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<tr>
<td>Author(s)</td>
<td>SAI TO, Kyoji; KOHNO, Toshitake</td>
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<td>Citation</td>
<td>数理解析研究所論論集 数理解析研究所論論集</td>
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<tr>
<td>Issue Date</td>
<td>1987-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/100094">http://hdl.handle.net/2433/100094</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
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<td>Textversion</td>
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Braid groups and their applications

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INTRODUCTION. The purpose of this note is to present a brief introduction on recent advances concerning Artin's braid groups principally from a point of view of complex analytic geometry. In Section 1, we recall the definition and basic properties of the braid groups. In Section 2, we study the operation of the braid groups on free groups. As an application we focus on a recent work of E. Brieskorn on the action of the braid groups on basis of quadratic forms. In Section 3, we give an alternative definition of the braid group as the fundamental group of the configuration space. Related topics such as the K(π,1) properties, the lower central series etc. are reported. Section 4 is devoted to introduce several approaches to construct linear representations of the braid groups containing a method to investigate integrable connections on the configuration space. We shall also explain an algebraic method using Iwahori's Hecke algebra and discuss their relations. In Section 5, we review recent works of V. Jones and several authors on new invariants of links. A complete account on these subjects is found in:

"Artin's braid groups" Proceeding of the conference held at Santa Cruz, to appear as a volume of Contemporary Math.
1. DEFINITION OF THE BRAID GROUP. Let us first recall the geometric definition proposed by E. Artin in 1925 [A]. We fix distinct $n$ points $a_1, \ldots, a_n$ in the Euclidean plane $\mathbb{R}^2$. We consider disjoint arcs $\mathcal{E}_i = (f_i(t), t)$ in $\mathbb{R}^2 \times I$, $1 \leq i \leq n$, such that $f_i(1) = a_{\sigma(i)}$ with some element $\sigma$ of the symmetric group $S_n$. Their union $\mathcal{E} = \mathcal{E}_1 \cup \ldots \cup \mathcal{E}_n$ is called a geometric braid (see Fig. 1). The arc $\mathcal{E}_i$ is called the $i$-th braid.

![Figure 1]

Now we define an equivalence relation between geometric braids. Let $\mathcal{E}$ and $\mathcal{E}'$ be geometric braids. We shall say that $\mathcal{E}$ and $\mathcal{E}'$ are equivalent if there exists an isotopic deformation $F_t$ of $\mathbb{R}^2 \times I$ which is the identity on $\mathbb{R}^2 \times (0)$ and $\mathbb{R}^2 \times (1)$ and for each $t \in [0,1]$, the image $F_t(\mathcal{E})$ satisfies the properties:

(i) $F_t(\mathcal{E})$ is a geometric braid.

(ii) $F_0(\mathcal{E}) = \mathcal{E}$ and $F_1(\mathcal{E}) = \mathcal{E}'$.

We denote by $B_n$ the above equivalence classes of the geometric braids. The set $B_n$ has a structure of a group by means of the composition of geometric braids. We call $B_n$ the Artin's braid.
group with \( n \) strings. Let \( \sigma_i \) be the element of \( B_n \) represented by the following geometric braid (see Fig. 2).

\[
\begin{array}{cccccc}
a_1 & a_{i-1} & a_i & a_{i+1} & a_{i+2} & a_n \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
\]

Figure 2

It was shown by E. Artin that \( B_n \) is generated by \( \sigma_i \), \( 1 \leq i \leq n-1 \), with relations:

\begin{align*}
\sigma_i \sigma_j &= \sigma_j \sigma_i \quad \text{if } |i-j| > 1 \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}
\end{align*}

Since each geometric braid determines a permutation of the set \( \{a_1, \ldots, a_n\} \) by associating the end points to its initial points we have a surjective homomorphism \( \pi : B_n \to \mathbb{S}_n \). The kernel of \( \pi \) is called the pure braid group with \( n \) strings which is denoted by \( \mathbb{P}_n \).

The presentation 1.1 leads us to the following generalization of the Artin's braid group (see [Br1]). Let \( M = (m_{ij}) \in \mathbb{M}_m \), \( \mathbb{N} \) be a Coxeter matrix (see [Bou]). We associate to \( M \) a group \( A_M \) with generators \( \varepsilon_1, \ldots, \varepsilon_m \) and relations:

\begin{align*}
\varepsilon_i \varepsilon_j \varepsilon_i \cdots &= \varepsilon_j \varepsilon_i \varepsilon_j \cdots \\
m_{ij} \text{ times} & \quad m_{ij} \text{ times}
\end{align*}

We call \( A_M \) the Artin group associated with \( M \). We observe that Artin's braid group \( B_n \) can be considered as the Artin group associated with the Coxeter matrix of type \( A_n \).
Several group theoretic problems such as the word problem, the conjugacy problem and the determination of the center for the Artin's braid group are solved in [A] and [G], where the center in an infinite cyclic group generated by

$$\Delta^2 = (\sigma_1 \ldots \sigma_{n-1})^n$$

Here $\Delta$ is called a fundamental element which represents a rotation of $180^\circ$ of the Euclidean plane. The same problems for Artin groups of finite type are solved in [BS] and [D].

2. OPERATION OF BRAID GROUPS ON FREE GROUPS. Let $F_n$ denote the free group generated by $n$ letters $x_1, \ldots, x_n$.

We define a homomorphism $\rho: B_n \to \text{Aut}(F_n)$ by

$$\rho(\sigma_i)(x_j) = \begin{cases} 
  x_j , & j \neq i , i+1 \\
  x_{i+1}x_i x_{i+1}^{-1} , & j = i+1 \\
  x_{i+1} , & j = i 
\end{cases}$$

This operation is visualized in the following way by identifying $F_n$ with the fundamental group $\pi_1(\mathbb{C} - \{a_1, \ldots, a_n\}, a_0)$ where the letters $x_i$ are realized by the paths turning around $a_i$ once positively without intersections except $a_0$.

**Figure 3**

```
Figure 3  a_1 a_2 \ldots a_n  \quad a_1 \ldots a_i a_{i+1} \ldots a_n

\begin{align*}
  &* &* &* &* \\
  &x_1 &x_2 &x_n &\rightarrow \\
  &* &* &* &* \\
  &a_0 &x_{i} &x_{i+1} &\rightarrow \\
\end{align*}
```
It is known that the above homomorphism $\rho$ is injective. The image $\rho(B_n)$ is characterized in the following way. An automorphism $\alpha$ of $F_n$ is an element of $\rho(B_n)$ if and only if the following two conditions are satisfied:

(i) $\alpha(x_1 x_2 \ldots x_n) = x_1 x_2 \ldots x_n$

(ii) $\alpha$ leaves invariant the set $(\bar{x}_1, \ldots, \bar{x}_n)$, where $\bar{x}_i$ denotes the image of $x_i$ in the abelianization $F_n^{ab} = F_n / [F_n, F_n]$.

Let $D'$ denote the commutator subgroup $[F_n, F_n]$ and let $D''$ denote the second commutator $[D', D']$. We see that $D'/D''$ is a free module over $Z[F_n^{ab}]$. Here $Z[F_n^{ab}]$ stands for the group algebra of $F_n^{ab}$. Identifying this algebra with $Z[t, t^{-1}]$ the above operation $\rho$ induces a linear representation $\Psi : B_n \to GL_n(Z[t, t^{-1}])$ defined by

$$\Psi(\sigma_i) = \begin{pmatrix} 1 & \ldots & 1 \\ 0 & 1 & \ldots \\ t & 1-t & \ldots \\ 1 & \ldots & 1 \end{pmatrix}$$

This representation is called the **Brauer representation**. It is known by Magnus and Peluso (see [B]) that $\Psi$ is injective if $n = 3$. It is an open question to determine whether $\Psi$ is faithful or not in the case $n \geq 4$.
An application (d'après Brieskorn [Br2] et al): operation of $B_n$ on basis of quadratic forms. Let us consider a proper flat family $\pi: X \to \Delta$ of complex varieties possibly with boundary over a disc $\Delta$ (with some additional condition on the boundary). Let $(a_1, \ldots, a_n) \subset \Delta$ be the set of critical values of $\pi$ and we assume that the fibre $X_{a_i}$ over $a_i$ has a single ordinary double point. By fixing a base point $a_0 \in \Delta - (a_1, \ldots, a_n)$ and choosing paths $x_i$ as before, we obtain generators of the free group $\pi_1(\Delta - (a_1, \ldots, a_n), a_0)$. On the other hand inside a fibre $X_t$, $t \in \Delta$, for $t$ near to $a_i$ there exists a middle homology class $e_i$, which vanishes to the singular point of $X_{a_i}$.

By transporting $e_i$ along the path $x_i$ to $a_0$, we obtain a cycle in $X_{a_0}$ denoted by $x_i^{*}e_i$. Let us now consider a lattice defined as a formal sum

$$L = \bigoplus_{i=1}^{n} \mathbb{Z} x_i^{*}e_i$$

with the natural intersection form $\langle , \rangle: L \times L \to \mathbb{Z}$. We obtain the monodromy representation

$$\theta: \pi_1(\Delta - (a_1, \ldots, a_n), a_0) \to \text{Aut}(L)$$

By the Picard-Lefschetz theorem the monodromy $\theta$ is expressed as

$$\theta(x_i)(u) = u \pm \frac{2 \langle u, x_i^{*}e_i \rangle}{\langle x_i^{*}e_i, x_j^{*}e_j \rangle}$$

for $u \in L$. We put $B = \langle x_i^{*}e_i, x_j^{*}e_j \rangle, i, j = 1, \ldots, n \in \mathbb{M}(n, \mathbb{Z})$.

Now we consider the operation on an element $b$ of $B_n$ on $\pi_1(\Delta - (a_1, \ldots, a_n), a_0)$ via $\rho$ to get another basis $x'_1, \ldots, x'_n$ (see Fig. 3). Thus we obtain the matrix $B' = \langle (x'_i)^{*}e_i, (x'_j)^{*}e_j \rangle$. 

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By means of this process we have an operation of \( B_n \) on \( M(n, \mathbb{Z}) \), which can be written down explicitly (see [Br2]). We should remark that this is not a linear representation.

For example in the case \( n=3 \) the space of the solutions of

\[
\text{det} \begin{bmatrix} 2 & x & z \\ x & 2 & y \\ z & y & 2 \end{bmatrix} = t , \quad x, y, z \in \mathbb{Z}
\]

admits an action of \( B_3 \). The orbits of this action were studied by Markov, Mordel, Brieskorn (see [Br2]), Naruki [N] and Saito [S2].

3. CONFIGURATION SPACE. Let \( H_{ij} \), \( 1 \leq i < j \leq n \), denote the hyperplanes in \( \mathbb{C}^n \) defined by \( H_{ij} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n ; z_i = z_j \} \).

The symmetric group \( S_n \) acts freely on the complementary space \( \mathbb{C}^n - \bigcup_{1 \leq i < j \leq n} H_{ij} \) by the permutation of the coordinates. The configuration space of distinct \( n \) points in \( \mathbb{C} \) is by definition \( \mathcal{E}_n(\mathbb{C}) = (\mathbb{C}^n - \bigcup_{1 \leq i < j \leq n} H_{ij}) / S_n \).

Let \( p : \mathbb{C}^n \to \mathbb{C}^n / S_n \) denote the natural projection. We may identify \( \mathbb{C}^n / S_n \) with \( \mathbb{C}^n \) via elementary symmetric polynomials \( \sigma_i \), \( 1 \leq i \leq n \). The image of the union \( \bigcup_{1 \leq i < j \leq n} H_{ij} \) by the projection \( p \) is called the discriminant which is a complex hypersurface

\[
\mathcal{D}_n = \{(\sigma_1, \ldots, \sigma_n) \in \mathbb{C}^n ; \Delta(\sigma_1, \ldots, \sigma_n) = 0 \}
\]

where \( \Delta(\sigma_1, \ldots, \sigma_n) = \prod_{i \neq j} (z_i - z_j) \) is the discriminant polynomial. Then \( p \) is an unramified covering outside \( \mathcal{D}_n \) and \( \mathcal{E}_n(\mathbb{C}) \) is also written as \( \mathbb{C}^n - \mathcal{D}_n \). The braid group \( B_n \) is isomorphic to the fundamental group \( \pi_1(\mathbb{C}^n - \mathcal{D}_n, \ast) \). It is
known by Fox and Neuwirth (see [Br1]) that

$$\pi_i(\mathbb{C}^n - \mathcal{D}_n, \ast) = 0 \text{ for } i \geq 2.$$  

This leads us to the isomorphisms:

$$H^*(\mathbb{P}_n) \cong H^*(\mathbb{C}^n - \mathcal{D}_n)$$

$$H^*(\mathbb{P}_n) \cong H^*(\mathbb{C}^n - \bigcup H_{ij})$$

An analogous statement for an arbitrary Artin group of finite type was proved by Deligne [D].

The structure of $H^*(\mathbb{C}^n - \bigcup H_{ij})$ was studied by Arnold [Ar] (see [Br1] and [OS] for a more extensive treatment). It is known that $H^*(\mathbb{C}^n - \bigcup H_{ij})$ is generated by $\omega_{ij} = \frac{d(z_i - z_j)}{z_i - z_j}$, $1 \leq i < j \leq n$,

with fundamental relations:

$$\omega_{ij} \omega_{jk} + \omega_{jk} \omega_{ik} + \omega_{ik} \omega_{ij} = 0, \quad 1 \leq i < j < k \leq n.$$  

The Poincaré polynomial $\sum_{p \geq 0} \text{rank } H^p(\mathbb{C}^n - \bigcup H_{ij}) t^p$ is given as

$$(1+t)(1+2t) \ldots (1+(n-1)t).$$

Let us remark here that the numbers $(1, 2, \ldots, n-1)$ appearing in the above formula are the Coxeter exponents of type $A_n$ (see [Br1]). Let us look at several properties relevant to these numbers. Let $\mathcal{D}_n$ denote the set of germs of the holomorphic vector fields at 0 tangent to $\mathcal{D}_n$. Then $\mathcal{D}_n$ is a free $\mathcal{O}_{\mathbb{C}^n,0}$-module generated by homogeneous vector fields $X_i$, $1 \leq i \leq n-1$, such that $\deg X_i = i$ (see [S]). This relation between the Poincaré polynomial and the logarithmic vector fields was investigated by Terao [Te] in a more general situation. Another property which we want to describe here is a relation with the lower central series.
$P_n$. Let $\Gamma_1 \supseteq \Gamma_2 \supseteq \ldots \supseteq \Gamma_j \supseteq \ldots$ be the lower central series of $P_n$ defined inductively by $\Gamma_1 = P_n$ and $\Gamma_{j+1} = [\Gamma_1, \Gamma_j]$. We put $\varphi_j = \text{rank } (\Gamma_j / \Gamma_{j+1})$. Then we have

$$\prod_{j=1}^{n-1} \prod_{p=1}^{\infty} (1 - \lambda t)^{\varphi_p} \quad \text{in } \mathbb{Z}[[t]]$$

(see [K] and [KO]).

4. LINEAR REPRESENTATIONS. By the Riemann-Hilbert correspondence, we know philosophically that the study of finite dimensional linear representations of $B_n$ is "equivalent" to the study of systems of linear differential equations with regular singularities along the discriminant $\mathcal{D}_n$. But at this stage we do not have a sufficient understanding of either of them. We list up several approaches in this direction.

(i) hypergeometric differential equations: Appell, Picard, Terada [Te] and Deligne-Mostow [DM]. Linear representations of the braid groups appear as the monodromy of the integrals of Pochhammer type.

(ii) total differential equations with logarithmic singularities: By means of solutions of classical Yang-Baxter equations we have a method to construct an integrable connection over the configuration space associated with any finite dimensional irreducible representation of a complex simple Lie algebra (see Kohno's note in this volume). Another approach using the integrals of difference products is due to Aomoto [Ao].
(iii) **Gauss-Manin connections**: K. Saito, Anbai ... . The reader may refer to Anbai's note in this volume.

(iv) **conformal field theory on \( \mathbb{P}^1 \)**: Belavin-Polyakov-Zamolodchikov [BPZ], Knizhnik-Zamolodchikov [KZ] and Tsuchiya-Kanie [TK] ... . Linear representations of \( B_n \) appear as the monodromy of \( n \)-point functions. These functions satisfy the total differential equations mentioned in (ii).

(v) **Iwahori's Hecke algebra representations (von Neumann algebra representations)**: Jones [Jo], Ocneanu [O] ... . These representations were used effectively to define new invariants of links (see the next Section).

(vi) **algebra with two parameters**: Birman-Wenzl [BW], J. Murakami [M1] ... . This new algebra was introduced motivated by Kauffman polynomial. It is an extension of Iwahori's Hecke algebra.

Let us recall here the definition of Iwahori's Hecke algebra \( H(n,q) \) of the symmetric group. Let us define \( H(n,q) \) as the algebra over \( \mathbb{Z}[q, q^{-1}] \) generated by \( 1, T_1, \ldots, T_{n-1} \) with relations:

\[
T_i T_j = T_j T_i \text{ if } |i-j| > 1
\]

\[
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}
\]

\[
(T_i + 1)(T_i - q) = 0 , \ i = 1, 2, \ldots, n-1
\]

We observe that \( H(n,q) \) is a free \( \mathbb{Z}[q, q^{-1}] \) - module of rank \( n \)!
and that $H(n,1)$ is identified with the group algebra of the symmetric group $S_n$. We have a natural homomorphism $\pi : \mathbb{Z}[B_n] \rightarrow H(n,q)$ defined by $\pi(\sigma_i) = T_i$. A linear representation of $B_n$ factoring through $\pi$ is called a Hecke algebra representation.

The above approaches (i) - (vi) are closely intricate.

Hecke algebra representations appear as the monodromy of the connection associated with the vector representation of $sl(m,\mathbb{C})$ (see [K2]). In [TK] it was shown that as the monodromy of n-point function in two dimensional conformal field theory with gauge symmetry of type $A_1^{(1)}$ we obtain unitarizable representations of $B_n$ factoring through the Jones algebra. The study of the monodromy of n-point functions with other types of symmetry is in progress. The monodromy of the connection associated with the vector representation of the other types of non-exceptional simple Lie algebras is related with solutions of quantum Yang-Baxter equation obtained by Jimbo [J1] and [J2] (see [K2]). They commute with the diagonal action of the q-analogue of the corresponding Lie algebra and factor through the algebra with two parameters due to Birman-Wenzl.

5. APPLICATION TO LINK POLYNOMIALS. Given a geometric braid $b$ we obtain an oriented link $b^-$ by means of the closing process illustrated in Fig.4.
Let \( \mathcal{L} \) denote the set of the isotopy classes of oriented links.

We introduce an equivalence relation in \( \bigcup_{n=1}^{\infty} B_n \) generated by the following Markov equivalence:

(MI) for \( b, c \in B_n \), \( b \sim c \) if \( b \) and \( c \) are conjugate in \( B_n \).

(MII) for \( b \in B_n \), \( b \sim b \sigma_n \).

It is known by Alexander and Yamada that the natural map

\[
\bigcup_{n=1}^{\infty} B_n / \sim \rightarrow \mathcal{L}
\]

obtained by the closing of braids is surjective. Yamada's argument gives us a stronger statement. Namely he asserts that the minimal number of strings of the geometric braid to obtain an oriented link \( L \) is equal to the minimal number of the Seifert circles of \( L \) (see [Y] for a precise statement).

Moreover we know by Markov (see [B] for a proof) that the above natural map is bijective.

Let us now proceed to review known link polynomials.

The Alexander polynomial \( \Delta_L(t) \) may be defined in the following way from a point of view of the braid group. Let \( \Psi : B_n \rightarrow \)
$GL_n(\mathbb{Z}[t, t^{-1}])$ be the Burau representation. Then the Alexander polynomial of $L = b^\lambda$, $b \in B_n$, is given by

$$\Delta_L(t) = (1 + t + \ldots + t^{n-1})^{-1} \det((\Psi(b) - 1)).$$

After the definition of the Jones polynomial $V_L(t)$ it was generalized to the two variable polynomial $X_L(q, \lambda)$ independently by Freyd-Yetter, Lickorish-Millet, Ocneanu and Hoste. Let us recall the definition. We know by Ocneanu [O] that there exists a linear map $\text{tr} : H(\infty, q) \to \mathbb{Q}(\sqrt{q}, z)$ uniquely defined by

(i) $\text{tr}(ab) = \text{tr}(ba)$

(ii) $\text{tr}(1) = 1$

(iii) $\text{tr}(XT_n) = z \text{tr}(x)$ for $x \in H(q, n)$

Here we denote by $H(\infty, q)$ the inductive limit of $H(n, q)$ with respect to the natural inclusion $H(n, q) \subset H(n+1, q)$. Putting $\lambda = (qz)^{-1}(1-q+z)$, we define $X_L(q, \lambda)$ by

$$X_L(q, \lambda) = (1 - \frac{q-\lambda}{\sqrt{\lambda(1-q)}})^{n-1}(\sqrt{\lambda})^e \text{tr}(\pi(\alpha))$$

where $\alpha \in B_n$ is any braid with $\alpha^\lambda = L$, $e$ denotes the exponent sum of $\alpha$ as a word on $\sigma_i$'s and $\pi$ is a natural homomorphism to $H(n, q)$ defined in the previous section. By means of Markov's theorem and the properties of $\text{tr}$ we verify that $X_L(q, \lambda)$ is an invariant of the isotopy type of $L$. We put $t = \sqrt{\lambda q}$, $x = \sqrt{q} - (\sqrt{q})^{-1}$ and $P_L(t, x) = X_L(q, \lambda)$. Then $P_L(t, x)$ satisfies the following "skein rule". If $L_+$, $L_-$ and $L_0$ are links which have projections identical, except in one crossing where they are as in Fig.5.
Figure 5

\[
\begin{align*}
L_- & \quad L_+ & \quad L_0
\end{align*}
\]

Then we have \( t^{-1}P_{L_+} - tP_{L_-} = xP_{L_0} \). The two-variable polynomial \( P_L \) is characterized by the above skein rule together with the property that \( P_L = 1 \) for a trivial knot \( L \).

Both the Alexander polynomial and the Jones polynomial are obtained as the following specializations.

\[
\begin{align*}
\Delta_L(t) &= X_L(t, t^{-1}) = P_L(1, \sqrt{t} - 1/\sqrt{t}) \\
V_L(t) &= X_L(t, t) = P_L(it^{-1}, -i(\sqrt{t} - 1/\sqrt{t}))
\end{align*}
\]

They also satisfy the skein rules:

\[
\begin{align*}
\Delta_{L_+} - \Delta_{L_-} + (\sqrt{t} - 1/\sqrt{t})\Delta_{L_0} &= 0 \quad \text{(Conway)} \\
tV_{L_+} - t^{-1}V_{L_-} + (\sqrt{t} - 1/\sqrt{t})V_{L_0} &= 0
\end{align*}
\]

Several results have been obtained for the meaning of special values of \( V_L(t) \). The reader may refer to [M2] and [Jo] for works in this direction.

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