\gamma \beta} \cdot \varepsilon_{\beta \alpha} = \varepsilon_{\gamma \alpha} \) for \( \gamma \subset \beta \subset \alpha \).

(2) \( f_{\alpha \alpha} = \varepsilon_{\alpha \alpha} = \text{id} \).

(3) \( \varepsilon_{\gamma \alpha} \cdot f_{\alpha \beta} = f_{\gamma \delta} \cdot \varepsilon_{\delta \beta} \) for \( \delta \subset \beta \subset \alpha \) and \( \delta \subset \gamma \subset \alpha \).

(4) \( 1 - \varepsilon_{\beta \alpha} \cdot f_{\alpha \beta} \) is invertible for \( \beta \subset \alpha \) and \( |\alpha| = |\beta| + 1 \).
We call an object in \( S(n) \) an \( n \)-cube.

**Example 3.3** Given a local system \( H \) on \( \Delta^* \) with unipotent monodromies. Then, if we denote a stalk of \( H \) by the same \( H \),

1) the minimal extension \( \pi_H \) of \( H \) to \( \Delta^n \) corresponds to an \( n \)-cube \( \langle \pi, H \rangle := (H_\alpha, f_\alpha \beta, \varepsilon_\beta \alpha) \) defined by \( H_\alpha = \text{Im } N_j \), \( f_\alpha \beta = N_{\alpha-\beta} \), \( \varepsilon_\beta \alpha = \text{id} \).

2) the perverse sheaf \( R_j \pi_* H \) corresponds to an \( n \)-cube \( \langle *, H \rangle \) defined by \( H_\alpha = H \), \( f_\alpha \beta = N_{\alpha-\beta} \), \( \varepsilon_\beta \alpha = \text{id} \).

3) the perverse sheaf \( j_! H \) corresponds to an \( n \)-cube \( \langle !, H \rangle \) defined by \( H_\alpha = H \), \( f_\alpha \beta = \text{id} \), \( \varepsilon_\beta \alpha = N_{\alpha-\beta} \).

For \( n = 1 \),

\[ \langle \pi, H \rangle = \left[ H \begin{array}{c} N_1 \\ 1 \end{array} \right] \rightarrow \text{Im } N_j \), \langle *, H \rangle = \left[ H \begin{array}{c} N_1 \\ 1 \end{array} \right] \rightarrow H \), \langle !, H \rangle = \left[ H \begin{array}{c} 1 \\ N_1 \end{array} \right] \rightarrow H \). \]

For \( n = 2 \),

\[ \langle \pi, H \rangle = \left( \begin{array}{c} \begin{array}{c} H \\ N_1 \end{array} \end{array} \begin{array}{c} 1 \\ N_1 \end{array} \rightarrow \text{Im } N_1 \) \) \), \langle *, H \rangle = \left( \begin{array}{c} \begin{array}{c} H \\ N_1 \end{array} \end{array} \begin{array}{c} 1 \\ N_1 \end{array} \rightarrow H \) \)

\[ \begin{array}{c} \begin{array}{c} N_2 \\ 1 \end{array} \end{array} \begin{array}{c} N_2 \\ 1 \end{array} \\ \begin{array}{c} \text{Im } N_2 \\ 1 \end{array} \end{array} \rightarrow \text{Im } N_1 N_2 \) \) \)

\[ \begin{array}{c} \begin{array}{c} N_2 \\ 1 \end{array} \end{array} \begin{array}{c} N_2 \\ 1 \end{array} \\ \begin{array}{c} \text{Im } N_2 \\ 1 \end{array} \end{array} \rightarrow \text{Im } N_1 N_2 \) \) \)

3.4 An \( n \)-cube \( \mathcal{H} \) is said to have the decomposition property if \( H_\beta \xrightarrow{f_\alpha \beta, \varepsilon_\beta \alpha} H_\alpha \) for all \( \beta \subset \alpha \) (or equivalently, for all \( \beta \subset \alpha \) such that \(|\alpha| = |\beta| + 1 \)). Here we write \( A \xrightarrow{f} B \) when \( B = \text{Im } f \oplus \text{Ker } g \) (cf. [K1, §2]).

The terminology is explained by the following lemma.

**Lemma 3.5** ([K1, (2.3.1)]) The following statements are equivalent.

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3.6 We recalled the notion of IMHM in (2.3), which was motivated by the infinitesimal study of admissible VMHS. For further study, Kashiwara introduced an object, a mixture of a cube and an IMHM, which may be regarded as the infinitesimal version of a mixed Hodge module of M. Saito [Sa2].

Definition ([Kl, (5.6)]) Let \((H_\alpha; f_{\alpha\beta}, \varepsilon_{\beta\alpha})\) be an n-cube, \(W_\alpha\) (resp. \(F_\alpha\)) an increasing (resp. decreasing) filtration on \(H_\alpha\) for \(\alpha \in \mathbb{I} = (1, \ldots, n)\). Then we call \(\mathcal{M} = (H_\alpha, W_\alpha, F_\alpha; f_{\alpha\beta}, \varepsilon_{\beta\alpha})\) an MH-cube if the following conditions hold.

1. \(f_{\alpha\beta} \cdot \alpha_\beta \subseteq F_{\alpha_\beta} - |\alpha_\beta|\), \(f_{\alpha\beta} \cdot W_{\alpha_\beta} \subseteq W_{\alpha_\beta} - |\alpha_\beta|\),
   \(\varepsilon_{\beta\alpha} \cdot F_{\beta_\alpha} \subseteq F_{\beta_\alpha}, \varepsilon_{\beta\alpha} \cdot W_{\beta_\alpha} \subseteq W_{\beta_\alpha} + |\alpha_\beta|\) for \(\beta \subseteq \alpha\).

2. \(N_j \in \text{End}(H_\alpha)\) is nilpotent for any \(j\), where \(N_j := \varepsilon_{\alpha, \alpha \cup \{j\}} f_{\alpha \cup \{j\}, \alpha} \) if \(j \in \alpha\), \(f_{\alpha, \alpha \setminus \{j\}}, \varepsilon_{\alpha \setminus \{j\}, \alpha} \) if \(j \in \alpha\).

3. For each \(\alpha\), \((H_\alpha, W_\alpha, F_\alpha; N_1, \ldots, N_n)\) is an IMHM.

4. The property \(\text{Gr}_{k}^{W_\alpha} \cong \text{Gr}_{k-|\alpha_\beta|}^{W_{\alpha_\beta}}\) holds for \(\beta \subseteq \alpha\).

The morphism between MH-cubes is obviously defined. The category of MH-cubes is an abelian category and is denoted by MH(\mathbb{I}).
We have the operations like dual, Tate twist, nearby and vanishing cycles in the category MH(I).

3.7 Let \((H,W,F;N_1,\ldots,N_n)\) be an IMHM. Then we defined cubes \(\langle \pi,H \rangle, \langle *,H \rangle, \langle !,H \rangle\) in (3.3). We recall the definition of filtrations \(W^\alpha, F^\alpha\) on \(H_\alpha\) giving a structure of MH-cube, according to [K1,(5.8)].

**Definition**

1) For \(\langle *,H \rangle\), put \(W^\alpha := N^*_\alpha \cdot W, F^\alpha := F\).

2) For \(\langle !,H \rangle\), put \(W^\alpha := N^\alpha \cdot W(-|\alpha|), F^\alpha := F\).

3) For \(\langle \pi,H \rangle\), put \(W^\alpha := N^\alpha \cdot \text{Im} N^\alpha, F^\alpha := F\).

(Here \(W(\ell)\) is defined by \(W(\ell)_k := W_{k-2\ell}\).)

These are MH-cubes by (2.6), (3.5).

Of course, \(\langle *,H \rangle\) and \(\langle !,H \rangle\) are dual to each other, and \(\langle \pi,H \rangle\) is a quotient of \(\langle !,H \rangle\) as well as a subobject of \(\langle *,H \rangle\).

We put

\[
P_k(\ast\beta) := P_\beta(\text{Gr}^W_k(\ast,H)) = \cap \text{Ker}(g_{\gamma\beta}: \text{Gr}^N_{k-|\beta|} \cdot W \rightarrow \text{Gr}^N_{k-|\gamma|} \cdot W),
\]

\[
P_k(!\beta) := P_\beta(\text{Gr}^W_k(\!,H)) = \cap \text{Ker}(g_{\gamma\beta}: \text{Gr}^N_{k-|\beta|} \cdot W \rightarrow \text{Gr}^N_{k-|\gamma|} \cdot W),
\]

(the "primitive part" cf.(3.5,c)). Then, since an MH-cube has the decomposition property (3.4) by definition, we get the following:

**Lemma 3.8** We have a decomposition:

\[
\text{Gr}^W_k(\ast,H) = \oplus \langle \pi,P_{k-|J|}(\ast I-J)\rangle_{I-J}
\]

(resp. \(\text{Gr}^W_k(\!,H) = \oplus \langle \pi,P_{k-|J|}(\!,I-J)\rangle_{I-J}\)).

Here \(\langle,\rangle_{I-J}\) denote the object in \(\mathcal{E}(I-J)\) regarded as in \(\mathcal{E}(I)\).
Example 3.9 We illustrate the case \( n = 1 \).

\[
W_k^{*,H} = \left[ \begin{array}{c}
W_k \\
I
\end{array} \right] \xrightarrow{N} W_k^{N,W}_{k-1}
\]

\[
Gr_k^{*,H} = \left[ \begin{array}{c}
Gr_k^{W} \\
Gr_k^{N,W}_{k-1}
\end{array} \right] \xrightarrow{N} Gr_k^{N,W}_{k-1}
\]

\[
= \left[ \begin{array}{c}
Gr_k^{W} \\
NGr_k^{W}
\end{array} \right] \oplus \left[ 0 \xrightarrow{\text{Coker}(N:W_{k-1}Gr_k^M \rightarrow W_{k-1}Gr_k^{M-1})} \right]
\]

\[
W_k^{!,H} = \left[ \begin{array}{c}
W_k \\
I
\end{array} \right] \xrightarrow{N} W_k^{N,W}_{k+1}
\]

\[
Gr_k^{!,H} = \left[ \begin{array}{c}
Gr_k^{W} \\
NGr_k^{W}
\end{array} \right] \oplus \left[ 0 \xrightarrow{\text{Ker}(N:Gr_k^M(H/W_k) \rightarrow Gr_k^M(H/W_k))} \right]
\]
§4. Proof of Theorem 1.2 : Construction of CMHC

4.0 We give a concrete expression of cohomological MH complexes (CMHC) alluded to in §1 using the result of §3. See (4.8) for another and simpler expression due to Kashiwara.

We use the same notation as in §1, except for \( H \) denoting an admissible VMHS on \( U \). Denote a general stalk of \( H \) by the same \( H \).

We will use some notation related to the integrable connection \( \mathcal{O}_U^{\otimes H} \). Let \( E_X(H) \) denote the (left) canonical extension of \( \mathcal{O}_U^{\otimes H} \) [KK2]. Perverse sheaves corresponding to regular holonomic \( \mathcal{O} \)-modules by the Riemann-Hilbert correspondence (see e.g. [K2]), we denote by \( \mathcal{M}(F) \) the \( \mathcal{O} \)-module associated to a perverse sheaf \( F \).

4.1 First of all, note that the existence of MHS is automatic, if we give a structure of CMHC upon \( Rj_*H \) (resp. \( j_!H \)). This is due to the formalism by Deligne [D, III(8.1)].

We don't recall the precise definition of a CMHM here, but only mention that the data needed are:

1. a filtration \( W \) on \( K_*Q \) (resp. \( K_!Q \)),
2. filtrations \( W, F \) on \( K_*C \) (resp. \( K_!C \)).

Here \( K_*A \) (resp. \( K_!A \)) denotes a complex quasi-isomorphic to \( Rj_*H_A \) (resp. \( j_!H_A \)), \( A = Q, C \), and we assume that these are compatible between \( Q \) and \( C \).

These data should satisfy the condition:

3. The CMHC \( (\text{Gr}^W_KQ, (\text{Gr}^W_KC, F)) \) are pure for all \( p \), i.e., the spectral sequence associated to the filtered complex \( (R\Gamma(X, \text{Gr}^W_KC), F) \) degenerates at \( E_1 \) and gives rise to the Hodge filtration on the abutements \( (K = K_* \text{ or } K_!) \).
4.2 We will define the filtration $W$ using MH-cubes in §3. So we take $Rj_\ast J$ (resp. $j_!J$) itself as $K_\ast Q$ (resp. $K_!Q$). To realize the filtration $F$, we need a complex of $\mathcal{O}$-coherent modules relevant to the canonical extension $E_X(H)$. Thus we take the logarithmic de Rham complex $\Omega_X(\log Y)\otimes E_X(H)$ as $K_\ast C$.

Lemma 4.2.1 To a subobject $F$ of $Rj_\ast J C$ in the category of perverse sheaves $\mathcal{G}(n)$ (3.1), there is a subcomplex of $\Omega_X(\log Y)\otimes E_X(H)$ quasi-isomorphic to $F$. Denote it by $\Omega_X, \log(F)$. 
Proof Take $\Omega_X, \log(F) := \Omega_X \otimes \mathcal{F} \cap \Omega_X(\log Y)\otimes E_X(H)$. We have only to remark that the argument in [KK, §4] (case $F = \pi_X$) is applicable to $F$.

4.3 As for $K_! C$, we use the following simplicial construction. The construction being local, we may assume $(X, U) = (\Delta^n, \Delta^n \times \Delta^n)$ and put $Y_j = (z_j = 0)$ (cf. (2.0)). Let $N_j$ be the logarithm of the monodromy $T_j$ along $Y_j$.

Lemma 4.3.1 (i) The cube $\langle !, H \rangle$ is quasi-isomorphic to the simple complex associated to

$$\begin{aligned}
( \oplus \langle \pi, \text{Ker} N_j \rangle_{I-J[p]} )_{0 \leq p \leq n}.
\end{aligned}$$

Here $\langle \pi, \text{Ker} N_j \rangle_{I-J}$ denotes the object (3.3,1) in $\mathcal{E}(I-J)$ regarded as in $\mathcal{E}(I)$, and $\text{Ker} N_j := \cap_{j \in J} \text{Ker} N_j$.

(ii) $j_! H C$ is quasi-isomorphic to the simplicial complex

$$\begin{aligned}
( \oplus \Omega_{Y_j, \log(\pi \text{Ker} N_j)} )_{0 \leq p \leq n}.
\end{aligned}$$

$Y_j = \cap_{j \in J} Y_j$

Proof (ii) is an immediate consequence of (i). For (i), consider the case $n = 1$. The following is an exact sequence in $\mathcal{E}(n)$.
Thus we get $<!,H> = [ <\pi,H>_1 \rightarrow <\pi,\text{KerN}>_0 [1] ]$. The general case is obtained similarly.

**Variant 4.3.2** There is a construction analogous to (4.3.1). To a subobject $F$ of $j_1H$ in $\mathcal{F}(n)$, we associate a subcomplex of the complex in (4.3.1,ii), quasi-isomorphic to $F$. Denote it by $\Omega_{\text{spl}}(F)$. The procedure is the same as (4.3.1).

**Example 4.3.4** We illustrate the case $n = 1$ for $W_k<!,H>$.

$$W_k<!,H> = [ \begin{bmatrix} W_k \\ N(N_1W)_k+1 \\ \text{Gr}_{k+1}(\text{ImN}) \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ \text{N}(N_1W)_{k+1} \cap \text{KerN} \\ \text{Gr}_{k+1}(\text{ImN}) \end{bmatrix} [1] ]$$

$$\text{Gr}_k^W<!,H> = [ \begin{bmatrix} \text{Gr}_k^W \\ \text{Gr}_{k+1}(\text{ImN}) \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ \text{Gr}_{k+1}(\text{ImN}) \end{bmatrix} [1] ]$$

**4.4 CMHC on $R_{j_1H}$ (resp. $j_1H$)**

In (3.7), we defined a filtration $W$ on $<*,H>$ (resp. $<!,H>$). By the construction (4.2) (resp. (4.3)), we get a filtration $W$ on $R_{j_1H}$ (resp. $j_1H$) as well as $\Omega_{\log}(R_{j_1H})$ (resp. $\Omega_{\text{spl}}(j_1H)$):

\[ W_k\Omega_{\log}(R_{j_1H}) := \Omega_{\log}(W_k(R_{j_1H})), \]

\[ W_k\Omega_{\text{spl}}(j_1H) := \Omega_{\text{spl}}(W_k(j_1H)). \]
By Schmid's theorem [Sc, §6], the canonical extension $E_X(H)$ (or $E_{Y,J}(\text{Ker} N_J)$) has a filtration $F$ which prolongs the Hodge filtration of $\mathcal{O}_U \otimes \mathcal{O}_H$. Therefore, $\Omega^*_\log (R_{j,H})$ (resp. $\Omega^*_\text{spl} (j,H)$) has a filtration $F$:

$$F^P \Omega^*_\log (R_{j,H}) := F^P E_X(H) \rightarrow \Omega^*_X(\log Y) \otimes F^P-1 E_X(H) \rightarrow \cdots$$

$$F^P \Omega^*_\text{spl} (j,H) := [ \bigoplus_{|J|=q} F^P \Omega^*_Y (\log (\pi \text{Ker} N) \log (\mathcal{O}_{\text{Ker} N})) ]_{0 \leq q \leq n}.$$  

We can write down explicitly $W$ on $\Omega^*_\log (R_{j,H})$ locally. If $X = \Delta^n$, $Y_J = (z_J = 0)$, then

$$(4.4.1) \quad W_{k} \Omega^*_\log (R_{j,H}) \big| \sum_{|J|=p} \frac{dz_J}{z_J} \prod_{j \in J} \Omega^*_Y (\log (\mathcal{O}_{\text{Ker} N}) (N_{j,H}) (-q) ]_{0 \leq q \leq n}.$$  

Here $dz_J = \Lambda dz_j$, $z_J = \prod_{j \in J} z_j$. For $N_{j,H}$, see (2.8).

As for $W_{k} \Omega^*_\text{spl} (j,H)$, we have an expression as

$$W_{k} \Omega^*_\text{spl} (j,H) = [ \bigoplus_{|J|=q} \Omega^*_Y (\log (\mathcal{O}_{k-q}(\pi \text{Ker} N) \log (\mathcal{O}_{\text{Ker} N})) (-q) ]_{0 \leq q \leq n},$$

(cf. (3.7)). To get it, we must use the equality:

$$N_J(N_{j,H})_{k+|J|} = (N_{j,H})_{k-|J|} \cap \text{Im} N_J.$$  

To see this, the author have to use [Sa2] (the case $|J| = 1$ is trivial). But we won't use it later.

Remark 4.4.2 We defined $W, F$ using §2,3, so that a à la e they have meaning only locally around $Y$. But on $U, R_{j,H}$ and $j,H$ reduce to $H$ itself. Hence there is no problem. ( $\Omega^*_U \otimes \mathcal{O}_H$ patches to $\Omega^*_\log (R_{j,H})$ and $\Omega^*_\text{spl} (j,H).$)

Lemma 4.5 For each $k$, $Gr^W_{k} \Omega^*_\log (R_{j,H})$ (resp. $Gr^W_{k} \Omega^*_\text{spl} (j,H)$) is a direct sum of pure CMHC (4.1,3).

Proof We calculate $Gr^W_{k}$ using the local expression (4.4.1).

Obviously $Gr^W_{k} \Omega^*_\log (R_{j,H})$ is mapped into

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\[
\Theta_{\mathcal{Q}(-|J|)}^{p-|J|} (\log Y \cap Y_{j}) \otimes F_{X}^{N_{j}*w_{j}}, |J| \leq p
\]
where \( Y \cap Y_{j} \) means the divisor on \( Y_{j} \) induced by \( Y \), and \( \mathcal{Q}(-|J|) \) is the Tate twist. Calculate the stalk of \( \text{Gr}_{k}^{w} \) in the same way as in [KK1, §3] (using the homotopy formula for the Euler operator). Then the calculation reduces to that of \( \text{Gr}_{k}^{w} <*, H > \). But we see by (3.8) that

\[
\text{Gr}_{k}^{w} \mathcal{Q}_{j}^{*} \log (\mathcal{R}_{j}^{*} H) = \Theta_{Y_{j}, \log (\mathcal{R}_{p}^{k-|J|} (\mathcal{S}^{*} - J)) (-|J|)) [-|J|].
\]

Since \( \text{Gr}_{k}^{N_{j}*w} \) is polarizable by (2.9), \( \text{Gr}_{k-|J|}^{N_{j}*w} \) is a PVHS associated to a nilpotent orbit (i.e. pure \( \mathcal{L} \text{MHM} \)) of weight \( k-|J| \). Thus the second term is a pure complex of weight \( k \) by the main theorem of [KK1] or [CKS].

Similar but more complicated reasoning shows that \( \text{Gr}_{k}^{w} \mathcal{Q}_{j}^{*} \text{spl} (j, H) \) is a pure complex of weight \( k \):

\[
\text{Gr}_{k}^{w} \mathcal{Q}_{j}^{*} \text{spl} (j, H) = \Theta_{Y_{j}, \log (\mathcal{R}_{p}^{k-|J|} (\mathcal{S} - J)) (-|J|)) [-|J|].
\]

Corollary 4.6 (including Theorem 1.2) Let \( X, Y \) be as in (1.1), but \( H \) be an admissible VMHSS. Then there exist MHS's on \( H^{i}(U, H) \) and \( H^{i}_{C}(U, H) \). If \( H \) is pure of weight \( w \), then the weights of \( H^{i}(U, H) \) (resp. \( H^{i}_{C}(U, H) \)) are \( \geq w+i \) (resp. \( \leq w+i \)).

Remark 4.7 1) The MHS constructed above is independent of the choice of compactification \( X \). We can see this using the language of filtered \( \mathcal{O} \)-modules [Sal, §2].

2) The case \( \text{dim } X = 1 \) was treated in [SZ, §4]. The description by \( 1 \)-cubes is essentially used there.

3) The MHS's on \( H^{i}(U, H) \) and \( H^{2d-i}_{C}(U, H^{*}) \) are dual to each other.
(d = \dim X, H^* = \hom C(H, C_X)). This can be proven by observing the
natural pairing between \( \Omega_{\log}(Rj^*_H) \) and \( \Omega_{\text{Sp}1}(j^*_1H) \) (cf. [SZ, (4.30)],
[Sh, (3.3)]).

4.8 We present here a more direct way of expressing \( Rj^*_H \) or \( j^*_1H \) as
CMHC due to Kashiwara.

In general, let \( g \) be a holomorphic function on a complex
manifold \( X, Y = (g = 0) \). Then, for a regular holonomic \( \mathcal{D} \)-module \( \mathcal{M} \)
on \( X \), there is a canonical filtration \( (V^\lambda \mathcal{M}) \) on \( \mathcal{M} \) ("V-filtration"),
which we don't recall here (cf. [K3], [Sal, §3]). According to
Kashiwara, we have a quasi-isomorphism :

\[
\begin{align*}
\text{DR}_X \mathcal{M} & := \mathcal{M} \rightarrow \Omega^1 \otimes \mathcal{M} \rightarrow \Omega^2 \otimes \mathcal{M} \rightarrow \cdots \\
\sim V_0 \text{DR}_X \mathcal{M} & := V_0 \mathcal{M} \rightarrow \Omega^1 \otimes V_1 \mathcal{M} \rightarrow \Omega^2 \otimes V_2 \mathcal{M} \rightarrow \cdots 
\end{align*}
\]

Each \( V^\lambda \mathcal{M} \) is coherent over \( V_0 \mathcal{D}_X \). Moreover, it is \( \mathcal{O}_Y \)-coherent if
\( \mathcal{M} \) has support in \( Y \).

In our setup (1.1), we use the graph construction : take \( X \times C \)
and \( \text{pr}_2 \) in place of \( X \) and \( g \) above. For \( \mathcal{M} \), we take the direct
image of a regular holonomic \( \mathcal{D}_X \)-module \( \mathcal{N} \), \( \mathcal{O}_X \)-coherent over a dense
open subset \( X^* \) of \( X \), by the graph map \( t_g : X \rightarrow X \times C \). Then \( V^\lambda \mathcal{M} \)
has support in \( t_g(X) \), is coherent over \( \mathcal{O}_X \) and is related to the
canonical extension of \( \mathcal{M}|_{X^*} \). Thus we get a complex of \( \mathcal{O}_X \)-coherent
modules \( V_0 \text{DR}_{X 	imes C}(t_g^* \mathcal{N}) \). Since the construction is functorial, we get
a representative of the filtered complex \( \text{DR}_X \mathcal{M}(Rj^*_H) \) or \( \text{DR}_X \mathcal{M}(j^*_1H) \)
(4.0). (Take \( \mathcal{M}(W_0Rj^*_H) \) or \( \mathcal{M}(W_0j^*_1H) \) as \( \mathcal{N} \).)
References


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