EXAMPLES OF SEMI-STABLE DEGENERATIONS OF KUNEV SURFACES

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0. This article consists of some examples of semi-stable degenerations of Kunev surfaces together with some remarks concerning about a compactification of their moduli space and the Torelli problem.

A Kunev surface $X$ is defined as a canonical surface with $\chi(\mathcal{O}_X) = 2$ and $(\omega_X)^2 = 1$ which has an involution $\sigma$ such that $X/\sigma$ is a K3 surface with rational double points (RDP, for short). Let $\hat{X}$ be the minimal model of a Kunev surface. Then it is known that $\hat{X}$ is a simply connected surface of general type with $p_g = c_1^2 = 1$ and its bicanonical map is a Galois cover of $\mathbb{P}^2$ with $\mathbb{G}a\ell \cong (\mathbb{Z}/2)^{\oplus 2}$ whose branch locus consists of two cubics and a line on $\mathbb{P}^2$, and that $\hat{X}$ has the following numerical invariants:

(0.1) \[ h^{2,0}(\hat{X}) = h^{0,2}(\hat{X}) = 1, \quad h^{1,1}_{\text{prim}}(\hat{X}) = 18. \]
\[ H^0(T_{\hat{X}}) = H^2(T_{\hat{X}}) = 0, \quad h^1(T_{\hat{X}}) = 18. \]
\[ h^{2,0}(\hat{X})^\sigma = h^{0,2}(\hat{X})^\sigma = 1, \quad h^{1,1}_{\text{prim}}(\hat{X})^\sigma = 10. \]
\[ h^1(T_{\hat{X}})^\sigma = 12. \]

It is also known that the moduli space $\mathcal{M}$ (resp. $\mathcal{R}$) of surfaces with $p_g = c_1^2 = 1$ (resp. Kunev surfaces) is irreducible and rational (resp. irreducible). (For the above facts, see [Ca.1], [Ca.2], [U.2], [T.2], [SU], [M].)

On the Hodge theoretic view-point, surfaces with $p_g = c_1^2 = 1$ or, as its subfamily, Kunev surfaces are interesting materials. After Kunev constructed an example of a Kunev surface as a
counterexample to the infinitesimal Torelli theorem, the following
cresults are known:

(0.2) The generic infinitesimal Torelli theorem holds for
surfaces in \( \mathbb{M} \) ([Ca.1]).

(0.3) The period map \( \Phi_2 \) of surfaces in \( \mathbb{M} \) has positive
dimensional fibers ([T.1], [U.1], [U.2]; [T.1] treats only Kunev
surfaces).

(0.4) \( R \) in \( \mathbb{M} \) is characterized by \( \dim \Phi_2^{-1}(\{X\}) = 2 \), which
is the maximal dimension of the fibers of \( \Phi_2 \) ([U.1]).

(0.5) The infinitesimal mixed Torelli theorem holds for pairs
\((X, C)\) of surfaces \( X \) in \( \mathbb{M} \) and their smooth canonical curves \( C \)
([U.3]).

(0.6) The generic mixed Torelli theorem holds for Kunev
surfaces ([L], [SSU]; there is a point about monodromy which is not
clear in [L]).

(0.7) There exists a Zariski open subset \( U \) of \( R \) such that
\( \Phi^{-1}(U) = U \), where \( \Phi : \mathbb{M} \rightarrow \Gamma \backslash \mathbb{D} \) is the mixed period map ([SSU]).
Hence, in order to solve the mixed Torelli problem for surfaces in \( \mathbb{M} \)
via Kunev locus \( R \), it is necessary to study the following:

(0.8) A compactification of the mixed period map \( \Phi : \mathbb{M} \rightarrow \Gamma \backslash \mathbb{D} \).

(0.9) The monodromy \( \Gamma \) in (0.7), where we used a geometric one.
(For a general reference of the above as well as for the terminology
such as mixed period map, mixed Torelli etc., see [SSU].)

We shall report here some results in an experiment concerning
about the problem (0.8). We shall construct some examples of
semi-stable degenerations of pairs of Kunev surfaces and their
canonical curves in the sequel. The examples in 1, 2 and 3 are of
type I with respect to the local monodromy of the pure second cohomology. The examples in 4 and 5 are of type II and type III respectively.

As for the problem of a compactification of the moduli space \( \mathcal{R} \) of Kuniev surfaces, Horikawa and Shah constructed a compactification of the moduli space of K3 surfaces of degree 2 as one of the moduli space of sexetic curves on \( \mathbb{P}^2 \) by the geometric invariant theory ([H], [Sh]). The latter contains a 10-dimensional subspace \( \mathcal{R} \) which is a compactification of
\[
\mathcal{R} = \{ \Sigma C_j \in \text{Sym}^2 | 0 \}_{\mathbb{P}^2(3)} | \Sigma C_j \text{ has only simple singularities} \} / \text{SL}_3.
\]
A "compactification" of \( \mathcal{R} \) sits over \( \mathcal{R} \).

We use the terminology a numerical K3 surface for a minimal surface with \( p_g = 1, \ q = 0 \) and \( c_i^2 = 0 \) (for this terminology, cf. also [K]). Numerical K3 surfaces appeared in this article have an elliptic fibration with one double fiber.

1. Let \( C_1 \) and \( C_2 \) be general cubics on \( \mathbb{P}^2 \). Denote by \( \tilde{C}_j \subset \tilde{\mathbb{P}}^2 \) the dual curve of \( C_j \subset \mathbb{P}^2 \), i.e., the image of the Gauss map. Then each \( \tilde{C}_j \) has nine cusps corresponding to nine inflexes on \( C_j \), \( \Sigma \tilde{C}_j \) has nine bitangents \( \tilde{D}_i \) with tangent points \( P_{i1} \) and \( P_{i2} \) \( (1 \leq i \leq 9) \) subjected to nine nodes of \( \Sigma C_j \), and we have two stratifications of \( \tilde{\mathbb{P}}^2 \) determined by \( \Sigma \tilde{C}_j \) and \( \Sigma \tilde{D}_i \):
\[
\tilde{\mathbb{P}}^2 = (\tilde{\mathbb{P}}^2 - \Sigma \tilde{C}_j) \sqcup (\Sigma \tilde{C}_j - (\Sigma \tilde{D}_i + \text{Sing}(\Sigma \tilde{C}_j))) \\
\sqcup (\Sigma \text{Sing}(\tilde{C}_j)) \sqcup (\Sigma \tilde{C}_j) \sqcup (\Sigma \tilde{D}_i)
\]
\[
= R_0 \sqcup R_1 \sqcup R'_1 \sqcup R_2 \sqcup R'_0.
\]
(1.1)

(1.2)
We denote by $Y$ the minimal K3 surface which is obtained as the minimal resolution of the double cover of $\mathbb{P}^2$ branched along $\Sigma C_j$.

Let $\alpha_j : Y \to \mathbb{P}^2$ be the projection and $E_i$ ($1 \leq i \leq 9$) be the exceptional curves for $\alpha_i$, i.e., (-2)-curves.

Case $t_0 \in S_1 \cap R_0$: We may assume $t_0 \in \bar{D}_i$. Let $\Delta$ be a small disc with center $0 = t_0$ intersecting transversely with $\bar{D}_i$ such that $\Delta^* := \Delta - \{0\} \subset S_0$. We denote by $\mathcal{L} \subset \Delta \times \mathbb{P}^2$ the total space of the family of lines $\{L_t\}_{t \in \Delta}$. We can construct a semi-stable degeneration of pairs of K3 surfaces and the canonical curves over $\Delta$ in the following way: (0) Set $\alpha = 1 \times \alpha_j : \Delta \times Y \to \Delta \times \mathbb{P}^2$ and $\delta_i = \Delta \times E_i$ ($1 \leq i \leq 9$). (i) Let $\beta : \mathbb{Y} \to \Delta \times Y$ be the blowing-up along $\alpha^{-1}\mathcal{L} \cap \delta_i$. Denote by $\mathcal{W}_\beta$ the exceptional divisor. (ii) Take the double cover $\gamma : \mathbb{X} \to \mathbb{Y}$ branched along $(\alpha\beta)^{-1}\mathcal{L} + \beta^{-1}(\Sigma \delta_i)$. (iii) Let $\delta : \mathbb{X} \to \mathbb{X}$ be the contraction of $(\beta\gamma)^{-1}(\Sigma \delta_i)$. (iv) Let $\mathbb{I} \to \mathbb{X}$ be the contraction of $\delta(\beta\gamma)^{-1}\mathcal{W}_\beta$.

(In the above notation, we use $\alpha^{-1}\mathcal{L}$ etc. as the proper transforms.)

Set $\mathcal{L}_\mathbb{I} = (\delta(\alpha\beta\gamma)^{-1}\mathcal{L}$ with reduced structure) and $\mathcal{W}_\mathbb{I} = \delta\gamma^{-1}\mathcal{W}_\beta$.

Then we can prove the following: (a) The projection $f : (\mathbb{X}, \mathcal{L}_\mathbb{X}) \to \Delta$ is a semi-stable degeneration of pairs (for the terminology, see [SSU]). $K_{\mathbb{X}} = \mathcal{L}_\mathbb{X} + \mathcal{W}_\mathbb{X}$. (b) The pair $(X_t, L_{X,t}) := f^{-1}(t)$ consists of a minimal K3 surfaces and its canonical divisor which is an ample, smooth curve of genus 2 for $t \in \Delta^*$. The central fiber of $f$ is as Figure 1. (c) $\mathbb{I}$ has an isolated singular point, raised from $C_i \cap C_j \cap L_0$, which is analytically isomorphic to the cone over the Veronese embedding of $\mathbb{P}^2 \subset \mathbb{P}^5$ by $|0\mathbb{P}^2(2)|$. $K_{\mathbb{I}}$ is
nef.

(1.4) Case \( t_0 \in R_0^c \): We may assume \( t_0 = \bar{p}_{11} \). We use the notation \( \Delta, \mathcal{E} \) etc. in the same sense as in (1.3). The construction (0)-(iv) in (1.3) yields a family of pairs \( f : (\mathcal{X}, \mathcal{E}_\mathcal{X}) \rightarrow \Delta \) and \( \bar{\mathcal{X}} \). We can prove the following: (a) The central fiber \( X_0 \) of \( f \) is not a divisor with normal crossings (see Figure 2). The total space \( \mathcal{X} \) is smooth and \( K_{\mathcal{X}} = \mathcal{E}_{\mathcal{X}} + W_{\mathcal{X}} \). (b) After a base extension \( \Delta_2 \rightarrow \Delta_1 \), \( s \mapsto t = s^2 \), we can get a semi-stable reduction \( f' : (\mathcal{X}', \mathcal{E}_{\mathcal{X}'}) \rightarrow \Delta_2 \) of \( f \) whose central fiber is as Figure 3. (Because of the limit of pages, we omit the details of the process of reduction and contraction.) (c) The same statement as (c) in (1.3) holds.

(1.5) Case \( t_0 \in S_0^c \): We may assume \( t_0 \in \bar{p}_1 \cap \bar{p}_2 \). We use the notation \( \Delta, \mathcal{E} \) etc. in a similar sense as in (1.3). An analogous construction (0)-(iv) as in (1.3) yields a family of pairs \( f : (\mathcal{X}, \mathcal{E}_\mathcal{X}) \rightarrow \Delta \) and \( \bar{\mathcal{X}} \) with the following properties: (a) The same statement as (a) in (1.3) holds. (b) The central fiber of \( f \) is as Figure 4. (c) \( \bar{\mathcal{X}} \) has two Veronese cone singularities raised from two points \( C_1 \cap C_2 \cap L_0 \). \( K_{\bar{\mathcal{X}}} \) is nef.

Remark. For fixed general cubics \( C_1 \) and \( C_2 \), we constructed a complete family of degenerations of Kunyev surfaces over \( \bar{P}^2 \) and described their fibers \( X_t, t \in \bar{P}^2 \), in [U.4,1], [U.5]. We point out here only that the stratification (1.1) controls RDP on the main components \( V_t \) of \( X_t \), whereas the stratification (1.2) controls their Kodaira dimensions. In fact, we know that the main component \( V_t \) is a (singular) Kunyev surface, numerical K3 surface with one double fiber, or K3 surface according to \( t \in S_0^c, S_1^c, \) or \( S_2^c \) (for ...
more general statement, see Remark in 3).

2. Next we consider the case that both cubics $C_1$ and $C_2$ degenerate into two pairs of concurrent three lines. Denote by $Q_j$ the triple point of $C_j$ ($j = 1, 2$). We deal with the general situation in this case, i.e., we assume moreover that $\Sigma C_j$ consists of six different lines and that $Q_1 \neq Q_2$. Denote by $\tilde{Q}_j$ the line on $\tilde{\mathbb{P}}^2$ corresponding to the pencil of lines through $Q_j$ on $\mathbb{P}^2$ ($j = 1, 2$). Then we have a stratification of $\tilde{\mathbb{P}}^2$:

$$(2.1) \quad \tilde{\mathbb{P}}^2 = (\tilde{\mathbb{P}}^2 - \Sigma \tilde{Q}_j) \cup (\Sigma \tilde{Q}_j - \tilde{Q}_1 \cap \tilde{Q}_2) \cup (\tilde{Q}_1 \cap \tilde{Q}_2)
=: T_0 \cup T_1 \cup T_2.$$

Let $Y$ be the minimal K3 surface which is obtained as the minimal resolution of the double cover of $\mathbb{P}^2$ branched along $\Sigma C_j$ and let $\alpha_j : Y \rightarrow \mathbb{P}^2$ be the projection. We denote by $E_i$ ($1 \leq i \leq 9$) the (-2)-curves on $Y$ which are mapped by $\alpha_j$ to the nine points $C_1 \cap C_2$. Notice that, beside the $E_i$, $Y$ carries two $D_4$-configurations of (-2)-curves over the triple points $Q_j$ of $C_j$.

(2.2) Case $t_0 \in T_1$ and the line $L_{t_0} \not\in \Sigma C_j$: In this case, the line $L_{t_0}$ passes one and only one of the two triple points, say $Q_1$. Let $\Delta$ be a small disc with center $0 = t_0$ intersecting transversely with $\tilde{Q}_1$ such that $\Delta^* \subset T_0 - (\Sigma C_j^*)^\circ$. Let $\mathcal{L} \subset \Delta \times \mathbb{P}^2$ be the total space of the family of lines as before. Similarly as in (1.3), we can construct a family of pairs $f : (\mathcal{L}, \mathcal{L}_q) \rightarrow \Delta$ which is a degeneration of pairs of Kunyev surfaces and the canonical curves in the following way: (0) Set $\alpha = 1 \times \alpha_1 : \Delta \times Y \rightarrow \Delta \times \mathbb{P}^2$, $E_i = \Delta \times E_i$ ($1 \leq i \leq 9$) and $\mathcal{Q}_j = \Delta \times \alpha_j^{-1}(Q_j)$ ($j = 1, 2$).
(i) Take the double cover $\beta : \mathfrak{X}_f \rightarrow \Delta \times Y$ branched along $\alpha^{-1}\mathbb{L} + \Sigma \varepsilon_i$. (ii) Let $\gamma : \mathfrak{X}_f \rightarrow \mathfrak{X}$ be the contraction of $\beta^{-1}(\Sigma \varepsilon_i)$. (iii) Let $\delta : \mathfrak{X} \rightarrow \mathfrak{X}$ be the contraction of $\gamma \beta^{-1}(\Sigma \vartheta_j)$.

Set $\mathfrak{L}_\mathfrak{X} = (\gamma(\alpha\beta)^{-1}\mathbb{L}$ with reduced structure) and $f : (\mathfrak{X}, \mathfrak{L}_\mathfrak{X}) \rightarrow \Delta$, the projection. Then we can prove: (a) $X_0$ of the central fiber $f^{-1}(0) = (X_0, L_X, 0)$ is irreducible. Rising from the cocurrent four lines $C_i + L$ on $\mathbb{P}^2$, $X_0$ has the singular locus which is a rational curve consisting of double points, on which there are four cuspidal points. The total space $\mathfrak{X}$ is smooth and $K_{\mathfrak{X}} = \mathfrak{L}_\mathfrak{X}$ which is nef. (b) After a base extension $\Delta_2 \rightarrow \Delta$, $s \mapsto t = s^2$, we get a semi-stable degeneration $f' : (\mathfrak{X}', \mathfrak{L}_{\mathfrak{X}'}) \rightarrow \Delta_2$ of pairs of Kneve surfaces and the canonical curves, whose central fiber is as Figure 6. (We omit the details of the process.) (c) Rising from the two triple points $Q_f$ and $Q_{2}^*$, $\mathfrak{X}$ has four compounds R.D.P. of type $D_4$, two of which coming from $Q_f$ clash to make up a simple elliptic singularity of type $E_8$ in the sense of K. Saito on the central fiber $\bar{X}_0$ with a local equation

$$z^2 + y(x^4 + y^2) = 0.$$

(2.3) Case $t_0 \in T_2$: We use the notation $\Delta, \mathfrak{L}, \varepsilon_i, \vartheta_j$ etc in (2.2). An analogous construction (0)-(iii) in (2.2) yields a family of pairs $f : (\mathfrak{X}, \mathfrak{L}_\mathfrak{X}) \rightarrow \Delta$ and $\mathfrak{X}$ with the following properties: (a) A similar statement of (a) in (2.2) holds but now the singular locus of $X_0$ consists of two copies of the rational curves as before. (b) The same statement as (b) in (2.2) holds and the central fiber of the semi-stable degeneration $f' : (\mathfrak{X}', \mathfrak{L}_{\mathfrak{X}'}) \rightarrow \Delta_2$ is as Figure 6. (c) A similar statement of (c) in (2.2) holds but now each pair of compounds R.D.P. of type $D_4$ on $\mathfrak{X}$, coming from
Q_j (j = 1, 2), clashes to make up a simple elliptic singularity of type E_8.

3. Consider now the case that C_1 is degenerating into a smooth conic Q and a line L. Assume that Q, L and a smooth cubic C_2 are in general position on P^2. Let \{C_t\} be a family of cubics on P^2 over a disc Δ such that C_0 = Q + L and that C_t is smooth and intersects transversely with C_2 for \( t \in Δ^* \).

Denote by \( E_t \) the total space of the family \( \{C_t\}_{t \in Δ} \), \( E_2 = Δ \times C_2 \) and \( E = Δ \times L \). Assume that \( E_t \) is smooth. Then we can construct a degeneration of pairs \( f: (X, L_X) \to Δ \) of Kneve surfaces and their canonical curves in an analogous way as (0)-(iv) in (1.3).

On this stage, \( X_0 \) of the central fiber \( f^{-1}(0) \) is not reduced, hence we need a semi-stable reduction by extending the base \( Δ_2 \to Δ \). \( s \to t = s^2 \). Figure 7 illustrates the central fiber \((X'_0, L_{X'_0}, 0) = f'^{-1}(0)\) of the resulting family \( f': (X', L_{X'}) \to Δ_2 \).

Contracting \( W_i \) \( (3 \leq i \leq 5, 9 \leq i \leq 11) \), \( W_2 \) and \( W_i \) \( (6 \leq i \leq 8) \) in this order, we get a simpler semi-stable degeneration \( f'' : (X'', L_{X''}, 0) = f^{-1}(0) \) is as Figure 8. \( K_{X''} \) is not nef. Let \( X'' \to \tilde{X} \) be the contraction of \( W_i \) \( (12 \leq i \leq 17) \). Then \( K_{\tilde{X}} \) becomes nef and \( \tilde{X} \) has six Veronese cone singularities.

Remark (cf.[U.4,11],[U.5]). Recall the notation \( X \) in \( \Theta \).

For any fixed \( \{Σ C_j\} \in X \), we define functions in \( t \in P^2 \) by

\[ m(t) = \sum_{P \in P^2} \min \{ I(P, L_t \cap C_j) \mid j = 1, 2 \}, \]

\[ n(t) = \# \{ \text{triple points of } C_j \text{ on } L_t \mid j = 1, 2 \}. \]
Notice that if $C_j$ has a triple point then $C_j$ consists of three distinct lines with a common point. These functions define two stratifications of $\tilde{\mathbb{P}}^2$:

$$\tilde{\mathbb{P}}^2 = S_0 \cup S_1 \cup S_2,$$
$$\tilde{\mathbb{P}}^2 = T_0 \cup T_1 \cup T_2,$$

where $S_m = \{ t \in \tilde{\mathbb{P}}^2 \mid m = \min\{2, m(t)\} \}$, and $T_n = \{ t \in \tilde{\mathbb{P}}^2 \mid n = n(t) \}$.

Notice that $\text{codim } S_m = m$, $\text{codim } T_0 = 0$, and $\text{codim } T_n = n$ if $T_n$ is non-empty ($n = 1, 2$). We can construct a complete family of degenerations of Kuniev surfaces $\mathcal{F} : \mathbb{K} \rightarrow \mathbb{P}^2$ and we can prove:

(3.1) The main component $V_t$ of the fiber $\mathcal{F}^{-1}(t)$ is a (singular) Kuniev surface, numerical K3 surface with one double fiber, K3 surface, elliptic surface with $p_g = q = 1$, or abelian surface according to $t \in S_0 \cap T_0$, $S_1$, $S_2$, $S_0 \cap T_1$, or $T_2$.

(3.2) If $t \notin S_0 \cap T_0$, the main component $V_t$ is an elliptic surface. $V_t$ has constant J-invariant if and only if $t \in T_1 \cap T_2$. If this is the case, the K3 surface $Y$ is a Kummer surface associated to a decomposable abelian surface $D_1 \times D_2$, where $D_j$ is an elliptic curve ($j = 1, 2$).

Combining with the Clemens-Schmid exact sequence ([Cl]), (3.1) yields a uniform explanation of the appearance of positive dimensional fibers of the period map of Kuniev surfaces and elliptic surfaces with $p_g = 1$ and $q = 0, 1$:

(3.3) $S_0 \cap T_0$, $S_1$, and $S_0 \cap T_1$ appear as positive dimensional fibers of the period map of the pure second cohomology of Kuniev surfaces, numerical K3 surfaces with one double fiber, and elliptic surfaces with $p_g = q = 1$ respectively. (For Kuniev surfaces and elliptic surfaces with $p_g = q = 1$, these phenomena were pointed out separately in [T.1], [U.1], [U.2] and [Sa].)
4. Next we consider the case that \( C_1 \) is approaching \( C_2 \). Let \( C_2 \) be a smooth cubic and \( L \) a line on \( \mathbb{P}^2 \). Assume that \( C_2 \) and \( L \) intersect transversely. Let \( \{ C_t \} \) be a smooth family of cubics on \( \mathbb{P}^2 \) over a disc \( \Delta \) such that \( C_0 = C_2 \) and that \( C_t \) intersects transversely with \( C_2 \) for \( t \in \Delta^* \). Denote by \( \mathcal{E}_t \) the total space of the family \( \{ C_t \}_{t \in \Delta} \), \( \mathcal{E}_2 = \Delta \times C_2 \) and \( \mathcal{L} = \Delta \times L \).

Then, as before, we can construct a degeneration of pairs \( f : (\mathfrak{X}, \mathcal{L}_{\mathfrak{X}}) \rightarrow \Delta \) of Kunene surfaces and their canonical curves. On this stage, \( X_0 \) of the central fiber \( f^{-1}(0) \) is not reduced, hence we need a semi-stable reduction by extending the base \( \Delta_2 \rightarrow \Delta \), \( s \mapsto t = s^2 \). The resulting family \( f' : (\mathfrak{X}', \mathcal{L}_{\mathfrak{X}'}) \rightarrow \Delta_2 \) has the central fiber illustrated as Figure 9. Contracting \( W_t \) (\( 2 \leq t \leq 7 \)) we get a nef terminal model \( \mathfrak{X} \) which has six Veronese cone singularities.

5. We consider here the case that \( C_1 \) and \( C_2 \) are degenerating into the same set of three distinct lines \( \Sigma M_i \). Let \( L \) be a line on \( \mathbb{P}^2 \). Assume that \( \Sigma M_i + L \) has no triple points. Let \( \{ C_1, t \} \) and \( \{ C_2, t \} \) be two families of cubics on \( \mathbb{P}^2 \) over a disc \( \Delta \) such that \( C_{1,0} = C_{2,0} = \Sigma M_i \) and that \( C_{1, t} \) and \( C_{2, t} \) are smooth and intersect transversely for \( t \in \Delta^* \). Denote by \( \mathcal{E}_1 \) and by \( \mathcal{E}_2 \) the total space of the families \( \{ C_{1, t} \}_{t \in \Delta} \) and \( \{ C_{2, t} \}_{t \in \Delta} \) respectively and \( \mathcal{L} = \Delta \times L \). We also assume that \( \mathcal{E}_j \) (\( j = 1, 2 \)) are smooth. Then, as before, we can construct a degeneration of pairs \( f : (\mathfrak{X}, \mathcal{L}_{\mathfrak{X}}) \rightarrow \Delta \) of Kunene surfaces and their canonical curves. Since \( X_0 \) of the central fiber \( f^{-1}(0) \) is not
reduced, we should perform a semi-stable reduction by extending the base $\Delta_2 \to \Delta$, $s \mapsto t = s^2$. The resulting family $f' : (X', \mathcal{L}_X') \to \Delta_2$ has the central fiber illustrated as Figure 10. Let $X' \to \overline{X}$ be the contraction of $W_i'$ ($19 \leq i \leq 24$). Then $K_{\overline{X}} = \mathcal{L}_{\overline{X}}$, which is nef, and $\overline{X}$ has six Veronese cone singularities.
Figure 1

\[ f^{-1}(0) = (X_0, L_X, 0), \quad X_0 = V + W_X \]

\[ K_X = \mathcal{L}_X + W_X \]

\[ V : \text{a homotopic K3 surface} \]

\[ K_V = \mathcal{L}_X|V : \text{an elliptic curve} \]

\[ W := W_X \cong \mathbb{P}^2, \quad \mathcal{L}_X|W_X : \text{a line} \]

Figure 2

\[ f^{-1}(0) = (X_0, L_X, 0), \quad X_0 = V + W_X \]

\[ K_X = \mathcal{L}_X + W_X \]

\[ V : \text{a homotopic K3 surface} \]

\[ W := W_X \cong \mathbb{P}^2 \]

Figure 3

\[ f'^{-1}(0) = (X'_0, L_{X'}, 0), \quad X'_0 = V' + \sum_{1}^{3} W'_k \]

\[ K_{X'} = \mathcal{L}_{X'} + \sum W'_k \]

\[ V' : \text{a numerical K3 surface with one double fiber} \]

\[ W'_k = F_1 : \text{a rational ruled surface (1 \leq k \leq 3)} \]

\[ \mathcal{L}_{X'} \cap V' : \text{an elliptic curve} \]

Figure 4

\[ f^{-1}(0) = (X_0, L_X, 0), \quad X_0 = V + \sum W_i \]

\[ K_X = \mathcal{L}_X + \sum W_i \]

\[ V : \text{a K3 surface} \]

\[ K_V = \mathcal{L}_X|V : \text{a } (-1) \text{-curve} \]

\[ W_i \cong \mathbb{P}^2 \quad (i = 1, 2) \]
$f^{-1}(0) = (X'_1, L'_{X'_1, 0}), \ X'_0 = V' + W', \ K_X = \mathcal{L}_{X'} + W'$

$V'$: an elliptic surface with $p_g = q = 1$, with two singular fibers of type $I_0^*$ and with four sections each of which is an elliptic curve with self-intersection $-1$

$W'$: One description is as follows. We start with a configuration below of two cubics $D_V$ and $D_L$ on $\mathbb{P}^2$ such that $D_V = V' \cap W'$ and $D_L = \mathcal{L}_{X'} \cap W'$. 

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where $A_0$ is a common inflex on $D_V$ and $D_L$ with intersection multiplicity $I(A_0, D_V \cap D_L) = 3$; there is unique abelian group structure on $D_V$ and $D_L$ such that $A_0$ is the zero element; $A_k$ ($0 \leq k \leq 3$) are the four 2-torsion points both on $D_V$ and on $D_L$ with respect to the above abelian group structures and $I(A_k, D_V \cap D_L) = 2$ ($1 \leq k \leq 3$). Blow-up twice at each of the four common points $D_V \cap D_L$ we get $W'$.

$V' \cap W' = D_V$, $\mathcal{L}_{X'} \cap V'$ and $\mathcal{L}_{X'} \cap W' = D_L$ are elliptic curves.

Figure 6

\[
f^{-1}(0) = (x'_0, L_{x'}, 0), \quad x'_0 = V' + W'_1 + W'_2, \quad K_{x'} = \mathcal{L}_{x'} + \Sigma W'_j
\]

$V'$: blowing-up one point on a decomposable abelian surface $D_1 \times D_2$, where $D_j$ is an elliptic curve ($j = 1, 2$).

$W'_j$ ($j = 1, 2$): a rational surface like $W'$ in Figure 5

$V' \cap W'_j$ and $\mathcal{L}_{X'} \cap W'_j$ are elliptic curves ($j = 1, 2$)

$\mathcal{L}_{X'} \cap V'$: a (-1)-curve
$X'_0 = V' + \sum_{i=1}^{17} W'_i$, \quad $K_{X'} = \omega_{X'} + (\Sigma_1 + 2\Sigma_2 + 4\Sigma_6 + \Sigma_{12}) W'_i$

$V'$: a K3 surface blown-up two points (come from $Q \cap L$)

$W'_i$: One description is 14 times blown-ups of $F_1$ in the following way.

Figure 7

3 copies
(come from $L \cap C_2$)

3 copies
(come from $L \cap C_1$)
$W'_2$: One description is six times blow-ups of $\mathbb{P}^1$-bundle of degree 7 over a curve of genus 3 in the following way.

$W'_i \cong F_2 \ (3 \leq i \leq 5), \quad W'_i \cong F_1 \ (6 \leq i \leq 8, \ 12 \leq i \leq 17), \quad W'_i \cong \mathbb{P}^2 \ (9 \leq i \leq 11)$

Figure 8

$X'' = V'' + (\Sigma_1 + \Sigma_{12}'') W''_i, \quad K_{X''} = \mathcal{L}_{X''} + (\Sigma_1 + \Sigma_{12}'') W''$

$V''$: a minimal $K3$ surface

$W''_i$: One description is

$W''_i \cong \mathbb{P}^2 \ (12 \leq i \leq 17), \quad L_{X'',0} : a \ curve \ of \ genus \ 2$
\[ X_0' = \sum_1^2 V_i' + (\sum_1^5 + \sum_1^9) W_i' \]

\[ K_{\mathcal{X}}' = \mathcal{O}_{\mathcal{X}} + \sum_1^9 W_i' \]

\( V_i' \): a rational surface. One description is six times blown-ups of \( F_1 \) in the following way: \( (i = 1, 2) \).

\( W_i' \): One description is 15 times blown-ups of an elliptic ruled surface of degree 6 in the following way.

\[ W_i' \cong \mathbb{P}^2 \quad (14 \leq i \leq 19) \]
$X'_0 = \sum_1^2 V'_i + (\sum_1^6 + \sum_1^{24}) W'_i$, \quad $K_{X'} = \sum_1^2 V'_i + \sum_1^{24} W'_i$

$V'_i$: a rational surface which is the minimal resolution of the double cover of $\mathbb{P}^2$ branched along $\Sigma H_i + L$

$W'_i$: a rational surface (cf. the above drawing) \quad (1 \leq i \leq 3)$

$W'_i \simeq F_2$ \quad (4 \leq i \leq 6), \quad $W'_i \simeq \mathbb{P}^2$ \quad (19 \leq i \leq 24)
REFERENCES


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