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# LINEAR REPRESENTATIONS OF BRAID GROUPS VIA LOGARITHMIC CONNECTIONS

by Toshitake KOHNO
(Nagoya University)

In this note we discuss monodromy representations of braid groups arising from certain connections with regular singularities. In Section 1, we give an expression of our monodromy by means of Chen's iterated integrals and show a Riemann-Hilbert type correspondence in our situation. In Section 2, we focus a typical example of our connection, which gives classical Burau representations. The integrability condition of our connection is revelent to the classical Yang-Baxter equations. In Section 3, we deal with the monodromy representations of our connection associated with the rational solutions of the classical Yang-Baxter equation which contains the Jones representation as the simplest case.

Recently, a remarkable relation between the monodromy representation of braid groups obtained in this way and the vertex operators on the conformal field theory was discovered by A. Tsuchiya and Y. Kanie [TK]. They obtaied the irreducible representations of Hecke algebras due to H. Wenzl [W] as the monodromy.

Notations:

$B_n$ : braid group on $n$ strings with generators $\sigma_i$ , $1 \le i \le n-1$ , represented by a braid interchanging strings $i$ and $i+1$.

$P_n$ : pure braid group on $n$ strings with generators $\gamma_{ij}$ = $\sigma_i \sigma_{i+1} \cdots \sigma_{j-1} \sigma_j^2 \sigma_{j-1}^{-1} \cdots \sigma_i^{-1}$ , $1 \le i < i \le n$ .

$X_n = \{ (z_1, \ldots, z_n) \in \mathbb{C}^n \ ; \ z_i \ne z_j \ \text{if} \ i \ne j \}$ .

## 1. Holonomy Lie algebras and a Riemann-Hilbert type correspondence

First we give an intrinsic definition of the holonomy Lie algebra for a simplicial complex $X$ . The cup product is considered to be a linear map $\cup : \Lambda^2 H^1(X;\mathbb{Q}) \longrightarrow H^2(X;\mathbb{Q})$ , which induces by duality the homomorphism $\eta : H_2(X;\mathbb{Q}) \longrightarrow \Lambda^2 H_1(X;\mathbb{Q})$ . Let $LH_1(X;\mathbb{Q})$ be the free Lie algebra generated by $H_1(X;\mathbb{Q})$ over $\mathbb{Q}$ . Let $J$ be the homogeneous ideal of $LH_1(X;\mathbb{Q})$ generated by image $\eta$ . Here we identify $\Lambda^2 H_1(X;\mathbb{Q})$ with the degree two part of $LH_1(X;\mathbb{Q})$ . The quotient Lie algebra $LH_1(X;\mathbb{Q})/J$ is called the *holonomy Lie algebra* of $X$ over $\mathbb{Q}$ , which we shall denote by $\mathfrak{G}_X$ .

Let us discuss the meaning of the holonomy Lie algebra in the case of the complement of a hypersuface in $\mathbb{C}^n$ . Let $D_i$ , $1 \le i \le s$ , be a finitely many family of irreducible hypersurfaces defined by $f_i$ . We consider a matrix valued 1-form $\Omega = \sum_i P_i \, d\log f_i$ with some constant matrices $P_i$ . Let $X$ be the complementary space of the union $\cup D_i$ . The 1-form $\Omega$ is considered to be a connection of a trivial vector bundle of rank $m$ over $X$ in a natural way. This connection is integrable if and only if the condition

$$\Omega \wedge \Omega = 0 \tag{1.1}$$

is satisfied. Let $e : H_1(X ;\mathbb{C}) \longrightarrow gl(m, \mathbb{C})$ be the linear map defined by $e(X_i) = P_i$ , $1 \le i \le s$ , where $X_i$ denotes the homology class of the hypersurface $D_i$ . Then we observe that the integrability condition (1.1) is satisfied if and only if $e$ can be lifted to a Lie algebra homomorphism of $\mathfrak{G}_X \otimes \mathbb{C}$ .

In the case of the complement of the union of finitely many complex hyperplanes $H_j$ , $1 \le j \le s$ , in $\mathbb{C}^n$ , the holonomy Lie algebra is generated by $X_j$ , $1 \le j \le s$ , with relations :

$$[\, X_{j_\nu} , X_{j_1} + \dots + X_{j_p} \,] = 0 \quad , \ 1 \le \nu \le p , \tag{1.2}$$

for any maximal family $\{ H_{j_1} , \dots, H_{j_p} \}$ such that $\mathrm{codim}_{\mathbb{C}} ( H_{j_1} \cap \dots \cap H_{j_p} ) = 2$ (see [K1]) .

To describe basic results on a relation between holonomy Lie algebras and fundamental groups, we recall several notations. Let $\Gamma_1\pi_1(X) \supset \dots \supset \Gamma_m\pi_1(X) \supset \dots$ denote the *lower central series* of $\pi_1(X)$ defined inductively by $\Gamma_1\pi_1(X) = \pi_1(X)$ , $\Gamma_{m+1}\pi_1(X) = [\, \pi_1(X), \Gamma_m\pi_1(X) \,]$ , $m \ge 2$ . The lower central series of $\mathfrak{G}_X$ is defined in the same way by using the bracket products. The graded Lie algebra $[\, gr\ \pi_1(X) \,] \otimes \mathbb{Q}$ is defined to be

$$\oplus_j \ [\, \Gamma_j\pi_1(X) \,/\, \Gamma_{j+1}\pi_1(X) \,] \otimes \mathbb{Q}$$

with the bracket products induced from the commutator produsts (see [MKS]). The graded Lie algebra $gr\ \mathfrak{G}_X$ is defined to be $\oplus_j\ [\ \Gamma_j\mathfrak{G}_X\ /\ \Gamma_{j+1}\mathfrak{G}_X\ ]$ .

Let us denote by $[\ \pi_1(X)/\ \Gamma_j\pi_1(X)\ ] \otimes \mathbb{Q}$ the *Malcev Lie algebra* associated with the nolpotent group $\pi_1(X)/\ \Gamma_j\pi_1(X)$ , which is a nilpotent Lie algebra over $\mathbb{Q}$ defined inductively by means of the central extensions :

$$\Gamma_j\pi_1(X)/\ \Gamma_{j+1}\pi_1(X) \longrightarrow \pi_1(X)/\ \Gamma_{j+1}\pi_1(X) \longrightarrow \pi_1(X)/\Gamma_j\pi_1(X)$$

(see [M] for a precise definition).

**Theorem 1.3** ([K1]). *Let $X$ be the complement of a complex hypersurface in $\mathbb{C}^n$ . Then we have the following isomorphism of nilpotent Lie algebras over $\mathbb{Q}$ :*

(1.3.1) $\quad [\ \pi_1(X)/\ \Gamma_j\pi_1(X)\ ] \otimes \mathbb{Q} \simeq \mathfrak{G}_X\ /\ \Gamma_j\mathfrak{G}_X \quad , j\geq 1$ .

*In paticular, we have an isomorphism :*

(1.3.2) $\quad [\ gr\ \pi_1(X)\ ] \otimes \mathbb{Q} \simeq gr\ \mathfrak{G}_X$ .

The proof is based on a Sullivan's result [S] which describes a relation between the nilpotent completion on fundamental groups and 1-minimal models and the Morgan's mixed Hodge structure on the 1-minimal model [M] .

Let us go back to the situation of an integrable connection $\Omega = \sum P_i d\log f_i$ on the trivial bundle $E$ on the complement of a hypersurface. This defines a foliation transversal to the fibers on the tatal space of $E$ , which gives a linear representation of the fundamental group

$$\theta : \pi_1(X, *) \longrightarrow \mathrm{Aut}(E_*)$$

as the holonomy. We recall the notion of *iterated integrals* due to K.T. Chen [C] , which gives an expression of our monodromy representation $\theta$ .

(1.4) *Definition.* Let $X$ be a smooth manifold and let $\omega_i$ , $1 \leq i \leq r$ , be matrix valued 1-forms on $X$ . For a path $\gamma : [0,1] \longrightarrow X$ we put $\gamma^*\omega_i = A_i(t)dt$ and we define the *iterated integral* by

$$\int_\gamma \omega_1\omega_2 \ldots \omega_r = \int_0^1 A_1(t_1)\int_0^{t_1} A_2(t_2) \ \ldots \int_0^{t_{r-1}} A_r(t_r)dt_r \ldots dt_1$$

Lemma 1.5 ( K.T. Chen [C]). *Let us suppose that the connection* $\Omega$ *is integrable. Then the monodromy representation* $\theta$ *is given by*

$$\theta(\gamma) = 1 + \int_\gamma \Omega + \int_\gamma \Omega\Omega + \int_\gamma \Omega\Omega\Omega + \ \ldots \ .$$

We see that for each $\gamma$ there exists $M>0$ such that $\| \int_\gamma \underbrace{\Omega \ \ldots \ \Omega}_{r} \| = O(M^r/r!)$ , therefore the above series is convergent.

As a universal expression of (1.5) , we obtain a homomorphism

$$\theta^\sim : \pi_1(X ,*) \longrightarrow U(\mathfrak{G}_X \otimes \mathbb{C})^\wedge$$

by the iterated integral of the Chen's formal connection

$$\Omega^\sim = \sum X_i \otimes d\log f_i \ .$$

Here $U(\mathfrak{G}_X \otimes \mathbb{C})^\wedge$ denotes the completion of the universal enveloping algebra of $\mathfrak{G}_X \otimes \mathbb{C}$ with respect to the natural filtration by degrees and $X_i$ is the homology class corresponding to the hypersurface $D_i$ .

For a group $G$ we denote by $\mathbb{C}[G]$ its group algebra over $\mathbb{C}$ . Let $\varepsilon : \mathbb{C}[G] \longrightarrow \mathbb{C}$ denote the augmentation homomorphism and we put $IG = \mathrm{Ker}\ \varepsilon$ . Let $\mathbb{C}[G]^\wedge$ be the completion of $\mathbb{C}[G]$ with respect to the topology defined by $\{I^kG\}_{k\geq 1}$ , where $I^kG$ signifies the k-th power of $IG$ . Let $j : G \longrightarrow \mathbb{C}[G]^\wedge$ be the natural homomorphism.

**Theorem 1.6** ( [A1][H] ). *We have an isomorphism of complete Hopf algebras*

$$\mathbb{C}[\ \pi_1(X)\ ]^\wedge \simeq U(\mathfrak{G}_X\otimes\mathbb{C})^\wedge$$

*such that the following diagram is commutative.*

$$\begin{tikzcd} & \mathbb{C}[\ \pi_1(X)\ ]^\wedge \arrow[d, "\wr"] \\ \pi_1(X) \arrow[ur, "j"] \arrow[r, "\theta^\sim"'] & U(\mathfrak{G}_X\otimes\mathbb{C})^\wedge \end{tikzcd}$$

(1.7) *Remark.* By taking the primitive part $P$ of $A = U(\mathfrak{G}_X\otimes\mathbb{C})^\wedge$ , i.e., $P = \{\ x \in A\ ;\ \Delta\ x = x\hat{\otimes}1 + 1\hat{\otimes}x\ \}$ , we obtain the Malcev Lie algebra of the nilpotent completion of $\pi_1(X)$ over $\mathbb{C}$ . This describes a relation between Theorem 1.3 and Theorem 1.6 .

In the case $X = \{\ (z_1, \ldots, z_n) \in \mathbb{C}^n\ ;\ z_i\neq z_j \text{ if } i\neq j\ \}$ , the fundamental group of $X$ is the *pure braid group on* $n$ *strings*. The holonomy Lie algebra $\mathfrak{G}_X$ is generated by $X_{ij}$ , $1\leq i<j\leq n$ , with the relations :

$$[\ X_{ij}\ ,\ X_{ik} + X_{jk}\ ] = [\ X_{ij} + X_{ik}\ ,\ X_{jk}\ ] = 0\ ,\quad i<j<k$$

$$[\ X_{ij}\ ,\ X_{k\ell}\ ] = 0\ ,\quad i,j,k,\ell \text{ distinct .}$$

The above relations are called the *infinitesimal pure braid relations*.

Let $\mathbb{C}\langle\langle X_{ij}\rangle\rangle$ denote the non-commutative formal power series ring with inderterminants $X_{ij}$ , $1\leq i<j\leq n$ . Let $I$ be the two-sided ideal of $\mathbb{C}\langle\langle X_{ij}\rangle\rangle$ generated by

$$[X_{ij}, X_{ik}+X_{jk}], \quad [X_{ij}+X_{ik}, X_{jk}], \quad i<j<k$$
$$[X_{ij}, X_{k\ell}], \quad i, j, k, \ell \text{ distinct.}$$

We denote by $A$ the quotient algebra $\mathbb{C}\langle\langle X_{ij}\rangle\rangle/I$ .

The main result of this section is the following.

**Theorem 1.8.** *Let* $\rho : P_n \longrightarrow GL(m, \mathbb{C})$ *be a linear representation such that each* $\| \theta(\gamma_{ij}) - 1 \|$ *is sufficiently small for* $1\leq i<j\leq n$ . *Then there exist constant matrices* $\Omega_{ij}$ , $1\leq i<j\leq n$ , *close to* $0$ , *satisfying the infinitesimal pure braid relations , such that the monodromy of the connection* $\Omega = \sum_{1\leq i<j\leq n}\Omega_{ij}d\log(z_i - z_j)$ *is the given* $\rho$ .

(1.9) *Remark.* In the case of unipotent representations, the above statement follows from Aomoto's theorem [A1] without the hypothesis on $\| \theta(\gamma_{ij}) - 1 \|$ ( see also [H]).

The key fact to prove Theorem 1.2.6 is the following.

**Proposition 1.10.** *The universal monodromy* $\tilde{\theta} : P_n \longrightarrow A$ *is injective.*

For the proof of the Proposition the reader may refer to [K5].

The proof of Theorem 1.10 is outlined in the following way. For an integrable connection $\Omega$ the monodromy matrices $\theta(\gamma_{ij})$ are given by (1.5). We put $M_{ij} = \log \theta(\gamma_{ij})$ . By taking the logarithm of (1.5), we obtain the expansion :

$$M_{ij} = \Omega_{ij} + \sum_{|I|\geq 2} a_I \Omega_I \quad , \quad a_I \in \mathbb{C} \ , \ 1\leq i<j\leq n \ .$$

Here $I$ stand for usual multi-indices. By means of the injectivity of $\vartheta$ , we can formally invert the power series :

$$Z_{ij} = X_{ij} + \sum_{|I|\geq 2} a_I X_I$$

to get the power series

$$X_{ij} = Z_{ij} + \sum_{|I|\geq 2} b_I Z_I \ , \quad b_I \in \mathbb{C} \ .$$

It follows from a work of Golubeva [G] that when each $\| \theta(\gamma_{ij})-1 \|$ is small enough the above power series converges absolutely for $Z_{ij} = \log \theta(\gamma_{ij})$ and that the monodromy of the connection thus obtained is the given $\theta$ .

## 2. Jordan-Pochhammer matrices and Burau representations

We start with the natural projection $\pi : X_{n+1} \longrightarrow X_n$ . In the followings, fix a complex number $\lambda$ such that $\lambda$ and $n\lambda$ are not integers. Let $\mathcal{L}$ be the local system over $X_{n+1}$ associated with the representation defined by sending $\gamma_{ij}$ to $e^{2\pi i\lambda}$ . The main object of this paragragh is to study the local system over $X_n$ defined by $\mathbb{R}^1\pi_*\mathcal{L}^*$ . Here $\mathcal{L}^*$ stands for the dual local system of $\mathcal{L}$ . Let $i : Z \hookrightarrow X_{n+1}$ denote the fiber of $\pi$ over $( z_1 , \dots , z_{n+1} )$ .

We put $\Phi = (\xi-z_1)^{\lambda} \ldots (\xi-z_n)^{\lambda}$ , where $\xi = z_{n+1}$ . for $w \in H_1(Z, i^*\mathcal{L})$ , let us consider the integral

$$F_i(z_1, \ldots, z_n) = \int_w \Phi \, d\log(\xi-z_i) , \quad 1\le i\le n$$

which are considered as multi-valued functions on $X_n$ .

**Proposition 2.1.** *We have the following equalities :*

$$dF_i = \sum_{j\neq i}(\lambda F_i - \lambda F_j)d\log(z_i - z_j) , \quad 1\le i\le n .$$

*Proof.* We see that

$$dF_i = -\sum_{j\neq i}\{ \int_w \lambda\Phi(\xi-z_i)^{-1}(\xi-z_j)^{-1}d\xi \}dz_j$$
$$- \{ \int_w (\lambda-1)\Phi(\xi-z_i)^{-2}d\xi \}dz_i .$$

The first term is equal to

$$-\sum_{j\neq i}(\lambda F_i - \lambda F_j)(z_i - z_j)^{-1}dz_j .$$

It suffices to show

$$(2.2) \qquad \int_w (\lambda-1)\Phi(\xi-z_i)^{-2}d\xi + \sum_{j\neq i}(\lambda F_i - \lambda F_j)(z_i - z_j)^{-1} = 0 .$$

The LHS of the above expression can be written as

$$\int_w \lambda\Phi(\xi-z_j) \sum_{j=1}^{n}(\xi-z_j)^{-1}d\xi - \int_w \Phi(\xi-z_i)^{-2}d\xi$$

which is equal to $\int_w d\{ \Phi(\xi-z_i)^{-1}\}$ for a fixed $(z_1, \ldots, z_n)$ . This proves (2.2) by means of Stokes theorem, which completes the proof.

Let $w_j$ , $1\le j\le n-1$ , be a basis of $H_1(Z, i^*\mathcal{L})$ chosen as in [DM] Section 1. We put

$$Y = \begin{bmatrix} \int_{w_1} \Phi \, d\log(\zeta - z_1) , & \ldots , & \int_{w_{n-1}} \Phi \, d\log(\zeta - z_1) , & 1 \\ \vdots & & \vdots & \vdots \\ \int_{w_1} \Phi \, d\log(\zeta - z_n) , & \ldots , & \int_{w_{n-1}} \Phi \, d\log(\zeta - z_n) , & 1 \end{bmatrix}$$

The following $n\times n$ matrix $J_{ij}$ , $1\le i<j\le n$ , is called the *Jordan-Pochhammer matrix.*

$$\begin{array}{c} \\ J_{ij} = \end{array} \begin{bmatrix} \ddots & & & & \\ & \lambda & \cdots & -\lambda & \\ & \vdots & & \vdots & \\ & -\lambda & \cdots & \lambda & \\ & & & & \ddots \end{bmatrix} \begin{array}{l} \\ (i \\ \\ (j \\ \\ \end{array}$$

Here all the other components are zero. By means of Proposition 2.1 we have

**Corollary 2.3.** *The matrix* $Y$ *defined above is a fundamental solution of the total differential equation*

$$dy = \sum_{1\le i<j\le n} J_{ij} \, d\log(z_i - z_j) . \, y \quad .$$

The 1-form $\sum_{1\le i<j\le n} J_{ij} \, d\log(z_i - z_j)$ defines an integrable connection on the trivial bundle $X_n \times \mathbb{C}^n$ . Let the symmetric group $S_n$ act diagonally on this bundle via the premutation of coordinates. The above connection is invariant by this action, hence it defines a local system over $Y_n = X_n / S_n$ , which is the direct sum of a local system of rank $n-1$ , say $\mathcal{J}_n$ , and a rank 1 trivial local system. We call $\mathcal{J}_n$ the *Jordan-Pochhammer system.*

Let us discuss the monodromy representation of $B_n$ associated with $\mathcal{J}_n$. For this purpose we recall the Burau representation (see [B]). Let $\beta_j$, $1 \leq j \leq n-1$, be the matrices with $\mathbb{Z}[t, t^{-1}]$ coefficients defined by

$$\beta_j = \begin{array}{cc} \left[\begin{array}{ccccccc} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 1-t & 1 & & \\ & & & t & 0 & & \\ & & & & & 1 & \\ & & & & & & \ddots \\ & & & & & & & 1 \end{array}\right] & \begin{array}{c} \\ \\ \\ (j \\ \\ \\ \\ \\ \end{array} \end{array}$$

The correspondence $\sigma_j \longrightarrow \beta_j$ defines a linear representation of $B_n$ called the *Burau representation.* Since this representation remains invariant the subspace such that the sum of the coordinates is zero, we get a $n-1$ dimensional representation

$$\psi_n : B_n \longrightarrow GL_{n-1}(\mathbb{Z}[t, t^{-1}])$$

which we call the *reduced Burau representation.*

Proposition 2.4. *The monodromy representation of* $B_n$ *associated with the Jordan-Pochhammer system* $\mathcal{J}_n$ *is the reduced Burau representation with* $t = e^{2\pi i\lambda}$.

*Proof.* Let us assume that $\lambda$ and $n\lambda$ are not integers. By Corollary 2.3, the proposition can be proved by looking at the monodromy of the integrals of Pochhammer type :

$$G_j = \int_{\gamma_j} (\xi - z_1)^{\lambda} \ldots (\lambda - z_n)^{\lambda} d\xi \quad , \quad 1 \leq j \leq n$$

where $\{\gamma_j\}_{1 \leq j \leq n}$ is a basis of $\pi_1(\mathbb{C} - \{z_1, \ldots, z_n\}, *)$.

By looking at the action of $B_n$ on the above free group (see [B] Cor. 1.8.3 ), we see that by the action of $\sigma_i$ the integrals $G_i$ 's are transformed by the rule :

$$G_i \longrightarrow (1-e^{2\pi i\lambda})\, G_i + e^{2\pi i\lambda} G_{i+1}$$
$$G_{i+1} \longrightarrow G_i$$
$$G_j \longrightarrow G_j \quad \text{if} \quad j \neq i,\ i+1 .$$

This is nothing but the Burau representation. By choosing a basis of $H_1(Z, i^*\mathcal{L})$ , we get the reduced Burau representation. We know by Lemma 1.5 that the monodromy representation is holomorphic with respect to $\lambda$ , hence by an analytic continuation we have proved the proposition for any $\lambda$ .

## 3. Braid groups, classical Yang-Baxter equations and Jones algebras

### 3.1. Review of classical Yang-Baxter equations

Let $V$ be a finite dimensional complex vector space. By the *classical Yang-Baxter equation* we mean the following functional equation for a matrix valued function $r(u) \in \mathrm{End}(V\otimes V)$ of $u \in \mathbb{C}$ :

(3.1.1)

$$[\, r_{12}(u),\ r_{13}(u+v)\, ] + [\, r_{12}(u),\ r_{23}(v)\, ] + [\, r_{13}(u+v),\ r_{23}(v)\, ] = 0$$

for any $u,\ v \in \mathbb{C}$ . Here $r_{ij}(u) \in \mathrm{End}(V_1\otimes V_2\otimes V_3)$, $V_1 = V_2 = V_3 = V$ , signifies the matrix $r(u)$ on the space $V_i \otimes V_j$ , acting as identity on the third space ; e.g., $r_{12}(u) = r(u)\otimes 1_V$ , $r_{23}(u) = 1_V\otimes r(u)$ . Since the equation 3.1.1 is written in terms of bracket products, it makes sense for a $\mathfrak{g} \otimes \mathfrak{g}$ - valued function $r(u)$ , with an abstract Lie algebra $\mathfrak{g}$ . To each such $\mathfrak{g} \otimes \mathfrak{g}$ - valued solution $r(u)$ , we may associate a solution of (3.1.1) $(\rho\otimes\rho)(r(u)) \in \mathrm{End}(V\otimes V)$ , by specifying an irreducible representation $\rho : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ . In the case $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$ , solutions of the classical Yang-Baxter equation have been classified by Belavin-Drinfel'd (see [BD] for the precise statement). In particular, we know the following rational solution.

Proposition 3.1.2 ( Belavin-Drinfel'd [BD] ). *Let* $\mathfrak{g}$ *be a simple Lie algebra over* $\mathbb{C}$ *and let* $\{I_\alpha\}$ *be an orthonormal basis of* $\mathfrak{g}$ *with respect to the Cartan-Killing form. We put* $\tau = \sum_\alpha I_\alpha \otimes I_\alpha \in \mathfrak{g} \otimes \mathfrak{g}$ . *Then* $r(u) = \tau / u$ *is a solution of the classical Yang-Baxter equation.*

*Proof.* We denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ . Let $c \in U(\mathfrak{g})$ be the Casimir element defined by $c = \sum_\alpha I_\alpha . I_\alpha$ . Let $\Delta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$ be the diagonal homomorphism defined by $\Delta(x) = x\otimes 1 + 1\otimes 1$ for $x \in \mathfrak{g}$ . Then $\tau$ can be written as

(3.1.3) $\qquad \tau = \Delta(c) - c\otimes 1 - 1\otimes c$ .

Let us recall the well-known fact that the Casimir element $c$ lies in the center of $U(\mathfrak{g})$ (see for example [Hu]). It follows from the expression 3.1.3 that

(3.1.4) $\qquad [\ \Delta(x), \tau\ ] = 0$ for any $x \in \mathfrak{g}$ .

In particular, we have

(3.1.5) $[\ \tau_{12}+\tau_{13}\ ,\ \tau_{23}\ ] = [\ \tau_{12}\ ,\ \tau_{13}+\tau_{23}\ ] = 0$ .

Here $\tau_{ij}$ is defined by $r_{ij}(u) = \tau_{ij}\ /\ u$ . By an elementary computation we check that the equation 3.1.5 signifies that $r(u)$ is a solution of the classical Yang-Baxter equation.

Let us observe that the equation 3.1.5 corresponds to the infinitesimal pure braid relations in the case $n=3$.
This leads us to the following construction. Let $\rho_i : \mathfrak{g} \longrightarrow V_i$ , $1 \le i \le n$ , be a family of irreducible representations of $\mathfrak{g}$ . We define $\tau_{ij} \in \mathrm{End}(\ V_1 \otimes \ldots \otimes V_n\ )$ by $\tau_{ij} = (\rho_i \otimes \rho_j)(\tau)$ , $1 \le i < j \le n$ . Here $\rho_i$ signifies the representation $\rho_i$ on $V_i$ , acting as identity on the other factors. Then we have

Lemma 3.1.6. *Let* $\tau_{ij} \in \mathrm{End}(\ V_1 \otimes \ldots \otimes V_n\ )$ *be the matrices defined above. Then these satisfy the infinitesimal pure braid relations.*

*Proof.* The relations 1.1.5 follows from 3.1.5 . The other relations are clear from the construction.

Now let us consider the connection

(3.1.7) $\Omega = \sum_{1 \le i < j \le n} \tau_{ij}\ d\log(z_i - z_j)$

on $X_n = \{\ (\ z_1\ ,\ \ldots,\ z_n\ ) \in \mathbb{C}^n\ ;\ \ z_i \ne z_j\ \ \text{if}\ \ i \ne j\ \}$ . By Lemmas 3.1.6 and Sect.1, the above connection is integrable and hence gives a linear representation of the pure braid group on n strings as the monodromy of the total differential equation $dy = \Omega y$ . Starting from a simple Lie algebra $\mathfrak{g}$ and a family of linear representations $\rho_i : \mathfrak{g} \longrightarrow \mathrm{End}(V_i)$ , $1 \leq i \leq n$ , we have obtained a linear representation

(3.1.8) $\qquad \theta : P_n \longrightarrow GL( V_1 \otimes \ldots \otimes V_n )$ .

Our problem is to give an explicit form of the above representation. Let us remark that the total differential equations of this type appear naturally as the differential equations satisfied by N-point functions in the two dimensional conformal field theory (see [BPZ][TK]).

3.2. The case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$

The object of this paragragh is to describe the monodromy representation $\theta : P_n \longrightarrow GL(V_1 \otimes \ldots \otimes V_n)$ defined in 3.1.8 in the case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and all $\rho_i : \mathfrak{g} \longrightarrow \mathrm{End}(V_i)$ , $1 \leq i \leq n$ , are two dimensional irreducible representations. We put

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \quad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} , \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} .$$

In our case, the element $\tau \in \mathfrak{g} \otimes \mathfrak{g}$ defined in 3.1 can be written in the form

(3.2.1) $\qquad \tau = 2^{-1} h \otimes h + e \otimes f + f \otimes e$ .

For $\lambda \in \mathbb{C}$ , we consider the following total differential equation :

(3.2.2) $\qquad dy = - \sum_{1 \leq i < j \leq n} \lambda \, \tau_{ij} d\log(z_i - z_j) . \, y$

on $X_n$ . Let us denote by $\theta_\lambda : P_n \longrightarrow GL(V_1 \otimes \ldots \otimes V_n)$ the corresponding one-parameter family of monodromy representations. We put $V_1 = V_2 = \ldots = V_n = V$ in the followings.

We put

$$\mathcal{F}_n = \otimes_{i=1}^{n} M_2(\mathbb{C}) \quad .$$

Let $e$ be an element of $\mathcal{F}_n$ defined by

$$e = \{(1+q)^{-1}q\ (e_{11}\otimes e_{22}) + (1+q)^{-1}\sqrt{q}\ (e_{12}\otimes e_{21} + e_{21}\otimes e_{12}) + (1+q)^{-1}(e_{22}\otimes e_{11})\}\otimes 1\otimes 1\otimes \dots \otimes 1 ,$$

where $q \in \mathbb{C}^*$ , and $e_{ij}$ are matrix units for $M_2(\mathbb{C})$ . The above $\mathcal{F}_n$ has an obvious shifting endomorphism $\sigma$ defined by

$$\sigma.(x_1\otimes \dots \otimes x_n) = x_n\otimes x_1\otimes \dots \otimes x_{n-1} .$$

We put $e_i = \sigma^{i-1}(e)$ , $1\le i\le n-1$ . We see that the correspondence $\sigma_i \longrightarrow T(\sigma_i, q) = qe_i - (1-e_i)$ defines a linear representation of the braid group, which is known as the Pimsner-Popa-Temperley-Lieb (PPTL) representation in the case $n \longrightarrow \infty$ (see [J]) . Our main theorem in this section is the following.

Theorem. 3.2.3. *Let* $\theta_\lambda : P_n \longrightarrow GL(V\otimes \dots \otimes V)$ *be the monodromy representation of the connection*

$$- \textstyle\sum_{1\le i<j\le n} \lambda\ \tau_{ij} d\log(z_i - z_j)$$

*defined above by means of* $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ *and its two dimensional irreducible representation. Then* $\theta_\lambda$ *is equivalent to the restriction of the linear representation of* $B_n$ *defined by*

$$\theta_\lambda(\sigma_i) = q^{1/4}\{ qe_i - (1-e_i) \} , \quad 1\le i\le n-1 ,$$

*where* $q = e^{2\pi i\lambda}$ .

The proof is given in [K5].

*Remark 3.2.4.* Let us recall that $e_i$ , $1 \leq i \leq n-1$ , appearing in the definition of the PPTL representation satisfy the relations :

$$e_i^2 = e_i$$
$$e_i e_{i\pm 1} e_i = \beta^{-1} e_i \ , \quad \beta = 2+q+q^{-1}$$
$$e_i e_j = e_j e_i \quad \text{if} \quad |i-j| \geq 2$$

Let $A_{\beta,n}$ denote the abstract $\mathbb{C}$ algebra generated by $1, e_1, \ldots, e_{n-1}$ with the above relations. In the case $\beta \geq 4$ , this is a kind of the *Jones algebra* (see [J]). We have shown that our monodromy representation factors through the natural homomorphisms :

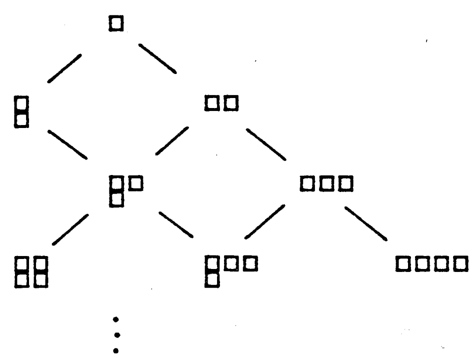
$$\mathbb{C}[B_n] \longrightarrow H(q,\ n) \longrightarrow A_{\beta,n}$$

where the second homomorphism is defined by the correspondence

$$g_i \longrightarrow qe_i - (1-e_i) \ .$$

By using a description of the representations of $A_{\beta,n}$ explained in [J] , our monodromy representation is decomposed in to the irreducible components parametrized by the following Young diagrams if $q$ is not a root of unity.

$B_2$
$B_3$
$B_4$



We denote by $\rho(d_1,\ d_2)$ the Hecke algebra representation of $B_n$ corresponding to the Young diagram of type $(d_1,\ d_2)$ ,

$d_1 \geq d_2 \geq 0$ , $d_1 + d_2 = n$ . As a corollary to our main theorem, we have the following.

**Corollary 3.2.5.** *If* $q$ *is not a root of unity, our monodromy representation 3.2.3 is a direct sum of* $\rho(d_1, d_2)$ , $d_1 \geq d_2 \geq 0$ , $d_1 + d_2 = n$ *with the multiplicity* $2d_1 - n + 1$ .

*Remark 3.2.6* : In the above decomposition, the Jordan-Pochhammer system $\mathcal{J}_n$ examined in Section 2 corresponds to the Young diagram of type (n-1, 1) .

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*Department of Mathematics*

*Faculty of Science*

*Nagoya University*

*Chikusa-ku, Nagoya 464 , Japan*