数理解析研究所講究録
635巻 1987年 178-192

ON THE STRUCTURE OF THE PARAMETERS SPACE OF THE HENON FAMILY

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1. *CONNECTIONS OF KNEADING SEQUENCES*

In this note, we show some trace diagrams of periodic orbits in the parameters space of the Hénon map drawn by a personal computer (NEC PC-9801M) and make some considerations on them. The Hénon map (which is also called the "Hénon family") is a 2-parameters family of polynomial maps of degree 2 on $\mathbb{R}^2$ such that;

$$H_{a,b}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y + 1 - ax^2 \\ bx \end{bmatrix} .$$

This is the simplest non-linear map on $\mathbb{R}^2$ and has constant Jacobian $-b$. When $b = 0$, $H_{a,b}$ is the standard unimodal map $f_a(x) = 1 - ax^2$ on x-axis. It is known that $\{f_a\}_{0\le a\le 2}$ has a self-similar bifurcation structure represented by symbolic sequences (Refer Th.III.1.1 in [C-E] and Exposé n°.VI,Cor.2 in [D-H]). How this structure extends in (a,b)-plane of the Henon

family ? Numerical studies were done by Hamouly-Mira[H-M1],[H-M2] and Ushiki[U] independently and they found an interesting phenomenon such that two periodic orbits of the unimodal maps with different kneading sequences were connected in (a,b)-plane of the Hénon familly. In this note, we develop Ushiki's calculations of bifurcation diagrams of the Hénon family using his algorithm.

Let $P(m,i)$ denote the i-th finite maximal sequence of length m of symbols R, L, C and $P(m,i) = A(m,i)C$ (Refer [C-E] for definitions) . For example, in fig.1, all finite maximal sequences of length 5,6,7 are shown. Symbolic sequences on the right of them are corresponding braids which will be stated later.

Let $\mathcal{O}$ be a periodic otbit of the standard unimodal map f_a of period m . It is known that the itinerary of some point of $\mathcal{O}$ is one of $P(m,i)$, $(A(m,i)L)^\infty$ or $(A(m,i)R)^\infty$ for some i . Therefore, we can define the following subsets of (a,b)-plane associated with the periodic orbits of the Hénon family which are obtained from the unimodal map by deformations.

Definition. For $\lambda \in \mathbb{R}$ and $m \in \mathbb{N}$, we put;

$$S_\lambda^m = \{ (a,b) \in \mathbb{R}^2 \mid H_{a,b} \text{ has a periodic point of period } m \text{ and trace } \lambda \},$$

where "trace" of periodic point p is the trace of the matrix $D_p(H_{a,b})^m$. Let $S^m_\lambda(i)$ be a branch of S^m_λ containning a parameter $(a,0)$ such that the corresponding periodic orbit of $H_{a,0}$ (i.e. $f_a(x)$ on x-axis) has a point whose itinerary is one of $P(m,i)$, $(A(m,i)R)^\infty$, $(A(m,i)L)^\infty$. (In [U], Ushiki used eigenvalues of periodic point. But, there were cases that the branches could not be continued because of the eigenvalue's complexification.)

Fig.2 is the case of period 5. $S^5_{\pm 0.9}(i)$ for $i = 1,2,3$ are drawn. It can be seen that $S^5_{\pm 0.9}(1)$ and $S^5_{\pm 0.9}(2)$ are twined making a delta region. Fig.3 is a magnification of that region. $S^5_\lambda(1)$, $S^5_\lambda(2)$ for $\lambda = \pm 0.9, \pm 0.7, \pm 0.5, \pm 0.3, \pm 0.1$ are drawn. There exists a saddle point of the function "trace" of periodic point. Moreover, a cusp singularity of a branch seems to exist. This is a tyipical feature of the "connection" of kneading sequences. We shall call this kind of region and connection a cusp region and a cusp connection in the parameters space of the Hénon family.

The same kind of phenomena are seen in case of higher periods. In fig.4, all $S^m_\lambda(i)$ for $m = 5,6,7$ and $\lambda = \pm 0.9$ except $S^7_{\pm 0.9}(9)$ are drawn. Our problem is; "What kind of relation is there in such

connections ?" . In [S], this problem is investigated using topological arguments. The result of [S] is that there exists a tidy conjugacy relation between braids associated with periodic orbits of the unimodal maps with "satellite sequences" as their kneading sequences, and this relation is realized by "topological" deformations. Our main interest in this note is a question that whether this relation is realized in the Hénon family or not.

2. *SATELLITE SEQUENCES*

For $\ell \geq 1$ and $m \geq 1$, satellite sequence of type (ℓ,m) is a finite sequence such that,

$$P = R\,L\cdots L\,A_1R\,A_2R\,\cdots\,A_mR\,C \quad ,$$

where the number of L's in the first sequence of L's is ℓ , and A_i is R or L .

Note that satellite sequence is always a maximal sequence. Therefore, for any satellite sequence, there exists an unimodal map f_a ($0 < a \leq 2$) whose kneading sequence is equal to that sequence. Fig.5 is satellite sequences of type (1,1),(1,2),(2,1),(2,2) and the relations stated in [S].

Right side relations indicate "r-conjugacy" of braids and correspond to orientation reversing case. Left side relations mean "conjugacy" which correspond to orientation preserving case. The number on the right of

each sequence indicates the order of the sequence counted upward from below in all finite maximal sequences of that length.

Fig.6, fig.7, fig,8 shows $S_\lambda^6(i)$, $S_\lambda^7(i)$, $S_\lambda^8(i)$ respectively and the same i in fig.5. It can be seen that in { b > 0 } (orientation reversing case) cusp connections exist. In { b < 0 } , expected connections ($S_\lambda^7(3)$ and $S_\lambda^7(4)$, $S_\lambda^8(10)$ and $S_\lambda^8(11)$) do not appear to exist. In reality, these twines surely exist, but they are realized in very high trace values. Fig.9 shows $S_\lambda^7(3)$, $S_\lambda^7(4)$ for $\lambda = \pm 180$, and, fig.10 shows $S_\lambda^8(10)$ for $\lambda = 380,780$, and $S_\lambda^8(11)$ for $\lambda = \pm 0.9$. Thus, it is confirmed that the conjugacy and r-conjugacy relations of braids associated with satellite sequences in fig.5 can be realized in the Hénon family. Similarly, we can confirm that the conjugacy relations of satellite sequences of type (3,1), (4,1), (1,3) are also realized in the Hénon family. Fig.11 shows branches of satellite sequences of type (1,1), (1,2), (1,3) with trace ±0.9.

3. *NON-SATELLITE SEQUENCES*

The tidy connection pattern in satellite sequences implies a certain close relation between hyperbolic set of period 2 and the bifurcation structure of the Hénon

family [S]. However, it is also sure that the "satellite" is not all in the bifurcation structure of the Hénon family. In fact, it can be seen that several pairs of non-satellite kneading sequences have connections in the parameters space of the Hénon family.

3.1 Orientation Preserving Case

Firstly, let us see the case of $b < 0$, that is the orientation preserving case. Fig.12 is $[0,8]\times[-1.2,0]$ in (a,b)-plane and drawn branches in it are $S^6_\lambda(2)$ and $S^6_\mu(3)$ for $\lambda = -20, -40, -60, -70, -80, -100$ and $\mu = -60, -80, -120$. A cusp connection exists. But in this case, the cusp region is very large and the absolute value of the trace of saddle point is also large (about $-70 \sim -80$). Fig.13 is a similar diagram of $S^7_\lambda(6)$ and $S^7_\mu(7)$ for $\lambda = -200, -450, -600, -620, -760$ and $\mu = -200, -400, -650$.

Let $B(P(m,i))$ be the braid associated with the periodic orbit of $H_{a,b}$ such that the kneading sequence of $H_{a,0}$ on x-axis is $P(m,i)$, $|b|$ is small and $b > 0$. Note that $B(P(m,i))$ is always a positive braid. Therefore if $B(P(m,i))$ and $B(P(m,j))$ have different length (i.e. they have different exponent sum), they cannot be conjugate each other and so cannot be connected in the parameters space of the Hénon family

(Refer (1.15)Remarks in [S]). Conversely, as far as I checked (up to period 9), all pairs of kneading sequences which have difference at only one symbol and the corresponding braids have the same length have always a cusp connection in the parameters space of the Hénon family. So, (although examples are not so many) the following conjecture seems to be true.

Conjecture. Let $P(m,i)$ and $P(m,j)$ have only one different symbol and the length of $B((P(m,i))$ and $B(P(m,j))$ are the same. Then, branches $S^m_\lambda(i)$ and $S^m_\mu(j)$ have a cusp connection in $\{ b < 0 \}$ of the parameters space of the Hénon family.

3.2 *Orientation Reversing Case*

Also in case of $b > 0$, there exist non-satellite connections. Fig.14 shows branches $S^6_\lambda(1)$, $S^6_\mu(3)$, $S^6_\nu(4)$ for $\lambda = \pm 0.9$, 20, 60, 70, 80, 100, 190, -5, -10, -100, -190, $\mu = \pm 0.9$, ± 20, ± 100, $\nu = \pm 0.9$, 40, 80, 190, -10, -20. It can be seen that branches of three periodic orbits twine each other. The same kind of phenomenon can be seen also in higher periods, and in [H-M2], the structure of the cusp region is investigated. But, concerning the connection rule in this orientation reversing case, I don't have any general conjecture as in

the orientation preserving case.

References

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[H-M2] ---, "Singularités dues au feuilletage du plan des bifurcations d'un diffeomorphisme bi-dimensionnel." Comptes rendus,294,série I,1982,387-390.

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[U] S.Ushiki, "Fine structure of bifurcation branches in Hénon's family of mappings." Rikigakukei riron no sogoteki kenkyu. Kirokushu. (in japanese), ed. K.Shiraiwa.(1981).

length 5

1	RLRRC	$\sigma_4\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_4$
2	RLLRC	$\sigma_4\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3$
3	RLLLC	$\sigma_4\sigma_3\sigma_2\sigma_1$

length 6

1	RLRRRC	$\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3\sigma_5$
2	RLLRLC	$\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2$
3	RLLRRC	$\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_5$
4	RLLLRC	$\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4$
5	RLLLLC	$\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1$

length 7

1	RLRRRRC	$\sigma_6\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1\sigma_6\sigma_5\sigma_4\sigma_3\sigma_2\sigma_6\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_6$
2	RLRRLRC	$\sigma_6\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1\sigma_6\sigma_5\sigma_4\sigma_3\sigma_2\sigma_6\sigma_5\sigma_4\sigma_3\sigma_6$
3	RLLRLRC	$\sigma_6\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1\sigma_6\sigma_5\sigma_4\sigma_3\sigma_2\sigma_6\sigma_5\sigma_4$
4	RLLRRRC	$\sigma_6\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1\sigma_6\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_6$
5	RLLRRLC	$\sigma_6\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1\sigma_6\sigma_5\sigma_4\sigma_3\sigma_2\sigma_6$
6	RLLLRLC	$\sigma_6\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1\sigma_6\sigma_5\sigma_4\sigma_3$
7	RLLLRRC	$\sigma_6\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1\sigma_6\sigma_5\sigma_4\sigma_6$
8	RLLLLRC	$\sigma_6\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1\sigma_6\sigma_5$
9	RLLLLLC	$\sigma_6\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1$

Fig.1.

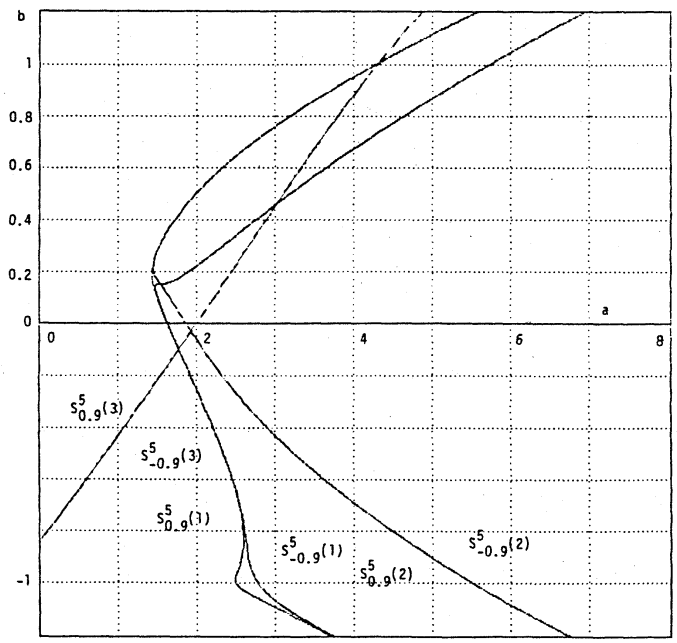


Fig.2.

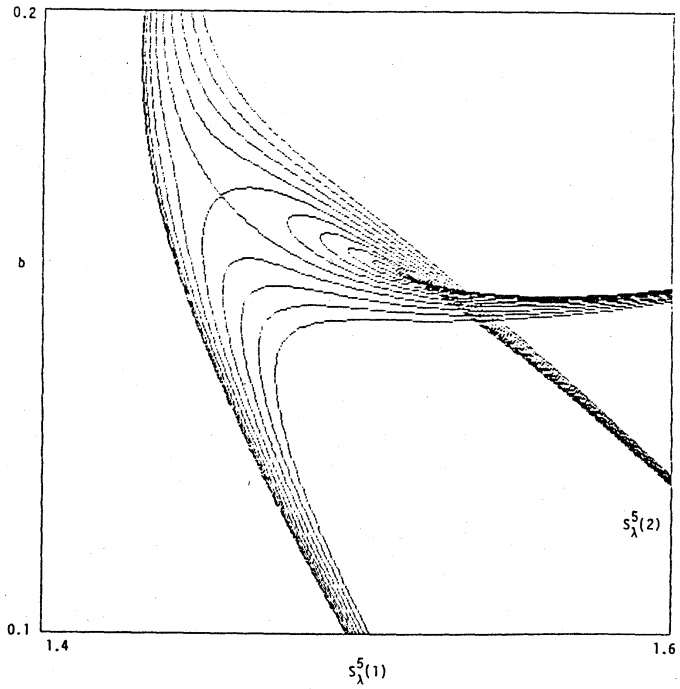


Fig.3.

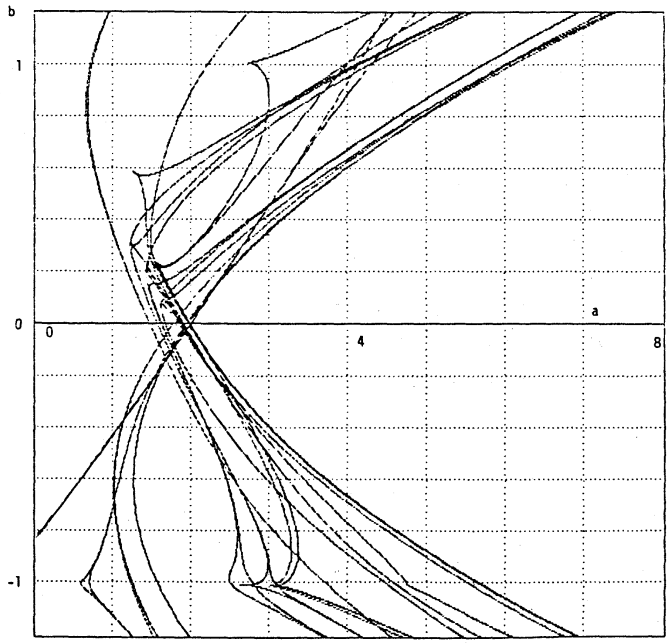


Fig.4.

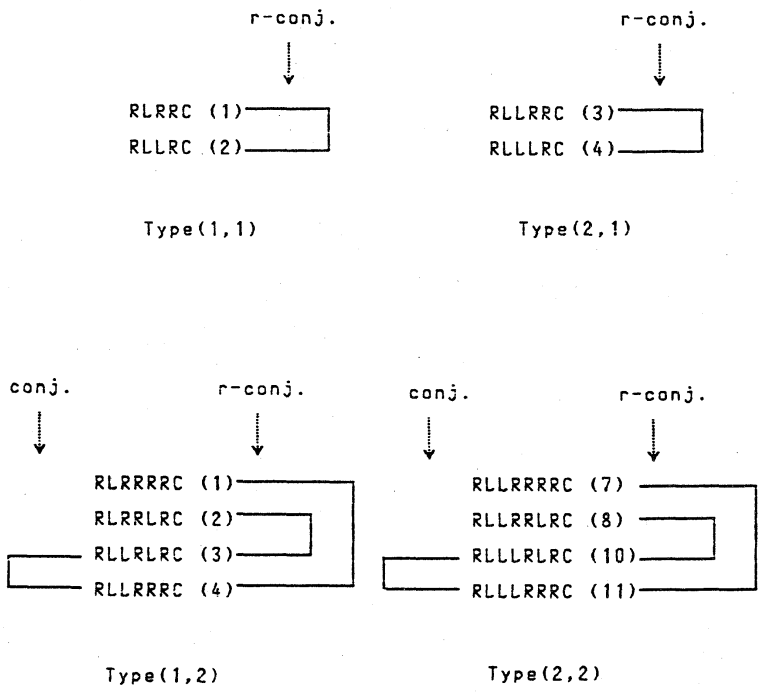

r-conj.
RLRRC (1)
RLLRC (2)
Type(1,1)
r-conj.
RLLRRC (3)
RLLLRC (4)
Type(2,1)
conj.
r-conj.
RLRRRRC (1)
RLRRLRC (2)
RLLRLRC (3)
RLLRRRC (4)
Type(1,2)
conj.
r-conj.
RLLRRRRC (7)
RLLRRLRC (8)
RLLLRLRC (10)
RLLLRRRC (11)
Type(2,2)

Fig.5.

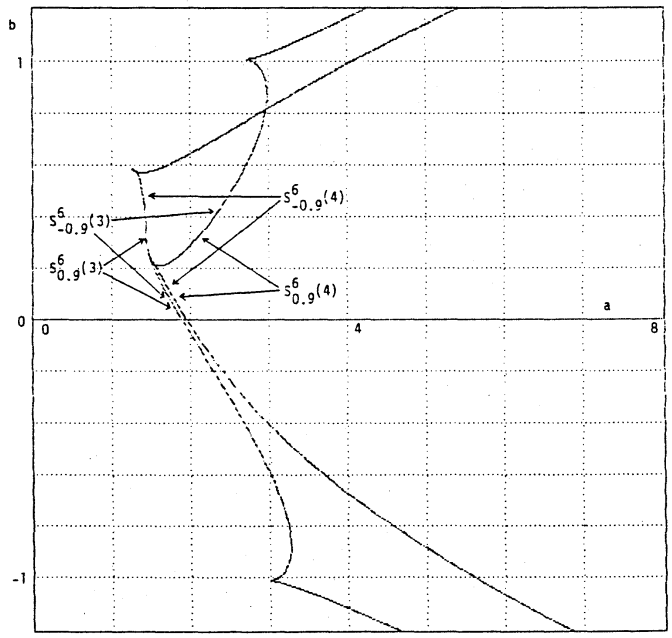

b
1
0
-1
0
4
8
a
$S^6_{-0.9}(3)$
$S^6_{0.9}(3)$
$S^6_{-0.9}(4)$
$S^6_{0.9}(4)$

Fig.6.

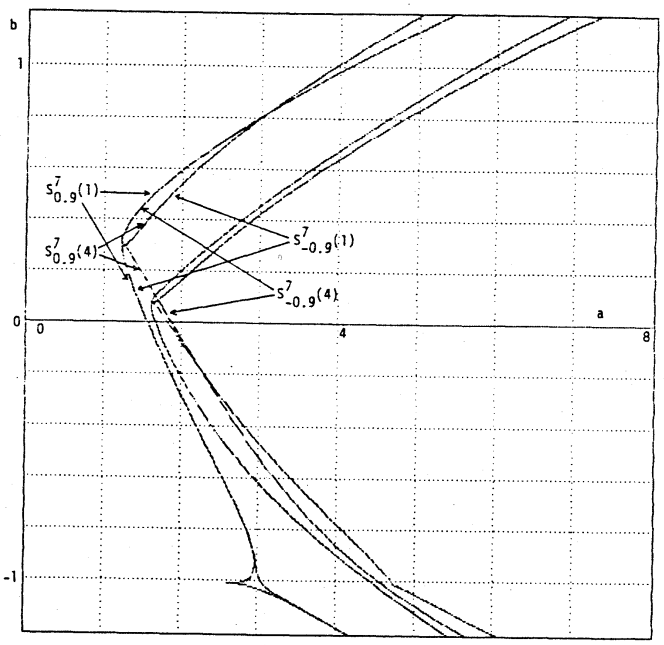

b
1
0
-1
0
4
8
a
$s^7_{0.9}(1)$
$s^7_{0.9}(4)$
$s^7_{-0.9}(1)$
$s^7_{-0.9}(4)$

Fig.7.

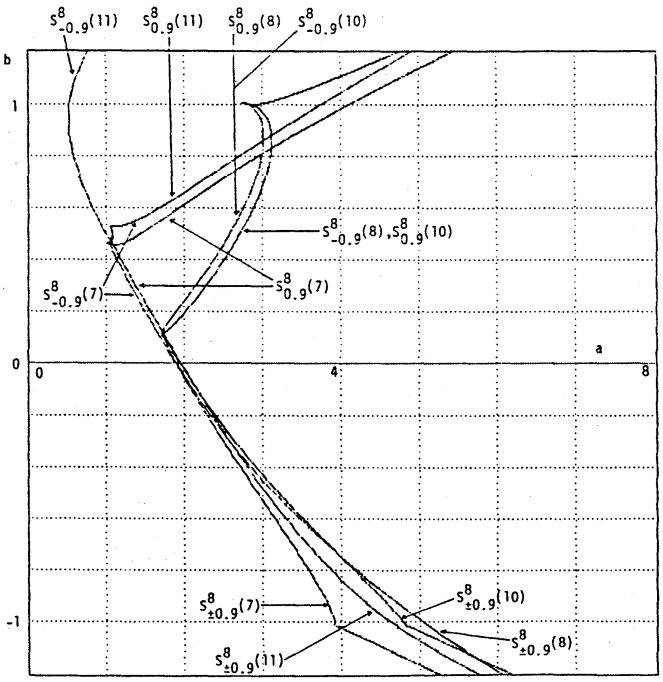

$s^8_{-0.9}(11)$
$s^8_{0.9}(11)$
$s^8_{0.9}(8)$
$s^8_{-0.9}(10)$
b
1
0
-1
0
4
8
a
$s^8_{-0.9}(8), s^8_{0.9}(10)$
$s^8_{-0.9}(7)$
$s^8_{0.9}(7)$
$s^8_{\pm 0.9}(7)$
$s^8_{\pm 0.9}(10)$
$s^8_{\pm 0.9}(8)$
$s^8_{\pm 0.9}(11)$

Fig.8.

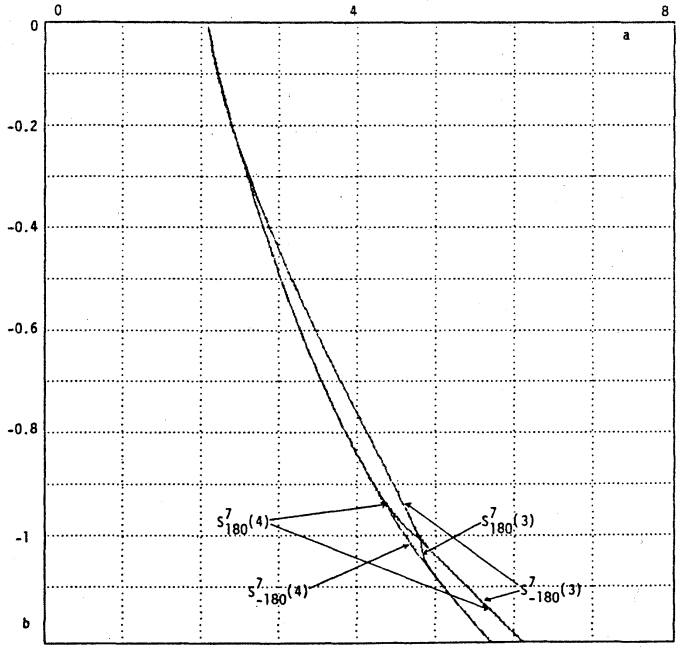

0
4
8
a
0
-0.2
-0.4
-0.6
-0.8
-1
b
$S^7_{180}(4)$
$S^7_{180}(3)$
$S^7_{-180}(4)$
$S^7_{-180}(3)$

Fig.9.

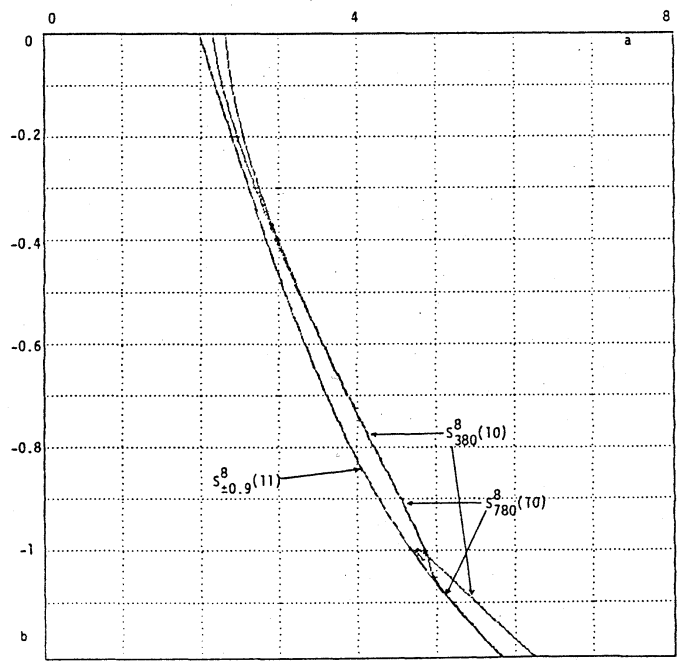

0
4
8
a
0
-0.2
-0.4
-0.6
-0.8
-1
b
$S^8_{380}(10)$
$S^8_{\pm 0.9}(11)$
$S^8_{780}(10)$

Fig.10.

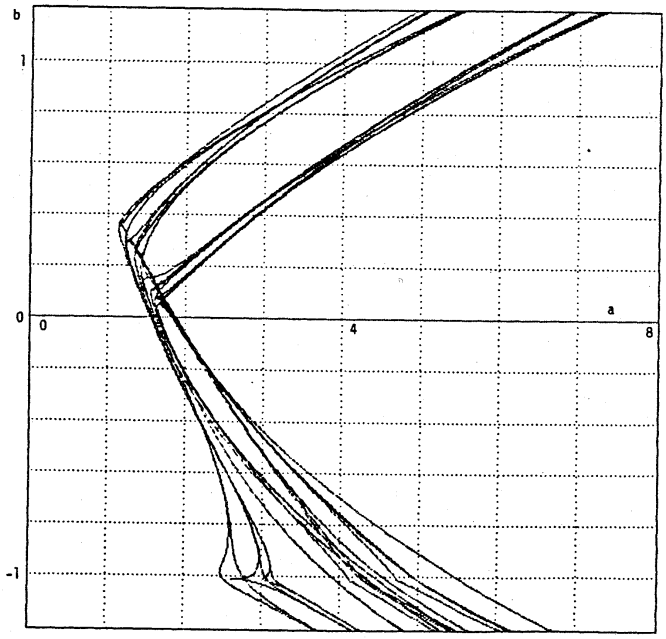
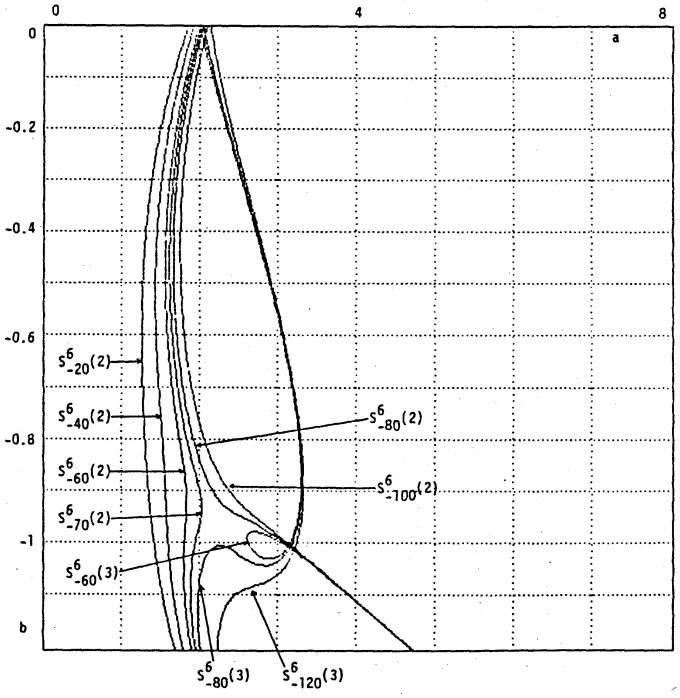

b
1
0
-1
0
4
8
a

Fig.11.


0
4
8
a
0
-0.2
-0.4
-0.6
-0.8
-1
b
$s^6_{-20}(2)$
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$s^6_{-60}(2)$
$s^6_{-70}(2)$
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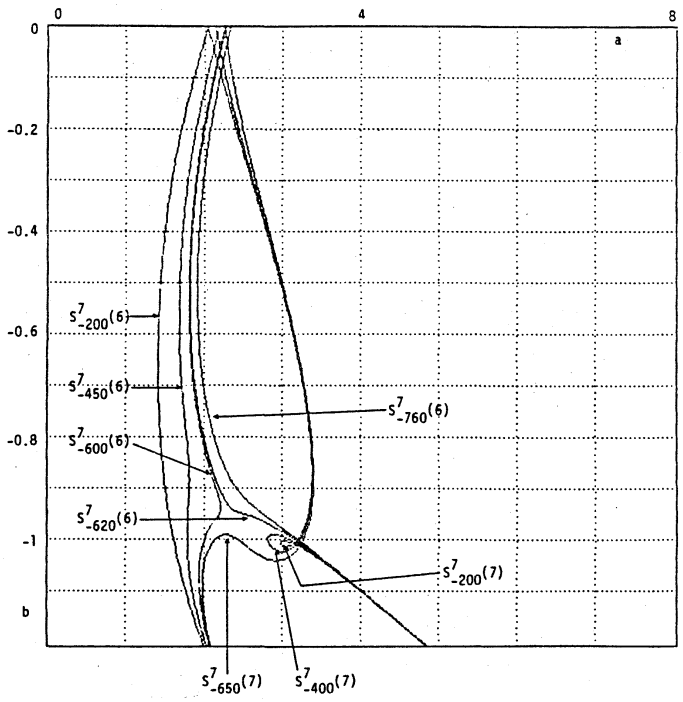
$s^6_{-80}(3)$
$s^6_{-120}(3)$

Fig.12.


0
4
8
a
0
-0.2
-0.4
-0.6
-0.8
-1
b
$s^7_{-200}(6)$
$s^7_{-450}(6)$
$s^7_{-600}(6)$
$s^7_{-620}(6)$
$s^7_{-760}(6)$
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$s^7_{-650}(7)$
$s^7_{-400}(7)$

Fig.13.

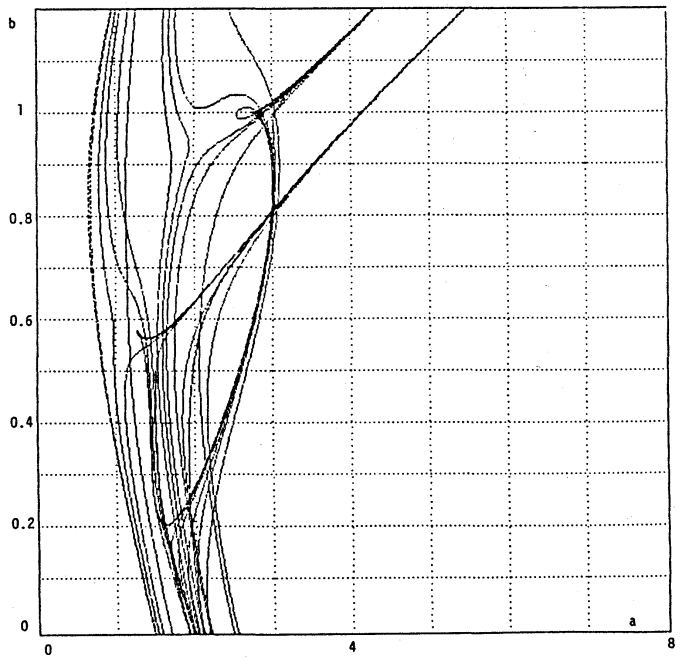
b
1
0.8
0.6
0.4
0.2
0
0
4
8
a

Fig.14.