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ON THE DEFORMATION OF A CERTAIN TYPE OF ALGEBRAIC VARIETIES

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Dedicated to Professor I. Tamura for his 60th birthday

§1. Introduction

Let $A = (a_{ij})$ $(1 \leq i,j \leq n)$ be an upper triangular integral matrix with a non-zero determinant and $a_{ij} \geq 0$ for each i,j. Let Δ be the n-simplex in $\mathbf{R}^n$ which is spun by $A_0 = \vec{0}$ and $A_i = (a_{i1},\dots,a_{in})$ $(i=1,\dots, n)$. Let $A_{n+1}, A_{n+2},\dots, A_\ell$ be the other integral points in Δ. For an integral vector $\nu = (\nu_1,\dots,\nu_n)$, we denote the monomial $y_1^{\nu_1}\dots y_n^{\nu_n}$ by $\mathbf{y}^\nu$. For $\mathbf{t} = (t_0,\dots,t_\ell)$ of $\mathbf{C}^{\ell+1}$, we define

(1.1) $$h(\mathbf{y},\mathbf{t}) = t_0 + \sum_{j=1}^{\ell} t_j \mathbf{y}^{A_j}$$

and let $M_{\mathbf{t}}^a$ be the affine variety in $\mathbf{C}^n$ defined by $h(\mathbf{y},\mathbf{t}) = 0$. There exists a toric variety W of dimension n which depends only on Δ and a Zariski open subset U of $\mathbf{C}^{\ell+1}$ such that $W \supset \mathbf{C}^n \supset M_{\mathbf{t}}^a$ and the closure $M_{\mathbf{t}}$ of $M_{\mathbf{t}}^a$ in W is non-singular for each $\mathbf{t}\in U$. This type of algebraic variety $M_{\mathbf{t}}$ appears as an exceptional divisor of a resolution of an

isolated hypersurface singularity ([12]). The purpose of this paper is to study this deformation $\{M_t\}$ in W.
In §5, we prove the surjectivity of the infinitesimal displacement map

$$\xi : T_tU \to H^0(M_t, \nu_t).$$

In §6, we give a criterion about the injectivity of the Kodaira-Spencer map

$$\delta \circ \xi^e : T_tU^e \longrightarrow H^1(M_t, \Theta_t).$$

In §7, we will apply the results in §§5,6 to construct a complete deformation of a Godeaux surface.

§2. **Infinitesimal displacement**

Let W be a compact complex manifold of dimension n and let $\{M_t\}$ $(t \in U)$ be an analytic family of non-singular hypersurfaces where U is an open set of $\mathbf{C}^{\ell+1}$. Let $\{(U_\alpha, z_\alpha)\}$ $(\alpha \in S)$ be local coordinate systems of W such that (i) $W = \bigcup_{\alpha\in S} U_\alpha$ and (ii) there exists analytic functions $f_\alpha(z_\alpha, t)$ on $U_\alpha \times U$ such that $M_t \cap U_\alpha = \{z_\alpha \in U_\alpha \; ; \; f_\alpha(z_\alpha, t) = 0\}$. Let $h_{\alpha\beta} = f_\alpha/f_\beta$. We may assume that $h_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. The line bundle $[M_t]$ is defined by the cocycle $\{h_{\alpha\beta}\}$ of $H^1(W, \mathcal{O}^*)$ and the normal bundle N_t of M_t in W is the restriction of $[M_t]$ to M_t. Let ν_t be the sheaf of the germs of the holomorphic sections of N_t. Take a holomorphic tangent vector $v \in T_tU$. As $f_\alpha = h_{\alpha\beta} f_\beta$, we have

(2.1) $$v(f_\alpha) = h_{\alpha\beta}\, v(f_\beta) \quad \text{on } U_\alpha \cap U_\beta \cap M_t.$$

This defines a canonical linear mapping

(2.2) $$\xi : T_tU \to H^0(M_t, \nu_t)$$

where $\xi(v) = \{v(f_\alpha)\}(\alpha \in S)$. $\xi(v)$ is called the infinitesimal displacement along v.

Let Θ_W and Θ_t be the sheaves of the germs of holomorphic vector fields of W and M_t respectively. We have the exact sequence of sheaves:

(2.3) $$0 \to \Theta_t \to \Theta_W|M_t \to \nu_t \to 0.$$

This induces the following exact sequence.

(2.4) $$0 \to H^0(M_t, \Theta_t) \to H^0(M_t, \Theta_W|M_t) \to H^0(M_t, \nu_t)$$
$$\xrightarrow{\delta} H^1(M_t, \Theta_t) \longrightarrow H^1(M_t, \Theta_W|M_t) \longrightarrow \cdots$$

The composition

(2.5) $$T_tU \xrightarrow{\xi} H^0(M_t, \nu_t) \xrightarrow{\delta} H^1(M_t, \Theta_t)$$

is equal to the infinitesimal deformation map. See Kodaira-Spencer [6] or Kodaira [7] for details.

§3. Resolution of a hypersurface singularity

We recall basic properties about the resolution of a hypersurface singularity through the toroidal embedding theory. We use the same notation as in [12]. Let $f(z_0, \ldots, z_n) = \sum_\nu a_\nu z^\nu$ be an analytic function defined in a neighborhood of the origin and we assume that $V = f^{-1}(0)$ has an isolated singular point at the origin. Let $\Gamma_+(f)$ be the convex hull of $\bigcup_{a_\nu \neq 0} \{\nu + (\mathbf{R}^+)^{n+1}\}$. The Newton boundary $\Gamma(f)$ is the union of the compact faces of $\Gamma_+(f)$. We assume that f is non-degenerate on each face Δ of $\Gamma(f)$. Let N be the dual space Hom $(\mathbf{R}^{n+1},\mathbf{R})$. We identify N with $\mathbf{R}^{n+1}$ through the standard inner product and we denote the dual vectors by column vectors to avoid confusion. Let N^+ be the set of non-negative dual vectors. We introduce an equivalence relation $\sim$ in N^+ by $P \sim Q$ if and only if $\Delta(P) = \Delta(Q)$. Here $\Delta(P)$ is the locus where the restriction of P on $\Gamma_+(f)$ takes its minimal value which we denote by $d(P)$. This induces a cone-like polyhedral decomposition of N^+ and we denote this by $\Gamma^*(f)$. Let Σ^* be a unimodular simplicial subdivision. For each n-simplex $\sigma = (P_0,\dots,P_n) = (p_{ij})$ which is a unimodular matrix, we associate an affine space $\mathbf{C}_\sigma^{n+1}$ with coordinate $\mathbf{y}_\sigma = (y_{\sigma 0},\dots,y_{\sigma n})$. Let $\pi_\sigma : \mathbf{C}_\sigma^{n+1} \to \mathbf{C}^{n+1}$ be the birational morphism defined by $\pi(\mathbf{y}_\sigma) = (z_0,\dots,z_n)$ where $z_i = \prod_{j=0}^{n} y_{\sigma j}^{p_{ij}}$. Let X be the complex manifold of dimension $n+1$ which is obtained by gluing the affine spaces $\mathbf{C}_\sigma^{n+1}$ where σ moves in the n-simplices of Σ^* and let $\hat{\pi} : X \to \mathbf{C}^{n+1}$ be the projection map. Let $\tilde{V}$ be the proper transform of V and let $\pi : \tilde{V} \to V$ be the restriction of $\hat{\pi}$ to $\tilde{V}$. By the non-degeneracy assumption, $\pi : \tilde{V} \to V$ is a good resolution of V. For each strictly positive vertex P of Σ^* with $\dim \Delta(P) \geq 1$, there are corresponding exceptional divisors $\hat{E}(P)$ and $E(P)$

of $\hat{\pi}$ and π respectively so that E(P) is a hypersurface in $\hat{E}(P)$. $\hat{E}(P)$ is a toric variety. Let $\sigma = (P_0,\ldots,P_n)$ with $P = P_0$. Then in the coordinate chart $\mathbf{C}_\sigma^{n+1}$, $\hat{E}(P)$ is defined by $y_{\sigma 0} = 0$ and E(P) is defined by $\hat{E}(P) \cap \{ h_\sigma(y_{\sigma 1},\ldots,y_{\sigma n}) = 0 \}$ where $h_\sigma(\mathbf{y}_\sigma)$ is defined by

$$(3.1) \qquad f_{\Delta(P)}(\pi_\sigma(\mathbf{y}_\sigma)) = \prod_{i=0}^{n} y_{\sigma i}^{d(P_i)} \, h_\sigma(y_{\sigma 1},\ldots,y_{\sigma n}).$$

§4. Compactification of $M_{\mathbf{t}}^a$.

Let $h(\mathbf{y},\mathbf{t})$ be as in (1.1). Let σ' be the unimodular matrix $(P,R_1,\ldots,R_n)$ where $P = {}^t(1,\ldots,1)$, $R_1 = {}^t(0,1,\ldots,0)$, Let $\pi_{\sigma'} : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$ be as in §3. Let $y_0,\ldots, y_n$ be the coordinate of the source. Then we have $z_0 = y_0$ and $z_i = y_0 y_i$ for $i = 1,\ldots, n$. Let k be the degree of h and we define $f_\Xi(\mathbf{z},\mathbf{t}) = h(\pi_{\sigma'}^{-1}(\mathbf{z},\mathbf{t}))\, z_0^k = h(z_1/z_0,\ldots,z_n/z_0,\mathbf{t})\, z_0^k$. Then $f_\Xi(\mathbf{z},\mathbf{t})$ is a homogeneous polynomial in $z_0,\ldots, z_n$ and we can write

$$(4.1) \qquad f_\Xi(\mathbf{z},\mathbf{t}) = \sum_{i=0}^{\ell} t_j \mathbf{z}^{B_j}$$

for some integral vectors $B_0,\ldots, B_\ell$. Note that $B_0 = (k,0,\ldots,0)$. Let $f(\mathbf{z},\mathbf{t}) = f_\Xi(\mathbf{z},\mathbf{t}) + \sum_{i=0}^{n} z_i^L$ for a sufficiently large L. The notation $f_\Xi(\mathbf{z})$ is the same as in [12] if we set $\Xi = \Delta(P)$. There exists a Zariski open subset U of $\mathbf{C}^{\ell+1}$ such that $f(z,\mathbf{t})$ has a non-degenerate Newton boundary for each $\mathbf{t} \in U$. Let $\sigma = (P,P+R_1,\ldots,P+R_n)$. If L is sufficiently large, $\Delta(P+P_i) \supset B_0$ for each $i = 1, \ldots, n$. Thus σ is an admissible simplex of $\Gamma^*(f)$. (σ' is not necessarily an admissible simplex.) Thus we can take a unimodular simplicial subdivision Σ^* which has σ as an n-simplex by §3 of [12].

Assertion. <u>The defining equation of</u> E(P) <u>in</u> $\mathbf{C}_\sigma^{n+1} \cap \{y_{\sigma 0} = 0\}$ <u>is equal to</u> $h(\mathbf{y}_\sigma, \mathbf{t}) = 0$.

<u>Proof</u>. E(P) is defined by $h_\sigma(\mathbf{y}_\sigma, \mathbf{t}) = 0$ where

$$h_\sigma(\mathbf{y}_\sigma, \mathbf{t}) = f_\Delta(\pi_\sigma(\mathbf{y}_\sigma), \mathbf{t}) \,/\, y_{\sigma 0}^{d(P)} \prod_{i=1}^{n} y_{\sigma i}^{d(P)+d(R_i)}$$

$$= f_\Delta(\pi_{\sigma'}(\mathbf{y})) \,/\, \{(y_{\sigma 0} \cdots y_{\sigma n})^{d(P)} \prod_{i=1}^{n} y_{\sigma i}^{d(R_i)}\}$$

$$= h(\mathbf{y}, \mathbf{t}) = h(\mathbf{y}_\sigma, \mathbf{t})$$

Here we have used the equality $\pi_{\sigma'}^{-1} \circ \pi_\sigma = \pi_{\sigma'^{-1}\sigma}$ and $y_0 = y_{\sigma 0} \cdots y_{\sigma n}$ and $y_i = y_{\sigma i}$ for $i = 1, \ldots, n$.

Thus we take E(P) as the compactification $M_{\mathbf{t}}$ of $M_{\mathbf{t}}^a$ and $\hat{E}(P)$ as W hereafter. Note that $\pi_1(M_{\mathbf{t}})$ is a finite cyclic group by Theorem (7.3) of [12]. Let S be the set of the n-simplex τ of Σ^* such that P is a vertex of τ. Then it is obvious that $\{\mathbf{C}_\sigma^n\}$ $(\sigma \in S)$ is an open covering of W where $\mathbf{C}_\sigma^n = \mathbf{C}_\sigma^{n+1} \cap \{y_{\sigma 0} = 0\}$.

Remark. To study the deformation of $M_{\mathbf{t}}$ in W, we only need the information about S.

§5. Main theorem

We are ready to state the main theorem. Let $\nu_{\mathbf{t}}$ be the sheaf of the germs of the holomorphic sections of the normal bundle $N_{\mathbf{t}}$ of $M_{\mathbf{t}}$ in W. Let ℓ be as in §1.

Theorem (5.1). (i) $\dim H^0(M_{\mathbf{t}},\nu_{\mathbf{t}}) = \ell$ *and the infinitesimal displacement map* $\xi : T_{\mathbf{t}}U \to H^0(M_{\mathbf{t}},\nu_{\mathbf{t}})$ *is surjective. The kernel of* ξ *is generated by* $\sum_{j=0}^{\ell} t_j \frac{\partial}{\partial t_j}$.

(ii) *Let* $\psi_1,\ldots, \psi_\ell$ *be a system of the generators of* $H^0(M_{\mathbf{t}},\nu_{\mathbf{t}})$ and let $\Psi : M_{\mathbf{t}} \to \mathbf{P}^{\ell-1}$ *be the associated mapping. Then* Ψ *is a birational morphism.*

Let $W = \hat{E}(P)$ and $M_{\mathbf{t}} = E(P)$ as in §4. For each n-simplex $\tau = (Q_0(\tau),\ldots,Q_n(\tau))$ of S, we may assume that

$$Q_0(\tau) = P. \tag{5.2}$$

Let $h_\tau(\mathbf{y}_\tau,\mathbf{t})$ be the defining polynomial of $M_{\mathbf{t}}$ in $\mathbf{C}_\tau^n = \mathbf{C}_\tau^{n+1} \cap \{y_{\tau 0} = 0\}$. h_τ is defined by the equality

$$f_\Delta(\pi_\tau(\mathbf{y}_\tau),\mathbf{t}) = \prod_{i=0}^{n} y_{\tau i}^{d(Q_i(\tau))} h_\tau(\mathbf{y}_\tau,\mathbf{t}). \tag{5.3}$$

Take two simplices α and β in S and let $\alpha^{-1}\beta = (\lambda_{ij})$ $(0 \leq i,j \leq n)$. By (5.2), we have $\lambda_{00} = 1$ and $\lambda_{i0} = 0$ for i = 1,..., n. Recall that $\mathbf{C}_\alpha^n$ and $\mathbf{C}_\beta^n$ are glued by

$$y_{\alpha i} = \prod_{j=1}^{n} y_{\beta j}^{\lambda_{ij}} \qquad (i = 1,\ldots,n). \tag{5.4}$$

Now we consider the line bundle $[M_{\mathbf{t}}]$ which is defined by the cocycle $\{h_{\alpha\beta}\}$ where $h_{\alpha\beta} = h_\alpha / h_\beta$. By (5.3), we have

$$(5.5) \qquad h_{\alpha\beta}(\mathbf{y}_\beta, \mathbf{t}) = \prod_{i=0}^{n} y_{\beta i}^{d(Q_i(\beta))} \Big/ \prod_{i=0}^{n} y_{\alpha i}^{d(Q_i(\alpha))} .$$

Here the right hand is considered as a monomial of $y_{\beta 1}, \ldots, \mathbf{y}_{\beta n}$ through (5.4). The exponent of $y_{\beta 0}$ is zero. We can write $h_\tau(\mathbf{y}_\tau, \mathbf{t})$ more explicitly as

$$(5.6) \qquad h_\tau(\mathbf{y}_\tau, \mathbf{t}) = \sum_{j=0}^{\ell} t_j \, \mathbf{y}_\tau^{A_j(\tau)}$$

where the positive integral vector $A_j(\tau)$ is characterized by

$$(5.7) \qquad \pi_\tau(\mathbf{y}_\tau)^{B_j} = \Big(\prod_{i=0}^{n} y_{\tau i}^{d(Q_i(\tau))}\Big) \, y_\tau^{A_j(\tau)} .$$

Combining (5.7) and (5.5), we obtain

$$(5.8) \qquad \mathbf{y}_\alpha^{A_j(\alpha)} = h_{\alpha\beta} \, \mathbf{y}_\beta^{A_j(\beta)} .$$

(5.8) says that $\{\mathbf{y}_\alpha^{A_j(\alpha)}\}(\alpha \in S)$ is an element of $H^0(W, \mathcal{O}([M_{\mathbf{t}}]))$. Thus we get the inequality $\dim H^0(W, \mathcal{O}([M_t])) \geq \ell + 1$. On the other hand, take a monomial $\mathbf{y}_\sigma^\mu$ where $\mu \neq A_j(\sigma)$ for $j = 0, \ldots, \ell$. (Here σ is fixed.) Let Π_k be the hyperplane which contains $\{A_i(\sigma) \; ; \; i \neq k, \; 0 \leq i \leq n\}$. Then there is an integer k $(0 \leq k \leq n)$ such that $A_k(\sigma)$ and μ are separated by Π_k. Take a simplex $\beta = (P, Q_1(\beta), \ldots, Q_n(\beta))$ such that

$$(5.9) \qquad B_i \in \Delta(Q_1(\beta)) \quad \text{for} \quad i \neq k, \quad i = 0, \ldots, n .$$

Assume that $y_\sigma^\mu = h_{\sigma\beta}\, y_\beta^\nu$ for $\nu = (\nu_1, \ldots, \nu_n)$. Then by the assumption, we have $\nu_1 < 0$. This implies that the section y_σ^μ of $H^0(\mathbf{C}_\sigma^n, ([M_t]))$ cannot be holomorphically extended to W. Thus using GAGA-principle [13], we have proved the following.

Lemma (5.10). dim $H^0(W, \mathcal{O}([M_t])) = \ell + 1$ <u>and</u> $\{y_\alpha^{A_j(\alpha)}\}(\alpha \in S)$, $(j = 0, \ldots, \ell)$ <u>gives a canonical basis</u>.

This is a special case of §6 of [1] and Lemma 2.3 of [10]. For the further geometry of the toric variety W, see [5, 2, 1, 9, 3].

We are ready to prove (i) of Theorem (5.1). From the exact sequence of sheaves on W :

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}([M_t]) \longrightarrow \nu_t \longrightarrow 0,$$

we have the exact sequence

$$(5.11) \quad 0 \to \mathbf{C} \longrightarrow H^0(W, \mathcal{O}([M_t])) \xrightarrow{\theta} H^0(M_t, \nu_t) \to 0.$$

Here we have used the fact that $H^1(W, \ ) = 0$ because W is simply connected ([1]). Thus dim $H^0(M_t, \nu_t) = \ell$ and $H^0(M_t, \nu_t)$ is generated by $\varphi_j = \{\mathbf{y}_\alpha^{A_j(\alpha)}\}_{\alpha \in S}$ $(j = 0, \ldots, \ell)$. They satisfy the obvious relation $\sum_{j=0}^{\ell} t_j \varphi_j = 0$. Now we study the infinitesimal displacement map $\xi : T_t U \to H^0(M_t, \nu_t)$. By the definition of ξ, we have

$$\xi\left(\frac{\partial}{\partial t_j}\right) = \left\{\frac{\partial h_\alpha}{\partial t_j}\right\}_{\alpha \in S} = \{\mathbf{y}_\alpha^{A_j(\alpha)}\}_{\alpha \in S} = \varphi_j.$$

Thus ξ is surjective and the kernel of ξ is generated by $\sum_{j=0}^{\ell} t_j \frac{\partial}{\partial t_j}$. This completes the proof of (i) of Theorem (5.1).

Now we will prove (ii) of Theorem (5.1). Let $\varphi_0, \ldots, \varphi_\ell$ be as above and define $\hat{\Psi} : W \to \mathbf{P}^\ell$ by $\hat{\Psi}(x) = [\varphi_0(x); \ldots; \varphi_\ell(x)]$. Let $\tau \in S$. As the polynomial $h_\tau(\mathbf{y}_\tau)$ contains a non-zero constant term, there exists an integer $0 \leq k \leq n$ such that $A_k(\tau) = (0, \ldots, 0)$. As $\hat{\Psi}(\mathbf{y}_\tau) = [\mathbf{y}_\tau^{A_0(\tau)}; \ldots; \mathbf{y}_\tau^{A_\ell(\tau)}]$ on $\mathbf{C}_\tau^n$, this implies that $\hat{\Psi}$ is a morphism. We have to prove that $\hat{\Psi}$ is generically injective. Note that $\{A_0(\tau), \ldots, A_\ell(\tau)\}$ is equal to the set of the integral points of the simplex spun by $A_j(\tau)$ ($j = 0, \ldots, n$). By Lemma (3.8) of [12], there exist $0 \leq i_1 < \ldots < i_n \leq \ell$ such that ${}^t\zeta = ({}^tA_{i_1}(\tau), \ldots, {}^tA_{i_n}(\tau))$ is a unimodular matrix. Let $\zeta^{-1} = (\zeta_{ij})$. The image of $\hat{\Psi}|\mathbf{C}_\tau^n$ is in the coordinate chart $U_k = \{X_k \neq 0\}$ of $\mathbf{P}^\ell$. Let $Y_j = X_j / X_k$ ($j \neq k$). Assume that $\hat{\Psi}(y_\tau) = (Y_j)_{j \neq k}$ for $y_\tau \in (\mathbf{C}_\tau^*)^n$. Then y_τ is determined by $y_{\tau m} = \prod_{j=1}^{n} Y_{i_j}^{\zeta_{mj}}$ ($m = 1, \ldots, n$). This proves that $\hat{\Psi}$ is injective on $(\mathbf{C}_\tau^*)^n$. Therefore the restriction of $\hat{\Psi}$ to $M_{\mathbf{t}}$ is also a morphism and is injective on $M_{\mathbf{t}} \cap (\mathbf{C}_\tau^*)^n$. The image of $\hat{\Psi}|M_{\mathbf{t}}$ is in the hyperplane H: $\sum_{j=0}^{\ell} t_j X_j = 0$ of $\mathbf{P}^\ell$. Identifying H with $\mathbf{P}^{\ell-1}$, we have $\hat{\Psi}|M_{\mathbf{t}} = \Psi$. This completes the proof of Theorem (5.1).

Remark. If $A = dI_n$, W is the projective space of dimension n and $\{M_t\}$ are projective hypersurfaces of degree d. This case is studied in [6].

§6. Canonical vector fields

Let $\tau \in S$. Then $\Theta_W | C_\tau^n$ is a free -module of rank n with a canonical basis $\{\frac{\partial}{\partial y_{\tau 1}}, \ldots, \frac{\partial}{\partial y_{\tau n}}\}$. We define $\frac{\tilde{\partial}}{\partial y_{\tau i}} = y_{\tau i} \frac{\partial}{\partial y_{\tau i}}$ for $i=1,\ldots,n$. Similarly we define $\tilde{d}y_{\tau i} = \frac{dy_{\tau i}}{y_{\tau i}}$. Let $\beta \in S$ and let $\beta^{-1}\tau = (\lambda_{ij})$ and let $(\mu_{ij}) = \tau^{-1}\beta$. Then we have

Proposition (6.1). (i) <u>We have the formula</u>

$$\frac{\tilde{\partial}}{\partial y_{\tau i}} = \sum_{j=1}^{n} \lambda_{ji} \frac{\tilde{\partial}}{\partial y_{\beta j}}, \qquad \tilde{d}y_{\tau i} = \sum_{j=1}^{n} \mu_{ij} \tilde{d}y_{\beta j}.$$

(ii) $\{ \frac{\tilde{\partial}}{\partial y_{\tau i}} ; i = 1,\ldots, n \}$ <u>can be holomorphically extended to W</u>.

<u>Proof</u>. Recall that $y_{\beta j} = \prod_{i=1}^{n} y_{\tau i}^{\lambda_{ji}}$. Thus the assertion (i) is obvious. The assertion (ii) follows from (i).

Definition (6.2). $\{\frac{\tilde{\partial}}{\partial y_{\tau 1}}, \ldots, \frac{\tilde{\partial}}{\partial y_{\tau n}}\}$ generates a subspace of dimension n of $H^0(W,\Theta_W)$ which we denote by $\mathrm{Can}(W,\Theta_W)$. The restriction of $\mathrm{Can}(W,\Theta_W)$ to $H^0(M_t,\Theta_W|M_t)$ is denoted by $\mathrm{Can}(M_t,\Theta_W)$. We call vector fields in $\mathrm{Can}(W,\Theta_W)$ or in $\mathrm{Can}(M_t,\Theta_W)$ <u>canonical vector fields</u>. These vector fields come from the torus action on W. It is easy to see that

$\dim \mathrm{Can}(M_t, \Theta_W) = n$.

Corollary (6.3). _We have the inequalities_ $\dim H^0(W, \Theta_W) \geq n$ _and_ $\dim H^0(M_t, \Theta_W | M_t) \geq n$.

Now we characterize the image of $\theta : \mathrm{Can}(M_t, \Theta_W) \to H^0(M_t, \nu_t)$. Let σ be the fixed simplex so that $h_\sigma(\mathbf{y}_\sigma, \mathbf{t}) = h(\mathbf{y}_\sigma, \mathbf{t})$ where h is as in (1.1). Let $X \in H^0(M_t, \Theta_W | M_t)$ and let $X = \sum_{i=1}^{n} X_{\tau i} \frac{\partial}{\partial y_{\tau i}}$ on $\mathbf{C}_\tau^n$. Then it is easy to see that

(6.4) $\theta(X) = (\theta(X)_\tau)_{\tau \in S}$ where $\theta(X)_\tau = \sum_{i=1}^{n} X_{\tau i} \frac{\partial h_\tau}{\partial y_{\tau i}}$.

Let $X^1, \ldots, X^n$ be the canonical vector fields defined by

(6.5) $$X^i = \frac{\tilde{\partial}}{\partial y_{\sigma i}} = y_{\sigma i} \frac{\partial}{\partial y_{\sigma i}} \text{ on } \mathbf{C}_\sigma^n \quad (i = 1, \ldots, n).$$

Then we have

(6.6) $$\theta(X^i)_\sigma = y_{\sigma i} \frac{\partial h}{\partial y_{\sigma i}} \quad (i = 1, \ldots n).$$

We claim that $\{\theta(X^i)\}$ $(i = 1, \ldots, n)$ _are linearly independent_. In fact, assume that $\sum_{i=1}^{n} \lambda_i \theta(X^i) = 0$. Then we must have $\sum_{j=1}^{\ell} t_j b_j \mathbf{y}_\sigma^{A_j} \equiv 0$ modulo $h(\mathbf{y}_\sigma, \mathbf{t})$ where $b_j = \sum_{i=1}^{n} \lambda_i a_{ji}$. This implies that $\lambda_i = 0$ for each i. Thus we have shown

Theorem (6.7). $\theta(X^1), \ldots, \theta(X^n)$ _are linearly independent. They are characterized by_

$$\theta(X^i)_\sigma = \frac{d}{ds} h(y_{\sigma 1}, \ldots, s y_{\sigma i}, \ldots, y_{\sigma n}, \mathbf{t})|_{s=1}.$$

Now we consider the following subfamily of $\{M_t\}$. Let $U^e = \{ \mathbf{t} \in U \ ; \ t_0 = \ldots = t_n = 1 \}$. We call $\{M_t\}$ $(t \in U^e)$ <u>the embedded deformation</u>. Let $\xi^e : T_t U^e \to H^0(M_t, \nu_t)$ be the restriction of ξ to $T_t U^e$. Then we have

Theorem (6.8). <u>Assume that</u> $H^0(M_t, \Theta_W | M_t) = \mathrm{Can}(M_t, \Theta_W)$. <u>Then the Kodaira-Spencer map</u> $\delta \circ \xi^e : T_t U^e \to H^1(M_t, \Theta_t)$ <u>is injective and</u> $H^0(M_t, \Theta_t) = 0$.

<u>Proof</u> The second assertion is immediate from Theorem (6.7), (2.4) and the assumption. Assume that $\delta \circ \xi^e(v) = 0$ where $v = \sum_{j=n+1}^{\ell} \lambda_j \frac{\partial}{\partial t_j}$. Then by (2.4), we can write $(\xi^e(v))_\sigma = \sum_{i=1}^{n} \mu_i \, y_{\sigma i} \frac{\partial h}{\partial y_{\sigma i}}$ for some complex $\mu_1, \ldots, \mu_n$. This implies that

$$\sum_{k=1}^{n} \left(\sum_{i=1}^{n} \mu_i \, a_{ki} \right) \mathbf{y}_\sigma^{A_k} + \sum_{k=n+1}^{\ell} \left(\lambda_k + \sum_{i=1}^{n} \mu_i \, a_{ki} \right) \mathbf{y}_\sigma^{A_k} \equiv 0$$

modulo $h(\mathbf{y}_\sigma, \mathbf{t})$. This implies that $\lambda_k = 0$ for $k = n+1, \ldots, \ell$ and $\mu_i = 0$ for $i=1,\ldots,n$, because the left side has no constant term. This completes the proof. It seems that the assumption in Theorem (6.8) is satisfied in many cases if W is not projective space P^n. The following is an example where the Kodaira-Spencer map is not injective.

Example (6.9). (Hashimoto- Oka[4]) Let M be the algebraic surface which is the compactification of $y_1 + y_1^9 y_2^{16} + y_1^3 y_3^4 + 1 = 0$. Then M has the following invariants: $K^2 = 0$, $p_g = 1$ and $\pi_1(M) = Z/2Z$. M has 27 dimensional effective deformation and dim $H^1(M,\Theta_W|M) = 20$. On the other hand, $H^0(M,\Theta_W|M) = 12$ and the dimension of the image of effective deformation is 18.

§7. Deformation of a Godeaux surface.

In this section, we study the case of $n = 3$. Recall that $\Xi = \Delta(P)$ is spun by $B_0, \ldots, B_3$. Let Ξ_i be the 2-face of Ξ with $B_i \notin \Xi_i$ for $i=0,\ldots,3$. Let $P_0,\ldots, P_3$ be the vertices of Σ^* which are adjacent to P such that $\Delta(P_i) \supset \Xi_i$. We define divisors $\hat{C}_i$ of W by $\hat{E}(P) \cap \hat{E}(P_i)$ and divisors C_i of M by $E(P) \cap E(P_i)$ for $i=0,\ldots, 3$. Let σ be as in §4 and we denote $y_{\sigma i}$ by y_i for simplicity. Let $A = \mathbf{C}[y_1, y_1^{-1}, \ldots, y_3, y_3^{-1}]$. For a polynomial $g(\mathbf{y})$ of A, we define an integer $\mathrm{ord}_{\hat{C}_i}\, g(\mathbf{y})$ by the order of the zeros (or poles) of $g(\mathbf{y})$ along the divisor $\hat{C}_i$. Similarly we define $\mathrm{ord}_{C_i}\, g(\mathbf{y})$ by the order of the zeros of $(g|M_{\mathbf{t}})$ along C_i. In general, we have the inequality $\mathrm{ord}_{\hat{C}_i}\, g \leq \mathrm{ord}_{C_i}\, g$.

Definition (7.1). We say that $g(\mathbf{y})$ has _a regular form on_ C_i if $\mathrm{ord}_{\hat{C}_i}\, g(\mathbf{y}) = \mathrm{ord}_{C_i}\, g(\mathbf{y})$.

We fix an index a for $0 \leq a \leq 3$. Let $\tau = (P, Q_1(\tau), Q_2(\tau), Q_3(\tau))$ be a simplex of S such that $Q_1(\tau) = P_a$ and let $\sigma^{-1}\cdot\tau = (\lambda_{ij})$. Then by the definition, we have $\mathrm{ord}_{\hat{C}_a}\, \mathbf{y}^{\nu} = \sum_{j=1}^{3} \nu_j \lambda_{j1}$. We define $h^a(\mathbf{y},\mathbf{t}) = \Sigma'\, t_j\, \mathbf{y}^{A_j}$

where the sum is taken for j such that $B_j \in \Xi_a$. Note that $h^a(\mathbf{y},\mathbf{t})$ is homogeneous with respect to the weight $(\lambda_{11},\lambda_{21},\lambda_{31})$ and

$$\text{(7.2)} \qquad \operatorname{ord}_{\hat{C}_a} \mathbf{y}^{A_a} > \operatorname{ord}_{\hat{C}_a} h^a.$$

Note also that h^a is irreducible in A, because C_a is an irreducible curve and the defining polynomial of C_a is h^a up to the multiplication of a monomial. Take $g \in A$. Let $k = \operatorname{ord}_{\hat{C}_i} g$ and let g_k be the leading term of g with respect to the above weight. Then we have

Lemma (7.3). g <u>has a regular form on</u> C_a <u>if and only if</u> g_k <u>is not zero modulo</u> h^a.

<u>Proof</u>. We can write $g_k(\mathbf{y}(\mathbf{y}_\tau)) = y_{\tau 1}^k\, g'(y_{\tau 2},y_{\tau 3})$. As $g(\mathbf{y}(\mathbf{y}_\tau)) \equiv g_k(\mathbf{y}(\mathbf{y}_\tau))$ modulo $(y_{\tau 1}^{k+1})$, it is easy to see that $g'|C_a \equiv 0$ iff $g_k \equiv 0$ modulo h^a.

Now let $X = \sum_{j=1}^{3} X_j \frac{\tilde{\partial}}{\partial y_j}$ be a rational vector field on W such that $X_j \in A$. We define $\operatorname{ord}_{\hat{C}_i} X = \underset{1\le j\le 3}{\text{minimum}} \operatorname{ord}_{\hat{C}_i} X_j$ and $\operatorname{ord}_{C_i} X = \underset{1\le j\le 3}{\text{minimum}} \operatorname{ord}_{C_i} X_j$. Let $X = \sum_{j=1}^{3} X_{\tau j} \frac{\tilde{\partial}}{\partial y_{\tau j}}$ on $\mathbf{C}_\tau^3$. Then we have $\underset{1\le j\le 3}{\text{minimum}} \operatorname{ord}_{C_a} X_{\tau j} = \underset{1\le j\le 3}{\text{minimum}} \operatorname{ord}_{C_a} X_j$ by Proposition (6.1). In particular, <u>if X is an element of</u> $H^0(M_{\mathbf{t}},\Theta_W|M_{\mathbf{t}})$, <u>we have</u> $\operatorname{ord}_{C_i} X \ge -1$ <u>for each i</u>. Similarly let $\omega = \sum_{j=1}^{3} Y_j\, \tilde{d}y_j$ be a rational 1-form such that $Y_j \in A$. We define $\mathrm{ord}_{\hat{C}_i}\,\omega$ and $\mathrm{ord}_{C_i}\,\omega$ in the same way. Then we have

Lemma (7.4). (i) Let X be as above and assume that $\{X_j\}$ $(j = 1,2,3)$ have regular forms on C_a and assume that $\mathrm{ord}_{C_a} X \le -2$ for some a. Then X is not a holomorphic section of Θ_W over $M_{\mathbf{t}}$.
(ii) Let $D = \sum_{i=0}^{3} n_i\, C_i + D'$ be a divisor on $M_{\mathbf{t}}$ such that the support of D' does not include any of C_i $(i=0,\dots,3)$. Let ω be as above. Assume that $\{Y_j\}$ $(j=1,2,3)$ have regular forms on C_a for some a. If $\mathrm{ord}_{C_a}\,\omega \le -n_a$, the restriction of ω to $M_{\mathbf{t}}$ is not contained in $H^0(M_{\mathbf{t}}, \Omega_W^1 | M_{\mathbf{t}}(D))$.

For the rest of the section, we consider the following example. Let

$$f_\Delta(\mathbf{z}) = z_0^2\, z_1\, z_2^4 + z_1^2\, z_2\, z_3^4 + z_2^2\, z_3\, z_0^4 + z_3^2\, z_0\, z_1^4$$

and let $f(\mathbf{z}) = f_\Delta(\mathbf{z}) + \sum_{i=0}^{3} z_i^{11}$. Let $P = {}^t(1,1.1,1)$. As $\Gamma^*(f)$ is invariant under the canonical $\mathbf{Z}/4\mathbf{Z}$-action, we can take Σ^* to be $\mathbf{Z}/4\mathbf{Z}$-invariant and Σ^* is canonical in the sense of [12]. Namely we have $P_0 = {}^t(1,2,3,1)$, $P_1 = {}^t(1,1,2,3)$, $P_2 = {}^t(3,1,1,2)$ and $P_3 = {}^t(2,3,1,1)$. Let $\sigma = (P, P_0, P_1, R)$ where $R = (P_2 + 2P_0 + 3P_1 + 2P) / 5 = {}^t(2,2,3,3)$. Let $M = E(P)$. The defining equation of M in $\mathbf{C}_\sigma^3$ is

$$h(\mathbf{y}) = y_1^5\, y_3^2 + y_2^5\, y_3^3 + y_3 + 1 = 0.$$

We have shown in Example (9.11) of [12] that $\pi_1(M) = \mathbf{Z}/5\mathbf{Z}$ and $q = p_g = 0$. This surface is known as a Godeaux surface. As ℓ is 11, the dimension of the embedded deformation is 8. The corresponding embedded monomials are: y_2y_3, $y_2^3y_3^2$, y_1y_3, $y_1y_2y_3$, $y_1y_2^2y_3^2$, $y_1^2y_3$, $y_1^2y_2^2y_3^2$ and $y_1^3y_2y_3^2$. See [11]. Let $h(\mathbf{y},\mathbf{t})$ be as before. As numerical data, we have $K \sim 2C_3 - C_2 \sim 2C_1 - C_0$ and $C_i^2 = 1$ and $K^2 = 1$. Here K is a canonical divisor. By the Riemann-Roch theorem, we have $\chi(\Theta_{\mathbf{t}}) = -8$. We will show that

Theorem (7.5). We have $H^0(M_{\mathbf{t}},\Theta_{\mathbf{t}}) = H^2(M_{\mathbf{t}},\Theta_{\mathbf{t}}) = 0$, $H^1(M_{\mathbf{t}},\Theta_{\mathbf{t}}) \cong \mathbf{C}^8$ and the Kodaira-Spencer map

$$\delta\circ\xi^e : T_tU^e \to H^1(M_{\mathbf{t}},\Theta_{\mathbf{t}})$$

is an isomorphism.

Compare with the construction of the moduli space of the Godeaux surfaces by Miyaoka [8]. Note that $\mathbf{Z}/4\mathbf{Z}$ acts canonically on U^e so that $M_{\mathbf{t}} \cong M_{g\mathbf{t}}$ for $g \in \mathbf{Z}/4\mathbf{Z}$.

Lemma (7.6). $H^0(M_{\mathbf{t}},\Theta_W|M_{\mathbf{t}}) \cong \mathbf{C}^3$ and $H^2(M_{\mathbf{t}},\Theta_W|M_{\mathbf{t}}) = 0$.

Proof. Let $\tau = (P,P_2,P_3,R')$ where $R' = {}^t(3,3,2,2)$. We denote $y_{\tau i}$ by u_i for simplicity. Then we have $y_1 = u_1^{-2}u_2$, $y_2 = u_1^{-3}u_2^2$ and $y_3 = u_1^5u_2^{-5}u_3^{-1}$. Let $X \in H^0(M_{\mathbf{t}},\Theta_W|M_{\mathbf{t}})$. By the GAGA-principle, X can be expressed in $\mathbf{C}_\sigma^3 \cap M_{\mathbf{t}}$ as $\sum_{j=1}^{3} X_j \frac{\tilde{\partial}}{\partial y_j}$ where $X_j \in A$.

Assertion. We can assume that X_j has a regular form

<u>on</u> C_2 <u>and</u> C_3 <u>simultaneously</u>.

<u>Proof</u>. We may first assume that $\mathrm{ord}_{\hat{C}_3} X_i = \mathrm{ord}_{C_3} X_i$, using the irreducibility of h^3 in A. Assume that X_i has not a regular form on C_2. We substitute $h^2(\mathbf{y})\mathbf{y}^\nu$ by $(h(\mathbf{y},\mathbf{t}) - h^2(\mathbf{y},\mathbf{t}))\mathbf{y}^\nu$ to change X_i in a regular form on C_2 in a finite steps. Note that this operation does not decrease $\mathrm{ord}_{\hat{C}_3} X_i$. Thus if we change X_i in a regular form X'_i on C_2, we have

$$\mathrm{ord}_{C_3} X_i = \mathrm{ord}_{C_3} X'_i \geq \mathrm{ord}_{\hat{C}_3} X'_i \geq \mathrm{ord}_{\hat{C}_3} X_i.$$

This implies that $\mathrm{ord}_{\hat{C}_3} X'_i = \mathrm{ord}_{C_3} X'_i$ by the regularity assumption on C_3. Assume that the monomial $\mathbf{y}^\nu$ has a non-zero coefficient in X_i. As we have

$$y^\nu = u_1^{-2\nu_1-3\nu_2+5\nu_3}\, u_2^{\nu_1+2\nu_2-5\nu_3}\, u_3^{-\nu_3},$$

we must have $\nu_1 + 2\nu_2 + 1 \geq 5\nu_3 \geq 2\nu_1 + 3\nu_2 - 1$. Combine this with $\nu_1 \geq -\delta_{i1}$, $\nu_2 \geq -\delta_{i2}$ where δ_{ij} is the Kronecker's symbol. The possible cases are $y_2^2\, y_3 \frac{\tilde{\partial}}{\partial y_i}$ (i=1,2,3), $y_1^2\, y_2^{-1} \frac{\tilde{\partial}}{\partial y_2}$, $y_1\, y_2^{-1} \frac{\tilde{\partial}}{\partial y_2}$, $y_1^{-1}\, y_2 \frac{\tilde{\partial}}{\partial y_1}$ and $\frac{\tilde{\partial}}{\partial y_i}$. After checking their linear combinations in detail, we conclude that $H^0(M_\mathbf{t},\Theta_W|M_\mathbf{t}) = \mathrm{Can}(M_\mathbf{t},\Theta_W)$.

Now we consider $H^2(M_\mathbf{t},\Theta_W|M_\mathbf{t})$. By the Serre duality, this is isormophic to $H^0(M_\mathbf{t},\Omega_W^1(K)) \cong H^0(M_\mathbf{t},\Omega_W^1|M_\mathbf{t}(2C_1-C_0))$

where Ω^1_W is the sheaf of the germs of 1-forms on W. Let $\omega = \sum_{i=1}^{3} Y_i \tilde{d}y_i$ be a rational 1-form with $Y_i \in A$ and assume that the restriction of ω is in $H^0(M_t, \Omega^1_W | M_t(2C_1 - C_0))$. Let $\mathbf{y}^\nu$ be a monomial with a non-zero coefficient in Y_i. Then by Lemma (7.4), we have $\nu_1 \geq -2 + \delta_{i1}$, $\nu_2 \geq 1 + \delta_{i2}$ and $\nu_1 + 2\nu_2 \geq 5\nu_3 \geq 2\nu_1 + 3\nu_2$. This has no integral solution. This implies that $H^2(M_t, \Theta_W | M_t) = 0$, completing the proof of Lemma (7.6).

Proof of Theorem (7.5). We consider the exact sequence (1.4). Considering the section φ of $H^0(M_t, \nu_t)$ such that $\varphi_\sigma = 1$, we see that $N_t = [5C_3]$. Thus by Riemann-Roch theorem, we have $\chi(\nu_t) = 11$, $\chi(\Theta_t) = -8$ and $\chi(\Theta_W | M_t) = 3$. This implies that $H^1(M_t, \Theta_W | M_t) = H^2(M_t, \nu_t) = 0$ and $H^2(M_t, \Theta_t) = H^0(M_t, \Theta_t) = 0$ and $H^1(M_t, \Theta_t) \cong \mathbf{C}^8$. This completes the proof by Theorem (6.8).

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