数理解析研究所講究録
635巻 1987年 101-128

BAUM-CONNES CONJECTURES AND THEIR APPLICATIONS

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§1 Introduction Interrelationship among different areas in mathematics gives a plenty of beneficence to themselves as numbers of results support its justification. Especially concerning both geometry and analysis, there is no doubt that Atiyah-Singer index theory has a crucial role to develope their fields simultaneously.

Recently, Connes[3] has initiated a new index theory for both dynamical systems and foliated manifolds, which seems to be useful to cases with pathological ambient spaces to which the index theory of Atiyah-Singer type is no longer applicable. The main idea of his theory is based on K-theory of boht C^*-algebras and twisted vector bundles, whose validity can be illustrated in several manuscripts due to Baum-Douglas, Connes, Kasparov, Miscenko, Pimsner-Voiculescu and Rosenberg etc. Among others, Baum-Connes[1] has conjectured the existance of a K-theoretic index formula between geometric and analytic K-theory of differential dynamical systems and foliated manifolds, which may be viewed as an ultimate form of a generalization of Atiyah -Singer index theorem. It has a quite important meaning involved as a central ingredient to research differential geometry, topology and C^*-algebras etc. More accurately, their

conjecture says that the geometric K-group is isomorphic to the analytic one under the K-index mapping for foliated manifolds and differential dynamics. If it is affirmative, as corollaries are deduced the conjectures due to Novikov, Gromov-Lawson-Rosenberg and Kadison etc in topology, differential geometry and C*-algebras respectively. As a matter of fact, no theorem from general sights has been obtained until now although various examples supporting the conjecture have been constructed by several persons.

In this paper, we shall state the construction of Baum-Connes conjectures, the results obtained and their some applications. In particular, we shall show their affirmation for generalized Anosov foliations on infra-homogeneous spaces. The basic refferences are due to Baum-Connes[1],[2], Connes[3], Kasparov[6] and Rosenberg[9]~[11].

§2 Construction Let (M,F) be a foliated smooth manifold and G its holonomy groupoid. Let $\Omega^{1/2}$ be the half density bundle over G tangential to F⊗F and denote by $C_c(\Omega^{1/2})$ the *-algebra consisting of all continuous sections of $\Omega^{1/2}$ over G with compact support by the following algebraic operations:

$$(fg)(\gamma) = \int_{\gamma=\gamma_1\gamma_2} f(\gamma_1)g(\gamma_2)$$

$$f^*(\gamma) = \bar{f}(\gamma^{-1})$$

for all $f,g \in C_c(\Omega^{1/2})$. Given any $x \in M$, let H_x be the Hilbert space consisting of all L^2- sections of $\Omega^{1/2}$ over G. Let us define a *-representation π_x of $C_c(\Omega^{1/2})$ on H_x by

$$(\pi_x(f)\xi)(\gamma) = \int_{\gamma=\gamma_1\gamma_2} f(\gamma_1)\xi(\gamma_2)$$

for all $f \in C_c(\Omega^{1/2})$ and $\xi \in H_x$. Then a C^*-norm $\|\cdot\|$ on $C_c(\Omega^{1/2})$ is defined by

$$\|f\| = \sup_{x\in M} \|\pi_x(f)\|$$

for all $f \in C_c(\Omega^{1/2})$. Let us denote by $C_r^*(M,F)$ the completion of $C_c(\Omega^{1/2})$ with respect to $\|\cdot\|$, which is called a foliation C^*-algebra associated to (M,F).

We then consider the K-theory $K_a(M,F)$ of $C_r^*(M,F)$, which is called the <u>analytic</u> K-theory of (M,F). On the other hand, we shall offer a pure geometric way of defining its K-theory. Let X be a proper G-manifold and $\tilde{\nu}^*$ the dual bundle of the normal bundle $\tilde{\nu}$ of the foliation of X determined by the G-orbits. We denote by ρ the canonical G-equivariant mapping from X to M and $\rho^*(\nu^*)$ the pull back of the dual bundle ν^* of the normal bundle ν of F. We consider a pair (X,ξ) of X and a G-vector bundle ξ over $\tilde{\nu}^*\oplus\rho^*(\nu^*)$, which is called a <u>K-cocycle</u> of (M,F). Denote by $\Gamma(M,F)$ be the set of all K-cocycles of (M,F). We then introduce an equivalence relation $\sim$ on $\Gamma(M,F)$ by the following way: $(X_1,\xi_1) \sim (X_2,\xi_2)$ if and only if there exist a proper G-manifold X and G-mappings ϕ_j from X_j to X such that

(i) $\rho_j = \rho\cdot\phi_j$ and (ii) $\phi_1!(\xi_1) = \phi_2!(\xi_2)$,

where ρ,ρ_j are the canonical G-mappings from X,X_j to M respectively, and $\phi_j!$ mean the Thom-Gysin mappings from G-vector bundles over $\tilde{\nu}_j^*\oplus\rho_j^*(\nu^*)$ to those over $\tilde{\nu}^*\oplus\rho^*(\nu^*)$. Denote by $K_g(M,F)$ the set of all equivalence classes in $\Gamma(M,F)$ with respect to $\sim$. Then it is an abelian group equipped with the disjoint union of G-vector bundles. We call it the <u>geometric</u> K-theory of (M,F).

In what folows, we shall explain the <u>K-index</u> mapping μ from $K_g(M,F)$ to $K_a(M,F)$. Given any $(X,\xi) \in \Gamma(M,F)$, let us consider the G-mapping j from X to $X\times M$ defined by $j(x) = (x,\rho(x))$ for $x \in X$.

Then $\rho = \pi \cdot j$ where π is the projection from $X \times M$ to M. Let $\tilde{j}$ be the canonical G-mapping from $\tilde{\nu}^*_X \oplus \rho^*(\nu^*)$ to $\tilde{\nu}^*_{X\times M} \oplus \pi^*(\nu^*)$ associated to j, which is a bundle projection whose fibers have a G-equivariant spin^c structure. By the Thom-Gysin's theorem, the group generated by all G-vector bundles over $\tilde{\nu}^*_X \oplus \rho^*(\nu^*)$ is isomorphic to that by those over $\tilde{\nu}^*_{X\times M} \oplus \pi^*(\nu^*)$ under the mapping $\tilde{j}!$ of $\tilde{j}$. Suppose ξ is a G-vector bundle over $\tilde{\nu}^*_X \oplus \rho^*(\nu^*)$ and put $\tilde{\xi} = \tilde{j}!(\xi)$. Then it is a G-vector bundle over $\tilde{\nu}^*_{X\times M} \oplus \pi^*(\nu^*)$ which is G-isomorphic to ξ. Let τ_m be the cotangent bundle $T^*(\pi^{-1}(m))$ of $\pi^{-1}(m)$ for $m \in M$ and put $\tau = \bigcup_{m\in M} \tau_m$. Since π is a submersion, the G-space $\tilde{\nu}^*_{X\times M} \oplus \pi^*(\nu^*)$ is the total space of the bundle over τ under the canonical projection $\tilde{\pi}$ whose fibers are $\nu^* \oplus \nu^*$. Therefore, $\tilde{\xi}$ can be viewed as a G-vector bundle over τ under the mapping $\tilde{\pi}!$ of $\tilde{\pi}$. Let $\tilde{\xi}_m = \tilde{\xi}|_{\tau_m}$ be the restriction of $\tilde{\xi}$ to τ_m. By the definition of $\tilde{\xi}$, there exist elliptic differential operators D_m on $\pi^{-1}(m)$ such that $\tilde{\xi}_m$ is the symbol $\sigma(D_m)$ of D_m. Let D be the G-equivariant field of $\{D_m\}_{m\in M}$. Then it is considered as a G-invariant differential operator on $X \times M$ such that

(i) D_m are elliptic on $\pi^{-1}(m)$,

(ii) $\tilde{\xi}$ is the symbol $\sigma(D)$ of D.

Let us take the K-theoretic index ind D of D in $K_a(M,F)$ as follows:

$$\text{ind } D = [\text{Ker } D] - [\text{Coker } D],$$

where $[\cdot]$ means a $C^*_r(M,F)$-module generated by $\cdot$. We then define $\mu(X,\xi) = \text{ind } D$. It depends only on the equivalence class of (X,ξ). Therefore, it determines a homomorphism from $K_g(M,F)$ to $K_a(M,F)$. We now state the first Baum-Connes conjecture as follows:

<u>Baum-Connes</u> <u>Conjecture</u> I. Given any foliated manifold (M,F), the K-index mapping μ is an isomorphism from $K_g(M,F)$ to $K_a(M,F)$.

On the other hand, suppose (M,G,φ) is a differential dynamical

system where φ is free. Then the family F of all G-orbits becomes a foliation of M, and its C^*-algebra $C^*_r(M,F)$ is nothing but the C^*-crossed product $C(M)\times_\varphi G$ of $C(M)$ by φ. Thus it implies that $K_a(M,F) = K(C(M)\times_\varphi G)$. Moreover, $K_g(M,F)$ is isomorphic to the abelian group $K_g(M,G)$ defined by the following way: Let us denote by $\Gamma(M,G)$ the set of all triples (X,ξ,π) where X is a proper G-manifold, π is a G-mapping from X to M and ξ is a G-vector bundle over $T^*(X)\oplus\pi^*(T^*(M))$.

Then it has a similar equivalence relation as before. In other words, $(X_1,\xi_1,\pi_1) \sim (X_2,\xi_2,\pi_2)$ if and only if there exist a proper G-manifold X and G-mappings π,ρ_j such that

$$\text{(i)}\quad \pi_j = \pi\cdot\rho_j \qquad \text{and} \qquad \text{(ii)}\quad \rho_1!(\xi_1) = \rho_2!(\xi_2)$$

where $\rho_j!$ are the Thom-Gysin mappings from the groups generated by all G-vector bundles over $T^*(X_j)\oplus\pi_j^*(T^*(M))$ to the group generated by those over $T^*(X)\oplus\pi^*(T^*(M))$. Denote by $K_g(M,G)$ the set of all equivalence classes in $\Gamma(M,G)$ with respect to $\sim$. Then it is an abelian group by the canonical sum. According to the conjecture I, we also offer the following conjecture due to Baum-Connes[1]:

<u>Baum-Connes Conjecture</u> II Given a differentiable dynamical system (M,G,φ), the K-index mapping μ is an isomorphism from $K_g(M,G)$ to $K_a(M,G) = K(C(M)\times_\varphi G)$.

<u>Remark</u>. Let BG be the classifying space of G and EG the total space of the universal principal G-bundle over BG. Let τ be the vector bundle over $EG\times_G M$ whose fibers are $T^*(M)$. If we denote by $K^\tau(EG\times_G M)$ the K-group $K(B\tau/S\tau)$ of the quotient space $B\tau/S\tau$ of the ball bundle $B\tau$ of τ by its sphere bundle $S\tau$, then there exists a homomorphism δ from $K^\tau(EG\times_G M)$ to $K_g(M,G)$ with the property that $\mu\cdot\delta$ is the Kasparov β-mapping if M is one point. Moreover suppose G is discrete, then δ is $\mathbb{Q}$-injective. If G is torsion-free, then δ is bijective.

If the conjectures I and II are affirmative, then so are those due to Novikov, Gromov-Lawson-Rosenberg and Kadison in topology, differential geometry and C^*-algebra theory respectively. We shall explain them in what follows.

Let M be a closed oriented manifold and let p_j be the rational j-Pontrjagin class of M in $H^{4j}(M,\mathbb{Q})$. Namely, $p_j = (-1)^j c_{2j}$ where c_j is the rational j-Chern class of $T(M)\otimes_{\mathbb{R}}\mathbb{C}$. As a well known fact, they are topological invariants by Novikov whereas their integral classes are no longer with the property by Milnor. Moreover they are without homotopy invariance by Tamura, Shimada and Thom though they are homotopy invariant for ambient manifolds with nonpositive sectional curvature. Let π be the fundamental group of M, and we then consider the total Hirzebruch L-class defined by

$$L(M) = 1 + p_1/3 + 1/45(7p_2 - p_1^2) + \cdots\cdots .$$

By definition, the higher signature $\sigma_x(M)$ of M for $x \in H^*(B\pi,\mathbb{Q})$ is formulated as

$$\sigma_x(M) = \langle L(M) \cup f^*(x), [M] \rangle$$

where f is the classifying mapping from M to $B\pi$, f^* is the lift of f from $H^*(B\pi,\mathbb{Q})$ to $H^*(M,\mathbb{Q})$ and [M] is the fundamental homology class of M. We then state the Novikov conjecture as follows:

<u>Novikov Conjecture</u> Given any oriented closed manifold M and $x \in H^*(B\pi,\mathbb{Q})$, the higher signature $\sigma_x(M)$ is a homotopy invariant of M.

If fact, if the Baum-Connes conjecture II is affirmative for M = {pt}, then so is the Novikov conjecture. We shall see it briefly in what follows. It suffices to show that $f_*(L(M)\hat{})$ in $H_*(B\pi,\mathbb{Q})$ is

a homotopy invariant of M, where $L(M)^\wedge$ is the Poincare dual $L(M)\wedge[M]$ of $L(M)$ in $H_*(M,\mathbb{Q})$. We may assume that dim M is even if necessary replacing M by $M\times S^1$. Let $\Lambda^*(M)$ be the Grassmann algebra of $T^*(M)$. For any $[\xi]\in K^0(M)$, let us consider the signature operator D_ξ on the tensor bundle $\Lambda^*(M)\otimes\xi$ of $\Lambda^*(M)$ and ξ. In other words, denoting by d_ξ the tensor product $d\otimes 1$ of the exterior derivative d of M and the trivial mapping of ξ, D_ξ is defined as $d_\xi + d_\xi^*$. Since it is elliptic, we can define the analytic index $\mathrm{ind}_a D_\xi$ of D_ξ, which is nothing but the Kasparov product $[\xi]\otimes_M[D]\in\mathbb{Z}$ of $[\xi]$ and $[D]$ for the signature operator D on $\Lambda^*(M)$, where the latter is described in the following way: let $L^2(\Lambda^*(M))$ be the Hilbert space consisting of all L^2-sections of $\Lambda^*(M)$ and λ the canonical representation of $C(M)$ on $L^2(\Lambda^*(M))$, then $[D] = [(L^2(\Lambda^*(M),\lambda,D(1+D^2)^{-1/2})]$ in $KK(M,pt)$.

We denote by $\mathrm{ind}_g D_\xi$ the geometric index of D_ξ. Then it equals $\langle L(M)\vee ch([\xi]),[M]\rangle$ where ch is the Chern character from $K^0(M)$ to $H^{ev}(M,\mathbb{Q})$. It follows from Atiyah-Singer index theorem that

$$[\xi]\otimes_M[D] = \langle L(M)\vee ch([\xi]),[M]\rangle$$

for all $[\xi]\in K^0(M)$. Since $ch_{\mathbb{Q}}$ is an isomorphism from $K^0(M)\otimes_{\mathbb{Z}}\mathbb{Q}$ to $H^{ev}(M,\mathbb{Q})$, it implies that

$$ch_{\mathbb{Q}}^{-1}\cdot f^*(x)\otimes_M[D] = \langle f^*(x),L(M)^\wedge\rangle$$

for all $x\in H^{ev}(B\pi,\mathbb{Q})$. As a known fact, it follows that

$$ch_{\mathbb{Q}}^{-1}\cdot f^* = f^*\cdot ch_{\mathbb{Q}}^{-1} \quad\text{and}\quad f^*(a)\otimes_Q b = a\otimes_R f_*(b)$$

for all $a\in KK(P,R)$ and $b\in KK(Q,R)$ where f is a continuous mapping from Q to R and f^*, f_* are the lifts of f from $KK(P,R)$, $KK(Q,R)$ to $KK(P,Q)$, $KK(R,R)$ respectively. We then see that

$$ch_{\mathbb{Q}}^{-1}(x)\otimes_{B\pi} f_*([D]) = \langle x, f_*(L(M)^\wedge)\rangle$$

for all $x\in H^*(B\pi,\mathbb{Q})$. Then the homotopy invariance of $f_*(L(M)^\wedge)$ is equivarent to that of $f_*([D])$ in $K_0(B\pi)\otimes_{\mathbb{Z}}\mathbb{Q} = \varinjlim_{X\subset B\pi} K_0(X)\otimes_{\mathbb{Z}}\mathbb{Q}$.

of $L(M)$ in $H_*(M,\mathbb{Q})$. We may assume that dim M is even if necessary replacing M by $M\times S^1$. Let $\Lambda^*(M)$ be the Grassmann algebra of $T^*(M)$.

For any $[\xi] \in K^0(M)$, let us consider the signature operator D_ξ on the tensor bundle $\Lambda^*(M)\otimes\xi$ of $\Lambda^*(M)$ and ξ. In other words, denoting by d_ξ the tensor product $d\otimes 1$ of the exterior derivative d of M and the trivial mapping of ξ, D_ξ is defined as $d_\xi + d_\xi^*$. Since it is elliptic, we can define the analytic index $\mathrm{ind}_a D_\xi$ of D_ξ, which is nothing but the Kasparov product $[\xi]\otimes_M[D] \in \mathbb{Z}$ of $[\xi]$ and $[D]$ for the signature operator D on $\Lambda^*(M)$, where the latter is described in the following way: let $L^2(\Lambda^*(M))$ be the Hilbert space consisting of all L^2-sections of $\Lambda^*(M)$ and λ the canonical representation of $C(M)$ on $L^2(\Lambda^*(M))$, then $[D] = [(L^2(\Lambda^*(M),\lambda,D(1+D^2)^{-1/2})]$ in $KK(M,pt)$.

We denote by $\mathrm{ind}_g D_\xi$ the geometric index of D_ξ. Then it equals $\langle L(M)\mathrm{vch}([\xi]),[M]\rangle$ where ch is the Chern character from $K^0(M)$ to $H^{ev}(M,\mathbb{Q})$. It follows from Atiyah-Singer index theorem that

$$[\xi]\otimes_M[D] = \langle L(M)\mathrm{vch}([\xi]),[M]\rangle$$

for all $[\xi] \in K^0(M)$. Since $\mathrm{ch}_{\mathbb{Q}}$ is an isomorphism from $K^0(M)\otimes_{\mathbb{Z}}\mathbb{Q}$ to $H^{ev}(M,\mathbb{Q})$, it implies that

$$\mathrm{ch}_{\mathbb{Q}}^{-1}\cdot f^*(x)\otimes_M[D] = \langle f^*(x),L(M)\hat{}\ \rangle$$

for all $x \in H^{ev}(B\pi,\mathbb{Q})$. As a known fact, it follows that

$$\mathrm{ch}_{\mathbb{Q}}^{-1}\cdot f^* = f^*\cdot\mathrm{ch}_{\mathbb{Q}}^{-1} \quad\text{and}\quad f^*(a)\otimes_Q b = a\otimes_R f_*(b)$$

for all $a \in KK(P,R)$ and $b \in KK(Q,R)$ where f is a continuous mapping from Q to R and f^*, f_* are the lifts of f from $KK(P,R)$, $KK(Q,R)$ to $KK(P,Q)$, $KK(R,R)$ respectively. We then see that

$$\mathrm{ch}_{\mathbb{Q}}^{-1}(x)\otimes_{B\pi} f_*([D]) = \langle x, f_*(L(M)\hat{}\ )\rangle$$

for all $x \in H^*(B\pi,\mathbb{Q})$. Then the homotopy invariance of $f_*(L(M)\hat{}\ )$ is equivarent to that of $f_*([D])$ in $K_0(B\pi)\otimes_{\mathbb{Z}}\mathbb{Q} = \varinjlim_{X\subset B\pi} K_0(X)\otimes_{\mathbb{Z}}\mathbb{Q}$.

Let us now define the Kasparov homomorphism β from $K_*(B\pi)$ to $K_*(C_r^*(\pi))$ by the following way. Given a compact subset X of $B\pi$, put $\tilde{X} = i_X^*(E\pi)$ for the natural imbedding i_X from X to $B\pi$. Then it is a regular covering space with the property that $X = \tilde{X}/\pi$. Let E_X be the set of all continuous mappings f from $\tilde{X}$ to $C_r^*(\pi)$ such that

$$f(gx) = \lambda(g)f(x)$$

for all $g \in \pi$ and $x \in \tilde{X}$. It becomes a Hilbert $C(X)\otimes C_r^*(\pi)$-module equipped with

$$(fa)(\tilde{x}) = f(\tilde{x})a\cdot p(\tilde{x}) \quad \text{and} \quad \langle f_1 \| f_2\rangle\cdot p(\tilde{x}) = f_1(\tilde{x})^* f_2(\tilde{x})$$

for all $f, f_j \in E_X$, $a \in C(X)\otimes C_r^*(\pi)$ and $\tilde{x} \in \tilde{X}$, where p means the projection from $\tilde{X}$ to X. We then denote by $[E_X]$ the homotopy class of $(E_X, 0)$ which belongs to $KK(\mathbb{C}, C(X)\otimes C_r^*(\pi)) = K_0(C(X)\otimes C_r^*(\pi))$. Let us define a homomorphism β_X from $K_0(X)$ to $K^0(C_r^*(\pi))$ by

$$\beta_X(\xi) = [E_X] \otimes_X \xi$$

for $\xi \in K_0(X)$. Moreover, put $\beta = \underline{\lim}_{X\subset B\pi} \beta_X$. Then it is a homomorphism from $K_0(B\pi)$ to $K_0(C_r^*(\pi))$ such that $\beta_X = \beta \| K_0(X)$. Due to Miscenko-Fomenko[8] and Kasparov[6], the image $\beta_{\mathbb{Q}}\cdot f_*([D])$ of $f_*([D])$ under $\beta_{\mathbb{Q}} = \beta\otimes 1_{\mathbb{Q}}$ is a homotopy invariant of M in $K_0(B\pi)\otimes_{\mathbb{Z}}\mathbb{Q}$. Thus if $\beta_{\mathbb{Q}}$ is a monomorphism from $K_0(B\pi)\otimes_{\mathbb{Z}}\mathbb{Q}$ to $K_0(C_r^*(\pi))\otimes_{\mathbb{Z}}\mathbb{Q}$, then $f_*([D])$ is also a homotopy invariant of M. Remembering the definition of β, δ and μ, one can see that $\beta = \mu\cdot\delta$. Therefore if the conjecture II is affirmative or $\mu_{\mathbb{Q}}$ is injective in more general, then so is $\beta_{\mathbb{Q}}$. This implies that the Novikov conjecture is affirmative ([6],[12]~[14]).

We shall next state the Gromov-Lawson-Rosenberg conjecture in differential topology in connection with the Baum-Connes conjecture II. Let M be a closed spin manifold and π its fundamental group. Taking the classifying map f from M to $B\pi$, let f^* the lift of f from $H^*(B\pi,\mathbb{Q})$ to $H^*(M,\mathbb{Q})$. We then define the Hirzebruch $\hat{A}$-class $\hat{A}(M)$ of M by

$$\hat{A}(M) = 1 - p_1/24 - 1/32\cdot 45(p_2 - 7/4\, p_1^2) - \cdots$$

where p_j are the rational Pontrjagin classes of M. The <u>higher $\hat{A}$-genus</u> $\rho_x(M)$ of M for all $x \in H^*(B\pi,\mathbb{Q})$ is defined by the following fashion:

$$\rho_x(M) = \langle \hat{A}(M)\vee f^*(x), [M] \rangle$$

where [M] is the fundamental homology class of M. It is obviously

differentially invariant of M. Let $\kappa_m(M)$ be the scalar curvature of M at $m \in M$, in other words

$$\kappa_m(M) = \Sigma_{i,j} \langle R(X_i,X_j)X_j \| X_i \rangle_m$$

where $\{X_j\}$ is a locally orthonormal frame of T(M) and R is the curvature tensor of M with respect to a given Riemannian metric.

The following conjecture was established by Gromov, Lawson and Rosenberg:

<u>Gromov-Lawson-Rosenberg Conjecture</u> Let M be a closed spin manifold. Suppose there exists a Riemannian metric of M for which the scalar curvature κ is positive, then the higher $\hat{A}$-genus $\rho_x(M)$ of M vanishes for every $x \in H^*(B\pi,\mathbb{Q})$.

This conjecture is affirmative if the Kasparov mapping $\beta_{\mathbb{Q}}$ is injective, which is satisfied if Baum-Connes Conjecture II holds. In fact, let ξ be the flat $C_r^*(\pi)$-bundle over M, namely $\xi = \tilde{M} \times_\pi C_r^*(\pi)$ where $\tilde{M}$ is the universal covering space of M. We may assume that dim M is even as before. Since M has a spin structure S, there exist half spinor bundles S^+, S^- of S. Let $C^\infty(S^+\otimes\xi)$, $C^\infty(S^-\otimes\xi)$ be the sets of all C^∞-sections of $S^+\otimes\xi$, $S^-\otimes\xi$ respectively. Denote by D^+ the Dirac operator from $C^\infty(S^+\otimes\xi)$ to $C^\infty(S^-\otimes\xi)$ with respect to the flat connection of ξ. Then there exists the conjugate operator D^- of D^+ from $C^\infty(S^-\otimes\xi)$ to $C^\infty(S^+\otimes\xi)$. We next study the Chern character of $ch(\xi)$ of ξ due to Miscenko-Solov'ev. Let ξ be a $C_r^*(\pi)$-bundle over M whose fibers are finitely generated projective left $C_r^*(\pi)$-modules. Then the classes $[\xi]$ of ξ by stable equivalence generate the K-group $K_0(C(M)\otimes C_r^*(\pi))$ of the C^*-tensor product $C(M)\otimes C_r^*(\pi)$ of $C(M)$ and $C_r^*(\pi)$. Using the ordinary Chern character and the Kunneth formula, we obtain the Chern character $ch([\xi])$ of $[\xi]$ as a homomorphism from $K_0(C(M)\otimes C_r^*(\pi))$ to $H^{ev}(M,\mathbb{Q})\otimes K_0(C_r^*(\pi)) \oplus H^{od}(M,\mathbb{Q})\otimes K_1(C_r^*(\pi))$

which is actually an isomorphism modulo torsion. Since $S^+\otimes\xi$ and $S^-\otimes\xi$ are smooth $C_r^*(\pi)$-vector bundles over M, and D^+ is an elliptic bounded $C_r^*(\pi)$-valued operator from a Sobolev $C_r^*(\pi)$-module $H^*(S^+\otimes\xi)$ of $S^+\otimes\xi$ to $H^*(S^-\otimes\xi)$ of $S^-\otimes\xi$, there exists a $C_r^*(\pi)$-compact operator C from $H^*(S^+\otimes\xi)$ to $H^*(S^-\otimes\xi)$ such that both $[\mathrm{Ker}(D^+ + C)]$ and $[\mathrm{Coker}(D^+ + C)]$ are finitely generated projective $C_r^*(\pi)$-modules. Therefore one can define the $C_r^*(\pi)$-index $\mathrm{ind}_{C_r^*(\pi)} D^+$ of D^+ by

$$\mathrm{ind}_{C_r^*(\pi)} D^+ = [\mathrm{Ker}(D^+ + C)] - [\mathrm{Coker}(D^- + C)] .$$

It follows from Miscenko-Fomenko[8] that

$$\mathrm{ind}_{C_r^*(\pi)} D^+ = \langle \mathrm{ch}\cdot\sigma(D^+)\vee \mathrm{Td}(M), [T^*(M)] \rangle$$

in $K_*(C_r^*(\pi))\otimes_{\mathbb{Z}}\mathbb{Q}$, where Td(M) is the Todd class of M and $[T^*(M)]$ is the fundamental class of $T^*(M)$. Since M has a spin structure, it implies that there exists a Thom isomorphism Th from $H^*(M,\mathbb{Q})$ onto $H_c^*(T^*(M),\mathbb{Q})$ where H_c means de Rham cohomology with compact support. It then follows that

$$\mathrm{ch}([\xi])\vee\hat{A}(M) = \mathrm{Th}^{-1}(\mathrm{ch}\cdot\sigma(D^+)\vee \mathrm{Td}(M)).$$

Therefore we have that

$$\mathrm{ind}_{C_r^*(\pi)} D^+ = \langle \mathrm{ch}([\xi])\vee\hat{A}(M), [M] \rangle.$$

On the Sobolev $C_r^*(\pi)$-module $H^*(S^+\otimes\xi)$, the operator D^-D^+ satisfies the generalized Bochner-Weizenbeck formula:

$$D^-D^+ = \nabla^*\nabla + \kappa/4.$$

where ∇ is the canonical flat connection of $S^+\otimes\xi$. Similarly, D^+D^- has the following equality:

$$D^+D^- = \nabla\nabla^* + \kappa/4.$$

By the assumption of κ, it follows from Kazdan-Warner's result that there exist a Riemannian metricof M and a positive constant c such that $\kappa_m(M) \geq c1$ for all $m \in M$. Thus D^-D^+ and D^+D^- have bounded inverse operators, which means that

$$\langle \mathrm{ch}([\xi])\vee\hat{A}(M), [M] \rangle = 0.$$

Let $\tilde{\xi}$ be the universal $C_r^*(\pi)$-bundle over $B\pi$ associated to ξ. Since $\mathrm{ind}_a D_{\tilde{\xi}} = \mathrm{ind}_g D_{\tilde{\xi}}$, it follows from the definition of β that

$$\begin{aligned}\beta_{\mathbb{Q}}(ch_{\mathbb{Q}}^{-1}\cdot f_*(\hat{A}(M)\wedge[M])) &= [\tilde{\xi}]\otimes_{B\pi} ch_{\mathbb{Q}}^{-1}\cdot f_*(\hat{A}(M)\wedge[M]) \\ &= \langle ch([\tilde{\xi}]), f_*(\hat{A}(M)\wedge[M]) \rangle \\ &= \langle f^*\cdot ch([\tilde{\xi}])\vee\hat{A}(M),[M] \rangle.\end{aligned}$$

Since ξ is the flat $C_r^*(\pi)$-bundle over M, it is the pull back $f^*(\tilde{\xi})$ of $\tilde{\xi}$ with respect to f. Therefore it implies that

$$\beta_{\mathbb{Q}}(ch^{-1}\cdot f_*(\hat{A}(M)\wedge[M])) = \langle ch([\xi])\vee\hat{A}(M),[M] \rangle = 0.$$

Suppose $\beta_{\mathbb{Q}}$ is injective, then we have that

$$ch_{\mathbb{Q}}^{-1}\cdot f_*(\hat{A}(M)\wedge[M]) = 0.$$

Since $ch_{\mathbb{Q}}$ is an isomorphism from $K_*(B\pi)\otimes_{\mathbb{Z}}\mathbb{Q}$ to $H_*(B\pi,\mathbb{Q})$, it follows that

$$f_*(\hat{A}(M)\wedge[M]) = 0.$$

By the definition of $\rho_x(M)$, we conclude that

$$\begin{aligned}\rho_x(M) &= \langle \hat{A}(M)\vee f^*(x),[M] \rangle \\ &= \langle x, f_*(\hat{A}(M)\wedge[M]) \rangle = 0\end{aligned}$$

for every $x \in H^*(B\pi,\mathbb{Q})$. Especially, if the Baum-Connes conjecture II is affirmative, so is the Gromov-Lawson-Rosenberg conjecture. For instance, as $\rho_1(K^4) = 2$ for the K3-surface K^4, there exists no Riemannian metric of K^4 which induces a positive scalar curvature.

As an application toward C^*-algebras, we state a generalized Kadison conjecture for the existance of nontrivial projections in group C^*-algebras:

<u>Generalized Kadison Conjecture</u> Suppose G is a torsion free discrete group, then the reduced group C^*-algebra $C_r^*(G)$ of G has no nontrivial projections.

In fact, let us consider the geometric K-theory $K_g(pt,G)$ for a dynamical system (pt,G). By the definition of the K-index mapping μ, given any $[(X,\xi)] \in K_g(pt,G)$ there exists a G-invariant elliptic differential operator D_ξ on X such that

$$\mu(X,\xi) = \mathrm{ind}_a D_\xi \quad \text{and} \quad \sigma(D_\xi) = \xi.$$

As G is torsion free, it acts on X freely. By Atiyah's result, it follows that

$$tr_*(\mathrm{ind}_a D_\xi) \in \mathbb{Z}$$

where tr_* is the lift of the canonical normalized trace tr of $C_r^*(G)$ to $K_0(C_r^*(G))$. Suppose μ is onto, it implies that

$$tr_*(K_0(C_r^*(G)) \subset \mathbb{Z}.$$

Therefore, $C_r^*(G)$ has no nontrivial projections. Summing up the argument discussed above, we obtain the following observation:

<u>Observation</u> Suppose the Baum-Connes conjecture II holds for one point manifold, then affirmative are all the conjectures due to Novikov, Kadison and Gromov-Lawson-Rosenberg.

<u>Remark</u> The generalized Kadison conjecture is affirmatively solved for the free groups with finite generators due to Pimsner-Voiculescu.

§3 <u>Miscellaneous results</u> Let (A,G,α) be a C^*-dynamical system where G is a simply connected solvable Lie group. Due to Iwasawa, G is the multisemidirect product of $\mathbb{R}$. Using the duality for C^*-crossed products, Connes showed that $K_j(A\times_\alpha\mathbb{R})$ is isomorphic to $K_{j+1}(A)$ under the Thom isomorphism. Since crossed products are compatible with semidirect products, it follows the next theorem:

Theorem 1. Let (A,G,α) be a C^*-dynamical system where G is a simply connected solvable Lie group. Then $K_j(A\times_\alpha G)$ is isomorphic to $K_{j+\dim G}(A)$ under the Thom isomorphism.

Given a differential dynamical system (M,G,ϕ) where G is simply connected solvable, it follows from Theorem 1 that $K_a(M,G)$ is equal to $K_{\dim G}(M)$ via the Thom isomorphism. On the other hand, since G has no torsion, it implies from Baum-Connes[1] that $K_g(M,G)$ is isomorphic to $K(B\tau/S\tau)$ where $\tau = EG\times_G T^*(M)$ and $B\tau, S\tau$ its ball, sphere bundle respectively. By the assumption of G, there exists a strong retraction from $EG\times_G M$ to $M\times\mathbb{R}^{\dim G}$ with respect to which $K_g(M,G)$ is isomorphic to $K^{\dim G}(M)$. Combining this with the previous theorem, we have the following proposition:

Proposition 2. Let (M,G,φ) be a differential dynamical system where G is simply connected solvable. Then the Baum-Connes conjecture II holds for the triplet.

Suppose G is a compact Lie group, then the situation is quite simple, namely the conjecture II is nothing but the Atiyah-Singer index theorem:

Proposition 3. Let (M,G,φ) be a differential dynamical system where G is compact. Then the Baum-Connes conjecture II is affirmative.

Due to the above propositions, we may restrict our interest to the case where G is a noncompact semisimple Lie group in the next stage. Let G be as above and K its maximal compact subgroup. If

G/K has a G-invariant spin^c structure, we know from Baum-Connes[1] that

$$K_g(M,G) = K_g^{\dim G/K}(M,K).$$

By Proposition 3, it follows that

$$K_g(M,K) = K_a(M,K)$$

up to the K-index mapping. Thus it suffices to show that

$$K_a(M,G) = K_a^{\dim G/K}(M,K).$$

The next result is one example supporting the above equality:

<u>Proposition</u> 4. Let G be a connected Lie group and K the maximal compact subgroup of G such that G/K has a G-invariant spin^c structure. If there exists an amenable normal subgroup H of G such that G/H is locally isomorphic to the finite product of $SO_0(n,1)$ and compact groups, then we have that

$$K_a(M,G) = K_a^{\dim G/K}(M,K)$$

(cf;[7]). Especially, suppose M is a point, then the Baum-Connes conjecture II is verified affirmatively for more wider classes of G:

<u>Proposition</u> 5. Let be a connected reductive Lie group and K as in Proposition 4. Then we have that

$$K_a(pt,G) = K_a^{\dim G/K}(pt,K).$$

When G is a discrete group, there is no theorem supporting the Baum-Connes conjecture II affirmatively at the present stage. The only nontrivial example is the following due to Natsume[10]:

<u>Observation</u> 6. $K_g(pt,SL(2,\mathbb{Z})) = K_a(pt,SL(2,\mathbb{Z}))$.

In fact, the above result is deduced from the fact that $SL(2,\mathbb{Z})$

is the amalgamated product of $\mathbb{Z}_4$ and $\mathbb{Z}_6$ with respect to $\mathbb{Z}_2$. Since $SL(n,\mathbb{Z})$ $(n \geq 3)$ have no such fashion, we may ask the following:

<u>Question</u> 1. Is it true that

$$K_g(pt, SL(n,\mathbb{Z})) = K_a(pt, SL(n,\mathbb{Z}))$$

for all $n \geq 3$? More generally, suppose G is a discrete subgroup of a connected Lie group, can we show that $K_g(pt,G) = K_a(pt,G)$?

We now discuss the Baum-Connes conjecture I, which is verified affirmatively only for few cases. In what follows, we shall list up several examples:

<u>Proposition</u> 7. The Reeb foliations on 2-torus or 3-sphere satisfy the Baum-Connes conjecture I affirmatively ([17]).

<u>Proposition</u> 8. The same result as Proposition 7 holds for the Anosov foliations on infra-homogeneous manifolds ([15]).

Suppose an ambient manifold has an Anosov foliation, its rank is 1 automatically. The next example is the case where the Baum-Connes conjecture I holds for foliated manifolds with an arbitrary rank:

<u>Proposition</u> 9. Given any $n \in \mathbb{N}$, there exists a foliated manifold (M_n, F_n) such that

(i) rank $M_n = n$ and (ii) $K_g(M_n,F_n) = K_a(M_n,F_n)$

(Section 4).

Although the foliations cited above have nontrivial holonomy in

general, the next two cases are without holonomy:

Proposition 10. The Baum-Connes conjecture I is true for all codimension 1 foliations without holonomy on smooth manifolds ([9]).

Observation 11. The K-index mapping is injective for all Anosov foliations by topologically transitive diffeomorphisms of any compact smooth manifold.

Inorder to verify the Baum-Connes conjecture I, the following questions are quite fundamental:

Question 2. Given a K-oriented foliation whose leaves are all contractible, does the conjecture I hold affirmatively ?

Question 3. Can we show the conjecture I for all foliated bundles ?

Question 4. Is it true that the conjecture I holds for any foliated manifold whose fundamental group is $SL(n,\mathbb{Z})$ $(n \geq 3)$?

§4 Generalized Anosov foliations In this section, we shall examine the Baum-Connes conjecture I for generalized Anosov foliations on infra-homogeneous manifolds.

Let (M,G,φ) be a differentiable dynamical system. The action φ is called Anosov if there exist anelement $g \in G$ and subbundles E^s, E^u, E^c of the tangent bundle $T(M)$ of M such that

(i) $T(M) = E^s \oplus E^u \oplus E^c$ and $d\varphi_g(E^j) = E^j$,

(ii) E^j are completely integrable and $E^c = T(\varphi(G))$, and

(iii) $\|d\varphi_g^n(\xi)\| \leq \lambda\|\xi\|$ $(\xi\in E^s)$, $\mu\|\xi\| < \|d\varphi_g^n(\xi)\|$ $(\xi\in E^u)$,

$\lambda\|\xi\| < \|d\varphi_g^n(\xi)\| < \mu\|\xi\|$ $(\xi\in E^c)$ for some $0<\lambda<1<\mu$.

Then there exist foliations F^s, F^u, F^c of M such that $T(F^j) = E^j$ for $j = s,u,c$. Each leaf $W_x^j \in F^j$ $(j = s,u,c)$ is given by the following fashion:

$$W_x^s = \{ y \in M \mid d(\varphi_g^n(x),\varphi_g^n(y))\, \lambda^{-n} \longrightarrow 0 \ (n\to\infty)\},$$

$$W_x^u = \{ y \in M \mid d(\varphi_g^{-n}(x),\varphi_g^{-n}(y))\, \mu^{n} \longrightarrow 0 \ (n\to\infty)\},$$

$$W_x^c = \varphi(G)x.$$

Let us now take a noncompact semisimple Lie group G with finite center and K its maximal compact subgroup. We denote by G' the Lie algebra of G. Let $G' = K' + P'$ be a Cartan decomposition of G' and A' a maximal abelian subalgebra of P'. If Λ is the root system of A', then we have the root space decomposition of G' with respect to Λ as follows:

$$G' = M' \oplus A' \oplus \Sigma_{\lambda\in\Lambda}\, G'_\lambda ,$$

where M' is the centralizer of A' in K' and G'_λ is the λ-eigen space of ad(A') in G'. Given a regular element $a \in A = \exp A'$, we define two subsets $\Lambda_a^{+(-)}$ of Λ by

$$\Lambda_a^{+(-)} = \{ \lambda \in \Lambda \mid \lambda(\log a) > 0 \ (< 0) \}$$

respectively, where log a is the element of A' such that exp(log a) = a. Let us define $N'_{+(-)}$ as the direct sum of G'_λ $(\lambda \in \Lambda_a^{+(-)})$ respectively. Let $M = \exp M'$ and consider the diffeomorphism φ_a of G/M defined by $\varphi_a(gM) = gaM$ $(g \in G)$. Then it follows that G/M defined by $\varphi_a(gM) = gaM$ $(g \in G)$. Then it follows that

$$d\varphi_a(\xi) = \Sigma_{\lambda\in\Lambda_a^j}\, e^{-\lambda(\log a)}\, \xi_\lambda$$

for all $\xi = \Sigma_{\lambda\in\Lambda_a^j}\, \xi_\lambda \in N'_j$ and $j = +,-$. Therefore there exists a

constant $c > 0$ such that

$$\|d\varphi_a(\xi)\| \leq e^{-c}\|\xi\| \quad (\xi \in N'_+), \quad \|d\varphi_a(\xi)\| \geq e^{c}\|\xi\| \quad (\xi \in N'_-).$$

As the tangent $T_M(G/M)$ of G/M at M is $N'_- \oplus A' \oplus N'_+$, it implies that φ is an Anosov action of A on G/M.

Remark. If $a \in A$ is singular, then the decomposition of G'/M' with respect to a is obtained as follows:

$$G'/M' = N'_- \oplus A' \oplus N'_+ \oplus \Sigma_{\lambda(\log a)=0} G'_\lambda.$$

Therefore $d\varphi_a$ has no Anosov condition in general.

Let Γ be a torsion free uniform lattice of G and define an action ψ of A on $\Gamma\backslash G/M$ by $\psi_a(\Gamma gM) = \Gamma\varphi_a(gM) = \Gamma gaM$ $(a \in A, g \in G)$. Then we have the following lemma:

Lemma 1. The action ψ is an Anosov action of A on $\Gamma\backslash G/M$.

Except the foliations F^j of $\Gamma\backslash G/M$ with respect to ψ $(j=s,u,c)$, there exist other foliations F^j $(j=cs,cu)$ such that

$$T(F^{cs}) = E^s \oplus E^c, \quad T(F^{cu}) = E^u \oplus E^c.$$

Each leaf $W^j_x \in F^j$ $(j=cs,cu)$ has the following form:

$$W^{cj}_x = \cup_{y \in \varphi(G)x} W^j_y \quad (j=s,u).$$

We now check the structure of leaves in F^j $(j=s,cs)$ on $\Gamma\backslash G/M$. For any $gM \in W^s_M$, there exists a smooth curve $g(t)$ in G such that

$$g(0) = e, \ g(1) = gM \ \text{ and } \ d/dt(g(t)M) \in E^s_{g(t)M}.$$

Putting

$$X(t) = d/ds(g(t)^{-1}g(s)M)|_{s=t} \in N'_+ \quad (t \in \mathbb{R}),$$

it follows that

$$d/dt(g(t)M) = d\varphi_{g(t)}(X(t)) , \quad g(0)M = M.$$

Let us define one parameter family $h(t)$ of $N^+ = \exp \mathfrak{n}_+$ by

$$h(t) = \exp \int_0^t X(t)\, dt.$$

Then we easily check that $g(t)M = h(t)M$ $(t \in \mathbb{R})$, which implies that

$$(g(t)K, g(t)P) = (h(t)K, h(t)P) = (h(t), P) \in (N^+K/K)\times\{P\}$$

for all $t \in \mathbb{R}$ where $P = MAN^+$. This means that

$$W_M^s \subset (N^+K/K)\times\{P\}.$$

Similarly, we obtain that

$$W_M^{cs} \subset (G/K)\times\{P\}.$$

Conversely, given a $gM \in G/M$ $(g \in P)$, there exist $a \in A$ and $n \in N$ such that $gaM = nM$. Therefore, $gaM \in W_M^s$ implies $gM \in W_M^{cs}$. The similar method as above is taken place for W_M^j $(j=u,cu)$ with respect to $G = N^-AK$, $P^- = N^-AM$ where $N^- = \exp \mathfrak{n}_-$. Let π_Γ be the canonical projection from G/M to $\Gamma\backslash G/M$. Identifying G/M with $(G/K)\times(G/P)$ by $gM \longrightarrow (gK, gP)$, we obtain the following lemma:

<u>Lemma</u> 2. The Anosov dynamical system $(\Gamma\backslash G/M, A, \phi)$ gifts five foliations F^j of $\Gamma\backslash G/M$ $(j=s,u,c,cs,cu)$ whose leaves $W_{\Gamma gM}^j$ are given by

$$W_{\Gamma gM}^s = \pi_\Gamma((N^+K/K)\times\{gP\}) = \Gamma\backslash((N^+K/K)\times\{gP\}),$$

$$W_{\Gamma gM}^u = \Gamma\backslash((N^-K/K)\times\{gP^-\}), \quad W_{\Gamma gM}^c = \phi(A)\Gamma gM,$$

$$W_{\Gamma gM}^{cs} = \Gamma\backslash((G/K)\times\{gP\}), \quad W_{\Gamma gM}^{cu} = \Gamma\backslash((G/K)\times\{gP^-\}).$$

<u>Remark</u>. The following observation can be viewed as a sort of geometric approach to the above lemma. According to Oshima[11], there exists a real analytic closed manifold $(G/K)^\sim$ containing G/K as an open submanifold and G/P as the boundary of G/K. For the

Iwasawa decomposition $G = N^-AK$, we know that $N^- \times \mathbb{R}^l$ is embedded in $(G/K)^\sim$ and $N^- \times \mathbb{R}^l_+$ is isomorphic to G/K by the mapping defined in the following way:

$$(n^-, \exp-\lambda_1(\log a), \cdots, \exp-\lambda_l(\log a)) \longrightarrow n^-aK$$

where $l = \mathrm{rank}_{\mathbb{R}} G$ and $\{\lambda_j\}_{j=1}^l$ is a restricted positive simple root system of Λ. Moreover, G/P can be identified with $N^- \times \{0\}^l$. Using the fact that $g \exp(t \log a)K \longrightarrow gP$ as $t \to \infty$, we can see that the geodesic half lines $\{g \exp(t \log a)K\}_{t \geq 0}$ and $\{h \exp(t \log a)K\}_{t \geq 0}$ are asymptotically approaching each other ($t \to \infty$) if and only if $hM \in W^{cs}_{gM}$. On the other hand, W^s_{gM} is interpreted as the horosphere whose boundary passes through gP. The leaves W^{cu}, W^u are similarly translated as W^{cs}, W^s.

We now study the foliations F^j ($j=s,u,cs,cu$) of $\Gamma\backslash G/M$ in more detail. As $G/M = G/K \times G/P$, it follows that $\Gamma\backslash G/M$ is a G/P -bundle over $\Gamma\backslash G/K$. Applying Lemma 2, we have the following lemma:

<u>Lemma</u> 3. The foliated manifold $(\Gamma\backslash G/M, F^{cs})$ is the foliated G/P -bundle over $\Gamma\backslash G/K$ whose holonomy group is the image of the left translation action of Γ on G/P. The same is true for $(\Gamma\backslash G/M, F^{cu})$ replacing P by P^-.

Let us consider the principal M-bundle $\Gamma\backslash G$ over $\Gamma\backslash G/M$ and π_M be the natural projection from $\Gamma\backslash G$ to $\Gamma\backslash G/M$. Then the following lemma is also verified:

<u>Lemma</u> 4. The pull back foliations $\pi_M^*(F^s), \pi_M^*(F^u)$ of F^s, F^u by π_M are MN, N^-M -orbital by the right translation action ρ of G on $\Gamma\backslash G$ respectively.

Since Hausdorff are the holonomy groupoids of F^j (j=s,u,cs,cu), it implies the following lemma by Lemma 3 and Natsume-Takai's result for foliated bundles[15]:

<u>Lemma</u> 5. Concerning $(\Gamma\backslash G/M, F^j)$ (j=cs,cu), we have that

$$C_r^*(\Gamma\backslash G/M, F^{cs}) = (C(G/P)\times_\lambda \Gamma)_r \otimes BC(L^2(\Gamma\backslash G/K)),$$

$$C_r^*(\Gamma\backslash G/M, F^{cu}) = (C(G/P^-)\times_\lambda \Gamma)_r \otimes BC(L^2(\Gamma\backslash G/K))$$

up to isomorphisms where $(\cdot\times\cdot)_r$ means reduced crossed products and BC(H) is the C^*-algebra of all compact operators on H.

By Rieffel's work on Morita equivalence, $C(G/P)\times_\lambda \Gamma)_r$ is stably isomorphic to $(C(\Gamma\backslash G)\times_\rho P)_r$, which is equal to $C(\Gamma\backslash G)\times_\rho P$. Since N^- is equal to $\theta(N^+)$ for the Cartan involution θ of G, it follows that $C_r^*(\Gamma\backslash G/M, F^{cs})$ is stably isomorphic to $C_r^*(\Gamma\backslash G/M, F^{cu})$. By Hilsum-Skandalis'result[18], we have the following lemma:

<u>Lemma</u> 6. $C_r^*(\Gamma\backslash G/M, F^{cs})$ is isomorphic to $C_r^*(\Gamma\backslash G/M, F^{cu})$ if F^j are nontrivial (j=cs,cu).

By Lemma 4, we also can see the following:

<u>Lemma</u> 7. Concerning $(\Gamma\backslash G, \pi_M^*(F^j))$ (j=s,u), we have that

$$C_r^*(\Gamma\backslash G, \pi_M^*(F^s)) = C(\Gamma\backslash G)\times_\rho NM \ , \ C_r^*(\Gamma\backslash G, \pi_M^*(F^u)) = C(\Gamma\backslash G)\times_\rho N^-M$$

up to isomorphism.

Let (X,F) be a foliated manifold and ξ be a bundle over X whose fibres are a compact manifold C. Consider the pull back $\pi^*(F)$ of F by the natural projection π from ξ to X. Then we can obtain the next lemma:

<u>Lemma</u> 8. $C^*_\Gamma(\xi,\pi^*(F)) = C^*_\Gamma(M,F) \otimes BC(L^2(X))$ up to isomorphisms.

Combining Lemmas 7 and 8, the following lemma is automatically deduced:

<u>Lemma</u> 9. Concerning $(\Gamma\backslash G/M,F^j)$ $(j=s,u)$, we have that

$$C^*_\Gamma(\Gamma\backslash G/M,F^s) \otimes BC(L^2(M) = C(\Gamma\backslash G)x_\rho N^+M ,$$

$$C^*_\Gamma(\Gamma\backslash G/M,F^u) \otimes BC(L^2(M)) = C(\Gamma\backslash G)x_\rho N^-M.$$

Applying Hilsum-Skandalis'result[18] again, it follows from Lemma 9 that

<u>Corollary</u> 10. $C^*_\Gamma(\Gamma\backslash G/M,F^s)$ is isomorphic to $C^*_\Gamma(\Gamma\backslash G/M,F^u)$ if F^j are nontrivial $(j=s,u)$.

We now compute the analytic K-theory $K_a(\Gamma\backslash G/M,F^j)$ of $(\Gamma\backslash G/M,F^j)$ $(j=s,u,cs,cu)$ using Lemmas 5~9. It certainly follows that

$$K_a(\Gamma\backslash G/M,F^s) = K_a(\Gamma\backslash G,N^+M) , \quad K_a(\Gamma\backslash G/M,F^u) = K_a(\Gamma\backslash G,N^-M),$$

$$K_a(\Gamma\backslash G/M,F^{cs}) = K_a(G/P,\Gamma) , \quad K_a(\Gamma\backslash G/M,F^{cu}) = K_a(G/P^-,\Gamma).$$

Since $(C(G/P)x_\lambda\Gamma)_r$, $(C(G/P^-)x_\lambda\Gamma)_r$ are stably isomorphic to $C(\Gamma\backslash G)x_\rho P$, $C(\Gamma\backslash G)x_\rho P^-$ respectively, it then means that

$$K_a(G/P,\Gamma) = K_a(\Gamma\backslash G,P) , \quad K_a(G/P^-,\Gamma) = K_a(\Gamma\backslash G,P^-).$$

To analyze the right hand sides of the above equalities, we prepare a generalized Thom isomorphism essentially due to Connes and Julg:

<u>Lemma</u> 11. Let (A,G,α) be a C^*-dynamical system where G is the semidirect product $\mathbb{R}^n x_s C$ of $\mathbb{R}^n$ by a compact group C. Then there exists a Thom isomorphism between $K_a(A,G)$ and $K^n_{a,C}(A)$ where $K_{a,C}(\cdot)$ means the analytic C-equivariant K-theory.

<u>Remark</u>. If C is trivial, the above lemma is due to Connes and if $n = 0$, it is thanks to Julg.

Since P is the semidirect product of N^+ by MA, it follows from Lemma 11 that

$$\begin{aligned} K_a(\Gamma\backslash G,P) &= K_a(C(\Gamma\backslash G)\times_\rho MN^+,A) \\ &= K_a^{\dim A}(\Gamma\backslash G,MN^+) \\ &= K_{a,M}^{\dim AN^+}(\Gamma\backslash G). \end{aligned}$$

As Γ is torsion free, it has no nontrivial intersection with M. Therefore, ρ is a free action of M on $\Gamma\backslash G$. By Segal's result[19], we have that

$$K_{a,M}^{\dim AN^+}(\Gamma\backslash G) = K^{\dim AN^+}(\Gamma\backslash G/M).$$

Consequently, it follows that

$$K_a(\Gamma\backslash G/M,F^{cs}) = K^{\dim AN^+}(\Gamma\backslash G/M).$$

We shall next compute the geometric K-theory $K_g(\Gamma\backslash G/M,F^{cs})$ of $(\Gamma\backslash G/M,F^{cs})$. Let us look at the leave structure of F^{cs} in what follows. Since we know that G/K is contractible and

$$W^{cs}_{\Gamma gM} = \pi_\Gamma((G/K)\times\{gP\}) \quad (g \in G),$$

it implies that all of them are $K(\pi,1)$-spaces. Since Γ is torsion free, so is $\tilde{G} = \mathrm{Hol}(F^{cs})$. Let τ is the vector bundle over $B\tilde{G}$ whose fibres are those of the dual normal bundle ν^* of F^{cs}. We obtain from Baum-Connes[1] the next lemma:

<u>Lemma</u> 12. $$K_g(\Gamma\backslash G/M,F^{cs}) = K^\tau(B\tilde{G}) .$$

by definition, we can see that $\tilde{G}$ is isomorphic to $((G/P)\times_\lambda\Gamma)\times(B\Gamma\times B\Gamma)$ as a Borel groupoid by Natsume-Takai. However they are no longer

isomorphic as a topological groupoid in general. Let us study this correspondence more closely. Consider the map Φ from $\tilde{G}$ to $B\Gamma \times B\Gamma$ by taking $\Phi(\gamma) = (\pi_{B\Gamma}(s(\gamma)), \pi_{B\Gamma}(r(\gamma)))$, $\gamma \in \tilde{G}$. Then we check that the groupoids $\Phi^{-1}(x,y)$ $(x,y \in B\Gamma)$ are isomorphic to the principal one $(G/P)\times_\lambda \Gamma$, namely we have that

$$(G/P)\times_\lambda \Gamma \xrightarrow{\iota} \tilde{G} \xrightarrow{\Phi} B\Gamma \times B\Gamma .$$

Taking the classifying spaces of the above spaces, we have that

$$B((G/P)\times_\lambda \Gamma) \xrightarrow{B\iota} B\tilde{G} \xrightarrow{B\Phi} B(B\Gamma \times B\Gamma) .$$

Since $B(B\Gamma \times B\Gamma)$ is homotopic to a point, we have that $B\tilde{G}$ is homotopic to $B((G/P)\times_\lambda \Gamma)$ under $B\iota$. Let us consider the pull back $\sigma = B\iota^*(\tau)$ of τ by $B\iota$. Since $B\iota$ is a homotopy isomorphism, we obtain the next lemma:

<u>Lemma</u> 13. $$K^\sigma(B((G/P)\times_\lambda \Gamma)) = K^\tau(B\tilde{G}) .$$

By definition, ν^* is equal to $T^*(G/P)$. Since Γ is torsion free, it implies from Baum-Connes[1] that

Since $(G/P)\times_\Gamma E\Gamma$ is the base space of a principal $(G/P)\times_\lambda \Gamma$ -bundle, there exists the classifying map f of $(G/P)\times_\Gamma E\Gamma$ to $B((G/P)\times_\lambda \Gamma)$ which realizes the above bundle. Let us take the pull back bundle $f^*(\sigma)$ of σ by f. By definition, $E((G/P)\times_\lambda \Gamma)$ is nothing but $(G/P)\times E\Gamma$ up to Γ-equivariant homotopy equivalence(cf:[20]). Therefore, f is homotopic to id. We then have the following lemma:

<u>Lemma</u> 15. $$K^\sigma(B((G/P)\times_\lambda \Gamma)) = K^{f^*(\sigma)}((G/P)\times_\Gamma E\Gamma) .$$

Since $\nu^* = T^*(G/P)$, it follows from the definition of δ that

<u>Lemma</u> 16. $$K^{f^*(\sigma)}((G/P)\times_\Gamma E\Gamma) = K^\delta((G/P)\times_\Gamma E\Gamma) .$$

Combining the lemmas 12 ~ 16, we obtain the following:

<u>Lemma</u> 17. $$K_g(\Gamma\backslash G/M, F^{cs}) = K_g(G/P, \Gamma) .$$

Let H_j be two closed subgroups of G (j=1,2). We compare the geometric K-groups $K_g(G/H_1, H_2)$ and $K_g(H_2\backslash G, H_1)$ of $(G/H_1, H_2, \lambda)$ and $(H_2\backslash G, H_1)$ respectively. By the same phenomenon as in the case of K_a, we can verify the following crucial lemma:

<u>Lemma</u> 18. If H_2 is torsion free and $H_1 \cap H_2 = \emptyset$, then

$$K_g(G/H_1, H_2) = K_g(H_2\backslash G, H_1) .$$

Applying the above lemma to $H_1 = P$ and $H_2 = \Gamma$, it implies from Lemma 17 that

<u>Lemma</u> 19. $$K_g(\Gamma\backslash G/M, F^{cs}) = K_g(\Gamma\backslash G, P) .$$

Since P is the semidirect product $N^+ x_s MA$ of N^+ by MA, it follows the next lemma:

<u>Lemma</u> 20. $$K_g(\Gamma\backslash G, P) = K_M^{\dim AN^+}(\Gamma\backslash G) .$$

Summing up the argument discussed above, we obtain the following main theorem:

<u>Theorem</u> 21. The Baum-Connes conjecture I is affirmative for the foliated manifolds $(\Gamma\backslash G/M, F^j)$ (j=s,u,c,cs,cu).

In fact, the similar method takes place to show the conjecture even in the case of j=s,u,c,cu.

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