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3次元多様体の基本群の表現 II

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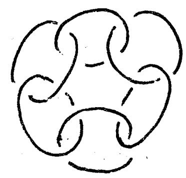

右図の様なlink L を考える。
Lのcomponentを順に、L_1, L_2, L_3, L_4, L_5, L_6 とする。Lのlink群を $\pi(L)$ とすると $\pi(L)$ のWirtinger表示は次の様になる。

$$\pi(L) \cong \langle x_1, x_2, x_3, x_4, x_5, x_6 : [x_i, x_{i-1}x_{i+1}^{-1}]=1, i=1,\cdots,6 (\mathrm{mod}\ 6)\rangle$$

$$\cong \langle x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, y_3, y_4, y_5, y_6 :$$

$$[x_i, y_i]=1,\ x_{i-1}x_{i+1}^{-1}y_i^{-1}=1,\ i=1,\cdots,6 (\mathrm{mod.}\ 6)\rangle$$

第2の表示で $y_1y_3y_5=1$, $y_2y_4y_6=1$ も成り立つ。
Lの各component L_i を p_i/r_i Dehn surgery して得られる閉多様体を

$$M = L(p_1/r_1, p_2/r_2, p_3/r_3, p_4/r_4, p_5/r_5, p_6/r_6)$$

とし、Mの基本群を $\pi_1(M)$ とすれば

$$\pi_1(M) \cong \langle x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, y_3, y_4, y_5, y_6 :$$

$[x_i, y_i]=1, x_{i-1}x_{i+1}^{-1}y_i^{-1}=1,\ x_i^p=y_i^r,\ i=1,\cdots,6 \pmod{6}>$

$\cong <a_1,a_2,a_3,a_4,a_5,a_6 : a_{i-1}^r a_{i+1}^{-r} a_i^{-p}=1,\ i=1,\cdots,6 \pmod{6}>$

となる。

π（L）のPSL（2，C）への表現が次の定理により構成される。
$z \varepsilon C$に対し　$c(z)=(z+1/z)/2$,　$s(z)=(z-1/z)/2$　とおく。

定理　$\lambda_1, \lambda_3, \lambda_5, \mu_1, \mu_3, \mu_5$　を $0, \pm 1$ と異なる６つの複素数とする。これに対し複素数　$\lambda_2, \lambda_4, \lambda_6, \mu_2, \mu_4, \mu_6 = 0, \pm 1$ を、次の条件を満たす様にとったとする　（$i=2,4,6 \pmod{6}$）。

$$c(\lambda_i)=c(\mu_{i-1})c(\mu_{i+1})$$

$$+(c(\lambda_{i-1})c(\lambda_{i+1})-c(\lambda_{i+3}))s(\mu_{i-1})s(\mu_{i+1})/s(\lambda_{i-1})s(\lambda_{i+1}),$$

$$s(\lambda_i)c(\mu_i)/s(\mu_i)=-c(\lambda_{i-1})c(\mu_{i+1})s(\mu_{i-1})/s(\lambda_{i-1})$$

$$-c(\mu_{i-1})c(\lambda_{i+1})s(\mu_{i+1})/s(\lambda_{i+1})$$

$$-c(\mu_{i+3})s(\lambda_{i+3})s(\mu_{i-1})s(\mu_{i+1})/(s(\mu_{i+3})s(\lambda_{i-1})s(\lambda_{i+1})).$$

このときSL（2，C）の元　A_i（$i=1,\cdots,6$）を適当にとると上の表示で

$$y_i \longmapsto A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} A_i^{-1}$$

$$x_i \longmapsto A_i \begin{pmatrix} \mu_i & 0 \\ 0 & \frac{1}{\mu_i} \end{pmatrix} A_i^{-1}$$

$(i=1,\cdots,6)$ によって　π（L）から PSL（2, C）への　non-abelian 表現が定まる。

証明　$i=1, 3, 5 \pmod{6}$ に対し

$$U_i = \begin{pmatrix} u_{i1} & u_{i2} \\ u_{i3} & u_{i4} \end{pmatrix} = \begin{pmatrix} \dfrac{\lambda_{i+2} - \lambda_i \lambda_{i-2}}{\lambda_{i+2}(\lambda_{i-2}^2 - 1)} & \lambda_1 \lambda_3 \lambda_5 - 1 \\ \dfrac{\lambda_i - \lambda_{i-2}\lambda_{i+2}}{\lambda_i(\lambda_{i-2}^2 - 1)(\lambda_{i+2}^2 - 1)} & \dfrac{\lambda_{i+2}(\lambda_{i-2} - \lambda_i \lambda_{i+2})}{\lambda_i(\lambda_{i+2}^2 - 1)} \end{pmatrix}$$

とおくと

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \frac{1}{\lambda_1} \end{pmatrix} U_5 \begin{pmatrix} \lambda_3 & 0 \\ 0 & \frac{1}{\lambda_3} \end{pmatrix} U_1 \begin{pmatrix} \lambda_5 & 0 \\ 0 & \frac{1}{\lambda_5} \end{pmatrix} U_3 = E$$

$$U_1 U_3 U_5 = 1$$

$$u_{i1} u_{i4} = \{C(\lambda_i) - C(\lambda_{i-2}) C(\lambda_{i+2}) + S(\lambda_{i-2}) S(\lambda_{i+2})\} / \{2S(\lambda_{i-2}) S(\lambda_{i+2})\},$$

$$u_{i2} u_{i3} = \{C(\lambda_i) - C(\lambda_{i-2}) C(\lambda_{i+2}) - S(\lambda_{i-2}) S(\lambda_{i+2})\} / \{2S(\lambda_{i-2}) S(\lambda_{i+2})\} \tag{1}$$

が成り立つ。 故に、$A_1 = E$, $A_3 = U_5$, $A_5 = U_3^{-1}$ とおくと

$i = 1, 3, 5 \pmod{6}$ に対し $U_i = A_{i+2}^{-1} A_{i-2}$ が成り立ち

$$\left\{A_1 \begin{pmatrix} \lambda_1 & 0 \\ 0 & \frac{1}{\lambda_1} \end{pmatrix} A_1^{-1}\right\}\left\{A_3 \begin{pmatrix} \lambda_3 & 0 \\ 0 & \frac{1}{\lambda_3} \end{pmatrix} A_3^{-1}\right\}\left\{A_5 \begin{pmatrix} \lambda_5 & 0 \\ 0 & \frac{1}{\lambda_5} \end{pmatrix} A_5^{-1}\right\} = E$$

となる。 即ち、$i = 1, 3, 5$ に対し

$$Y_i = A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} A_i^{-1} \qquad X_i = A_i \begin{pmatrix} \mu_i & 0 \\ 0 & \frac{1}{\mu_i} \end{pmatrix} A_i^{-1}$$

とおけば $Y_1 Y_3 Y_5 = E$ となる。 これは、関係式 $y_1 y_3 y_5 = 1$ に対応する。 つぎに、$i = 2, 4, 6$ に対し、$X_{i-1} X_{i+1}^{-1}$ の trace を計算すると

$$\mathrm{Tr}(X_{i-1} X_{i+1}^{-1}) = \mathrm{Tr}\left(A_{i-1} \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & \frac{1}{\mu_{i-1}} \end{pmatrix} A_{i-1}^{-1} A_{i+1} \begin{pmatrix} \frac{1}{\mu_{i+1}} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} A_{i+1}^{-1}\right)$$

$$= \mathrm{Tr}\left(\begin{pmatrix} \mu_{i-1} & 0 \\ 0 & \frac{1}{\mu_{i-1}} \end{pmatrix} A_{i-1}^{-1} A_{i+1} \begin{pmatrix} \frac{1}{\mu_{i+1}} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} A_{i+1}^{-1} A_{i-1}\right)$$

$$= \mathrm{Tr}\left(\begin{pmatrix} \mu_{i-1} & 0 \\ 0 & \frac{1}{\mu_{i-1}} \end{pmatrix} U_{i+3} \begin{pmatrix} \frac{1}{\mu_{i+1}} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} U_{i+3}^{-1}\right)$$

ところで

$$\begin{pmatrix}\mu_{i-1} & 0 \\ 0 & \frac{1}{\mu_{i-1}}\end{pmatrix} U_{i+3} \begin{pmatrix}\frac{1}{\mu_{i+1}} & 0 \\ 0 & \mu_{i+1}\end{pmatrix} U_{i+3}^{-1}$$

$$= \begin{pmatrix} \mu_{i-1}\left(\frac{1}{\mu_{i+1}} u_{i+3\,1} u_{i+3\,4} - \mu_{i+1} u_{i+3\,2} u_{i+3\,3}\right) & \mu_{i-1}\left(\mu_{i+1} - \frac{1}{\mu_{i+1}}\right) u_{i+3\,1} u_{i+3\,2} \\ \frac{1}{\mu_{i-1}}\left(\frac{1}{\mu_{i+1}} - \mu_{i+1}\right) u_{i+3\,3} u_{i+3\,4} & \frac{1}{\mu_{i-1}}\left(\mu_{i+1} u_{i+3\,1} u_{i+3\,4} - \frac{1}{\mu_{i+1}} u_{i+3\,2} u_{i+3\,3}\right) \end{pmatrix} \tag{2}$$

であるから

$$Tr\,(X_{i-1} X_{i+1}^{-1}) =$$

$$\mu_{i-1}\left(\frac{1}{\mu_{i+1}} u_{i+3\,1} u_{i+3\,4} - \mu_{i+1} u_{i+3\,2} u_{i+3\,3}\right) + \frac{1}{\mu_{i-1}}\left(\mu_{i+1} u_{i+3\,1} u_{i+3\,4} - \frac{1}{\mu_{i+1}} u_{i+3\,2} u_{i+3\,3}\right)$$

$$= \left(\frac{\mu_{i-1}}{\mu_{i+1}} + \frac{\mu_{i+1}}{\mu_{i-1}}\right) u_{i+3\,1} u_{i+3\,4} - \left(\mu_{i-1}\mu_{i+1} + \frac{1}{\mu_{i-1}\mu_{i+1}}\right) u_{i+3\,2} u_{i+3\,3}$$

$$= \frac{(\mu_{i-1}/\mu_{i+1} + \mu_{i+1}/\mu_{i-1})\{C(\lambda_{i+3}) - C(\lambda_{i+1})C(\lambda_{i-1}) + S(\lambda_{i+1})S(\lambda_{i-1})\}}{2S(\lambda_{i+1})S(\lambda_{i-1})}$$

$$- \frac{(\mu_{i-1}\mu_{i+1} + 1/(\mu_{i-1}\mu_{i+1}))\{C(\lambda_{i+3}) - C(\lambda_{i+1})C(\lambda_{i-1}) - S(\lambda_{i+1})S(\lambda_{i-1})\}}{2S(\lambda_{i+1})S(\lambda_{i-1})}$$

（１）による。

ここで

$$\frac{\mu_{i-1}}{\mu_{i+1}} + \frac{\mu_{i+1}}{\mu_{i-1}} = 2\,(C(\mu_{i-1})C(\mu_{i+1}) - S(\mu_{i-1})S(\mu_{i+1}))$$

$$\mu_{i-1}\mu_{i+1} + \frac{1}{\mu_{i-1}\mu_{i+1}} = 2\,(C(\mu_{i-1})C(\mu_{i+1}) + S(\mu_{i-1})S(\mu_{i+1}))$$

であるから

$$Tr(X_{i-1}X_{i+1}^{-1}) = 2(C(\mu_{i-1})C(\mu_{i+1})-S(\mu_{i-1})S(\mu_{i+1}))$$

$$(C(\lambda_{i+3})-C(\lambda_{i+1})C(\lambda_{i-1})+S(\lambda_{i+1})S(\lambda_{i-1}))/\{2S(\lambda_{i+1})S(\lambda_{i-1})\}$$

$$-2(C(\mu_{i-1})C(\mu_{i+1})+S(\mu_{i-1})S(\mu_{i+1}))(C(\lambda_{i+3})-$$

$$C(\lambda_{i+1})C(\lambda_{i-1})-S(\lambda_{i+1})S(\lambda_{i-1}))/\{2S(\lambda_{i+1})S(\lambda_{i-1})\}$$

$$=2\{C(\mu_{i-1})C(\mu_{i+1})+(C(\lambda_{i-1})C(\lambda_{i+1})-C(\lambda_{i+3}))S(\mu_{i-1})$$

$$S(\mu_{i+1})/(S(\lambda_{i-1})S(\lambda_{i+1}))$$

これは、定理の仮定から、$2C(\lambda_i) = \lambda_i + \dfrac{1}{\lambda_i}$ に等しい。そして

$\neq \pm 2$ 。$X_{i-1}X_{i+1}^{-1}$ の trace がこれに等しいのであるから

Jordan標準形の定理により

$$X_{i-1}X_{i+1}^{-1} = A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} A_i^{-1}$$

となる A_i が　SL（2，C）のなかに存在する。故に $i=2,4,6$ に対しても

$$Y_i = A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} A_i^{-1}, \qquad X_i = A_i \begin{pmatrix} \mu_i & 0 \\ 0 & \frac{1}{\mu_i} \end{pmatrix} A_i^{-1}$$

とおけば　$Y_i = X_{i-1} X_{i+1}^{-1}$ となる。これは関係式　$y_i = x_{i-1} x_{i+1}^{-1}$ に対応する。また、明らかに、$i=1,2,3,4,5,6$ に対し、$[X_i, Y_i] = E$ であるから、あとは $i=1,3,5$ に対して

$Y_i \approx X_{i-1} X_{i+1}^{-1}$ を示せばよいことになる。但し $\approx$ はスカラー倍を除いて等しいことを表わす。即ち

$$A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} A_i^{-1} \approx \left\{ A_{i-1} \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & \frac{1}{\mu_{i-1}} \end{pmatrix} A_{i-1}^{-1} \right\} \left\{ A_{i+1} \begin{pmatrix} \frac{1}{\mu_{i+1}} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} A_{i+1}^{-1} \right\}$$

であるが、これは　$V_i = A_{i-1}^{-1} A_i$　とおけば

$$\begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} V_{i+1} \begin{pmatrix} \mu_{i+1} & 0 \\ 0 & \frac{1}{\mu_{i+1}} \end{pmatrix} V_{i+1}^{-1} \approx V_i^{-1} \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & \frac{1}{\mu_{i-1}} \end{pmatrix} V_i \qquad (3)$$

と同値である。

これを示す為の準備をする。　$i=2,4,6$　に対して

$$X_{i-1}X_{i+1}^{-1} = A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} A_i^{-1} \qquad (4)$$

であるから、この両辺の左から A_{i-1}^{-1} 右から A_{i-1} を掛けると

$$A_{i-1}^{-1}X_{i-1}X_{i+1}^{-1}A_{i-1} = A_{i-1}^{-1}A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} A_i^{-1}A_{i-1}$$

$$= V_i \begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} V_i^{-1}$$

また

$$A_{i-1}^{-1}X_{i-1}X_{i+1}^{-1}A_{i-1} = \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & \frac{1}{\mu_{i-1}} \end{pmatrix} A_{i-1}^{-1}A_{i+1} \begin{pmatrix} \frac{1}{\mu_{i+1}} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} A_{i+1}^{-1}A_{i-1}$$

$$= \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & \frac{1}{\mu_{i-1}} \end{pmatrix} U_{i+3} \begin{pmatrix} \frac{1}{\mu_{i+1}} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} U_{i+3}^{-1}$$

ゆえに

$$V_i \begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} V_i^{-1} = \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & \frac{1}{\mu_{i-1}} \end{pmatrix} U_{i+3} \begin{pmatrix} \frac{1}{\mu_{i+1}} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} U_{i+3}^{-1}$$

となる。 $V_i = \begin{pmatrix} v_{i1} & v_{i2} \\ v_{i3} & v_{i4} \end{pmatrix}$ とおけば左辺は

$$\begin{pmatrix} \lambda_i v_{i1}v_{i4} - \frac{1}{\lambda_i}v_{i2}v_{i3} & \left(\frac{1}{\lambda_i} - \lambda_i\right)v_{i1}v_{i2} \\ \left(\lambda_i - \frac{1}{\lambda_i}\right)v_{i3}v_{i4} & \frac{1}{\lambda_i}v_{i1}v_{i4} - \lambda_i v_{i2}v_{i3} \end{pmatrix}$$

となる。右辺は （２）により

$$\begin{pmatrix} \mu_{i-1}\left(\dfrac{1}{\mu_{i+1}}u_{i+3\,1}u_{i+3\,4}-\mu_{i+1}u_{i+3\,2}u_{i+3\,3}\right) & \mu_{i-1}\left(\mu_{i+1}-\dfrac{1}{\mu_{i+1}}\right)u_{i+3\,1}u_{i+3\,2} \\ \dfrac{1}{\mu_{i-1}}\left(\dfrac{1}{\mu_{i+1}}-\mu_{i+1}\right)u_{i+3\,3}u_{i+3\,4} & \dfrac{1}{\mu_{i-1}}\left(\mu_{i+1}u_{i+3\,1}u_{i+3\,4}-\dfrac{1}{\mu_{i+1}}u_{i+3\,2}u_{i+3\,3}\right) \end{pmatrix}$$

に等しい。 故に

$$V_{i1}V_{i2}=-\frac{S(\mu_{i+1})\mu_{i-1}(\lambda_{i-1}-\lambda_{i+3}\lambda_{i+1})(\lambda_i\lambda_3\lambda_5-1)}{S(\lambda_i)\lambda_{i-1}(\lambda_{i+1}{}^2-1)}$$

$$V_{i3}V_{i4}=-\frac{S(\mu_{i+1})\lambda_{i-1}(\lambda_{i+3}-\lambda_{i-1}\lambda_{i+1})(\lambda_{i+1}-\lambda_{i-1}\lambda_{i+3})}{\mu_{i-1}S(\lambda_i)\lambda_{i+3}{}^2(\lambda_{i-1}{}^2-1)^2(\lambda_{i+1}{}^2-1)}$$

$$V_{i1}V_{i4}+V_{i2}V_{i3}$$

$$=\frac{C(\mu_{i-1})S(\mu_{i+1})\{C(\lambda_{i+1})C(\lambda_{i-1})-C(\lambda_{i+3})\}}{S(\lambda_i)S(\lambda_{i+1})S(\lambda_{i-1})}+\frac{S(\mu_{i-1})C(\mu_{i+1})}{S(\lambda_i)}$$

となる。 これは i=2,4,6 に対してだから i=1,3,5 に対しては

$$V_{i+1\,1}V_{i+1\,2}=-\frac{S(\mu_{i+2})\mu_i(\lambda_i-\lambda_{i-2}\lambda_{i+2})(\lambda_1\lambda_3\lambda_5-1)}{S(\lambda_{i+1})\lambda_i(\lambda_{i+2}{}^2-1)}$$

$$V_{i+13}V_{i+14} = -\frac{S(\mu_{i+2})\lambda_i(\lambda_{i-2}-\lambda_i\lambda_{i+2})(\lambda_{i+2}-\lambda_i\lambda_{i-2})}{\mu_i S(\lambda_{i+1})\lambda_{i-2}^2(\lambda_i^2-1)^2(\lambda_{i+2}^2-1)}$$

$$V_{i+11}V_{i+14}+V_{i+12}V_{i+13} = \frac{C(\mu_i)S(\mu_{i+2})\{C(\lambda_{i+2})C(\lambda_i)-C(\lambda_{i-2})\}}{S(\lambda_{i+1})S(\lambda_{i+2})S(\lambda_i)}$$

$$+\frac{S(\mu_i)C(\mu_{i+2})}{S(\lambda_{i+1})} \qquad (5)$$

となる。次に i=2,4,6 に対して（４）の両辺の左から A_{i+1}^{-1} 、右から A_{i+1} を掛けると

$$A_{i+1}^{-1}X_{i-1}X_{i+1}^{-1}A_{i+1} = A_{i+1}^{-1}A_i\begin{pmatrix}\lambda_i & 0\\ 0 & \frac{1}{\lambda_i}\end{pmatrix}A_i^{-1}A_{i+1}$$

$$= V_{i+1}^{-1}\begin{pmatrix}\lambda_i & 0\\ 0 & \frac{1}{\lambda_i}\end{pmatrix}V_{i+1}$$

また

$$A_{i+1}^{-1}X_{i-1}X_{i+1}^{-1}A_{i+1} = A_{i+1}^{-1}A_{i-1}\begin{pmatrix}\mu_{i-1} & 0\\ 0 & \frac{1}{\mu_{i-1}}\end{pmatrix}A_{i-1}^{-1}A_{i+1}\begin{pmatrix}\frac{1}{\mu_{i+1}} & 0\\ 0 & \mu_{i+1}\end{pmatrix}$$

$$= U_{i+3}^{-1}\begin{pmatrix}\mu_{i-1} & 0\\ 0 & \frac{1}{\mu_{i-1}}\end{pmatrix}U_{i+3}\begin{pmatrix}\frac{1}{\mu_{i+1}} & 0\\ 0 & \mu_{i+1}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1}{\mu_{i+1}}\left(u_{i+31}u_{i+34}\mu_{i-1}-u_{i+32}u_{i+33}\frac{1}{\mu_{i-1}}\right) & \mu_{i+1}\left(\mu_{i-1}-\frac{1}{\mu_{i-1}}\right)u_{i+32}u_{i+34}\\ \frac{1}{\mu_{i+1}}\left(\frac{1}{\mu_{i-1}}-\mu_{i-1}\right)u_{i+31}u_{i+33} & \mu_{i+1}\left(u_{i+31}u_{i+34}\frac{1}{\mu_{i-1}}-u_{i+32}u_{i+33}\mu_{i-1}\right)\end{pmatrix}$$

であるから（1）を用いて

$$v_{i+12}\,v_{i+14}=\frac{S(\mu_{i-1})\,\mu_{i+1}\,\lambda_{i-1}(\lambda_{i+1}-\lambda_{i-1}\,\lambda_{i+3})(\lambda_1\,\lambda_3\,\lambda_5-1)}{S(\lambda_i)\,\lambda_{i+3}(\lambda_{i-1}^2-1)}$$

$$v_{i+11}\,v_{i+13}=\frac{S(\mu_{i-1})(\lambda_{i-1}-\lambda_{i+3}\lambda_{i+1})(\lambda_{i+3}-\lambda_{i-1}\lambda_{i+1})}{S(\lambda_i)\,\mu_{i+1}\lambda_{i-1}\,\lambda_{i+3}(\lambda_{i-1}^2-1)(\lambda_{i+1}^2-1)^2}$$

$$v_{i+11}\,v_{i+14}+v_{i+12}\,v_{i+13}=\frac{C(\mu_{i+1})S(\mu_{i-1})\{C(\lambda_{i+3})-C(\lambda_{i+1})C(\lambda_{i-1})\}}{S(\lambda_i)S(\lambda_{i+1})S(\lambda_{i-1})}-\frac{S(\mu_{i+1})C(\mu_{i-1})}{S(\lambda_i)}$$

これは i=2,4,6 に対してだから i=1,3,5 に対しては

$$v_{i2}\,v_{i4}=\frac{S(\mu_{i-2})\,\mu_i\,\lambda_{i-2}\,(\lambda_i-\lambda_{i-2}\lambda_{i+2})(\lambda_1\,\lambda_3\,\lambda_5-1)}{S(\lambda_{i-1})\,\lambda_{i+2}(\lambda_{i-2}^2-1)}$$

$$v_{i1}\,v_{i3}=\frac{S(\mu_{i-2})(\lambda_{i-2}-\lambda_{i+2}\,\lambda_i)(\lambda_{i+2}-\lambda_{i-2}\,\lambda_i)}{S(\lambda_{i-1})\,\mu_i\,\lambda_{i-2}\lambda_{i+2}(\lambda_{i-2}^2-1)(\lambda_i^2-1)^2}\qquad(6)$$

$$v_{i1}\,v_{i4}+v_{i2}\,v_{i3} = \frac{C(\mu_i)S(\mu_{i-2})\{C(\lambda_{i+2})-C(\lambda_i)C(\lambda_{i-2})\}}{S(\lambda_{i-1})S(\lambda_i)S(\lambda_{i-2})}$$

$$-\ \frac{S(\mu_i)C(\mu_{i-2})}{S(\lambda_{i-1})}$$

となる。 さて （３）を示そう。

$$\begin{pmatrix}\lambda_i & 0\\ 0 & \frac{1}{\lambda_i}\end{pmatrix} V_{i+1} \begin{pmatrix}\mu_{i+1} & 0\\ 0 & \frac{1}{\mu_{i+1}}\end{pmatrix} V_{i+1}^{-1} = \begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix}$$

$$V_i^{-1}\begin{pmatrix}\mu_{i-1} & 0\\ 0 & \frac{1}{\mu_{i-1}}\end{pmatrix} V_i = \begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix}$$

と置き

$$\begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix} \approx \begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix}$$

を示す。

$$a_1 = \lambda_i(\,v_{i+1\,1}\,v_{i+1\,4}\,\mu_{i+1} - v_{i+1\,2}\,v_{i+1\,3}\frac{1}{\mu_{i+1}})$$

$$b_1 = \lambda_i \left(\frac{1}{\mu_{i+1}} - \mu_{i+1} \right) v_{i+1\,1}\ v_{i+1\,2}$$

$$c_1 = \frac{1}{\lambda_i} \left(\mu_{i+1} - \frac{1}{\mu_{i+1}} \right) v_{i+1\,3}\ \ v_{i+1\,4}$$

$$d_1 = \frac{1}{\lambda_i} \left(v_{i+1\,1}\ v_{i+1\,4} \frac{1}{\mu_{i+1}} - v_{i+1\,2}\ v_{i+1\,3}\ \mu_{i+1} \right)$$

$$a_2 = v_{i1}\ \ v_{i4}\ \ \mu_{i-1} - v_{i2}\ \ v_{i3} \frac{1}{\mu_{i-1}}$$

$$b_2 = \left(\mu_{i-1} - \frac{1}{\mu_{i-1}} \right) v_{i2}\ \ v_{i4}$$

$$c_2 = \left(\frac{1}{\mu_{i-1}} - \mu_{i-1} \right) v_{i1}\ \ v_{i3}$$

$$d_2 = v_{i1}\ v_{i4} \frac{1}{\mu_{i-1}} - v_{i2}\ v_{i3}\ \mu_{i-1}$$

である。（５）と（６）を用いて

$$b_1 = \lambda_i (-2S(\mu_{i+1})) \frac{-S(\mu_{i+2})\,\mu_i(\lambda_i - \lambda_{i-2}\lambda_{i+2})(\lambda_1\lambda_3\lambda_5 - 1)}{S(\lambda_{i+1})\,\lambda_i(\lambda_{i+2}^2 - 1)}$$

$$= \frac{S(\mu_{i+1})S(\mu_{i+2})}{S(\lambda_{i+1})S(\lambda_{i+2})} \quad \frac{\mu_i(\lambda_i - \lambda_{i-2}\lambda_{i+2})(\lambda_1\ \lambda_3\ \lambda_5 - 1)}{\lambda_{i+2}}$$

$$b_2 = 2S(\mu_{i-1}) \frac{S(\mu_{i-2})\,\mu_i\,\lambda_{i-2}(\lambda_i - \lambda_{i-2}\lambda_{i+2})(\lambda_1\ \lambda_3\ \lambda_5 - 1)}{S(\lambda_{i-1})\,\lambda_{i+2}(\lambda_{i-2}^2 - 1)}$$

$$= \frac{S(\mu_{i-1})S(\mu_{i-2})}{S(\lambda_{i-1})S(\lambda_{i-2})} \quad \frac{\mu_i(\lambda_i - \lambda_{i-2}\lambda_{i+2})(\lambda_1\lambda_3\lambda_5 - 1)}{\lambda_{i+2}}$$

であるから

$$b = \mu_i(\lambda_i - \lambda_{i-2}\lambda_{i+2})(\lambda_1\lambda_3\lambda_5 - 1)/\lambda_{i+2}$$

と置けば

$$b_1 = \frac{S(\mu_{i+1})S(\mu_{i+2})}{S(\lambda_{i+1})S(\lambda_{i+2})} \; b_3 \;,\; b_2 = \frac{S(\mu_{i-1})S(\mu_{i-2})}{S(\lambda_{i-1})S(\lambda_{i-2})} \; b_3 \quad (7)$$

となる。 同様な計算を c_1 , c_2 について行い、

$$c_3 = - \frac{(\lambda_{i-2} - \lambda_i\lambda_{i+2})(\lambda_{i+2} - \lambda_i\lambda_{i-2})}{\mu_i\lambda_{i-2}^2\lambda_{i+2}(\lambda_i^2 - 1)^2}$$

と置けば

$$c_1 = \frac{S(\mu_{i+1})S(\mu_{i+2})}{S(\lambda_{i+1})S(\lambda_{i+2})} \; c_3, \; c_2 = \frac{S(\mu_{i-1})S(\mu_{i-2})}{S(\lambda_{i-1})S(\lambda_{i-2})} \; c_3 \quad (8)$$

となる。 次に a_1 と a_2 を計算する。

$$a_1 = \lambda_i (v_{i+11} v_{i+14} \mu_{i+1} - v_{i+12} v_{i+13} / \mu_{i+1})$$

$$= \lambda_i \{(v_{i+11} v_{i+14} + v_{i+12} v_{i+13}) S(\mu_{i+1})$$

$$+ (v_{i+11} v_{i+14} - v_{i+12} v_{i+13}) C(\mu_{i+1})\}$$

$$= \lambda_i \{(v_{i+11} v_{i+14} + v_{i+12} v_{i+13}) S(\mu_{i+1}) + C(\mu_{i+1})\}$$

$$= \lambda_i S(\mu_{i+1}) \{(v_{i+11} v_{i+14} + v_{i+12} v_{i+13}) + C(\mu_{i+1}) / S(\mu_{i+1})\}$$

$$= \lambda_i S(\mu_{i+1}) \{ \frac{C(\mu_i) S(\mu_{i+2}) \{C(\lambda_{i+2}) C(\lambda_i) - C(\lambda_{i-2})\}}{S(\lambda_{i+1}) S(\lambda_{i+2}) S(\lambda_i)}$$

$$+ \frac{S(\mu_i) C(\mu_{i+2})}{S(\lambda_{i+1})} + \frac{C(\mu_{i+1})}{S(\mu_{i+1})} \}$$

$$= \lambda_i \frac{S(\mu_{i+1})}{S(\lambda_{i+1})} \{ \frac{C(\mu_i) S(\mu_{i+2}) \{C(\lambda_{i+2}) C(\lambda_i) - C(\lambda_{i-2})\}}{S(\lambda_{i+2}) S(\lambda_i)}$$

$$+ S(\mu_i) C(\mu_{i+2}) + \frac{S(\lambda_{i+1}) C(\mu_{i+1})}{S(\mu_{i+1})} \}$$

（５）を用いた。ここで、定理の仮定より

$$S(\lambda_{i+1})C(\mu_{i+1})/S(\mu_{i+1})=-S(\mu_i)C(\lambda_i)C(\mu_{i+2})/S(\lambda_i)$$

$$-S(\mu_{i+2})C(\mu_i)C(\lambda_{i+2})/S(\lambda_{i+2})$$

$$-S(\lambda_{i-2})S(\mu_i)S(\mu_{i+2})C(\mu_{i-2})/(S(\mu_{i-2})S(\lambda_i)S(\lambda_{i+2}))$$

であるから

$$a_1 = \lambda_i \frac{S(\mu_{i+1})}{S(\lambda_{i+1})} \{ \frac{C(\mu_i)S(\mu_{i+2})\{C(\lambda_{i+2})C(\lambda_i)-C(\lambda_{i-2})\}}{S(\lambda_{i+2})S(\lambda_i)}$$

$$+S(\mu_i)C(\mu_{i+2})-\frac{S(\mu_i)}{S(\lambda_i)}C(\lambda_i)C(\mu_{i+2})-\frac{S(\mu_{i+2})}{S(\lambda_{i+2})}C(\mu_i)C(\lambda_{i+2})$$

$$-\frac{S(\lambda_{i-2})S(\mu_i)S(\mu_{i+2})}{S(\mu_{i-2})S(\lambda_i)S(\lambda_{i+2})}C(\mu_{i-2})\}$$

$$=\frac{S(\mu_{i+1})S(\mu_{i+2})\lambda_i}{S(\lambda_{i+1})S(\lambda_{i+2})S(\lambda_i)}\{C(\mu_i)C(\lambda_{i+2})C(\lambda_i)-C(\mu_i)C(\lambda_{i-2})$$

$$+S(\mu_i)\frac{C(\mu_{i+2})S(\lambda_{i+2})S(\lambda_i)}{S(\mu_{i+2})}-\frac{S(\mu_i)C(\lambda_i)C(\mu_{i+2})S(\lambda_{i+2})}{S(\mu_{i+2})}$$

$-C(\mu_i)C(\lambda_{i+2})S(\lambda_i)-S(\lambda_{i-2})S(\mu_i)C(\mu_{i-2})/S(\mu_{i-2})\}$

$$= \frac{S(\mu_{i+1})S(\mu_{i+2})\lambda_i}{S(\lambda_{i+1})S(\lambda_{i+2})S(\lambda_i)} \{C(\mu_i)C(\lambda_{i+2})\{C(\lambda_i)-S(\lambda_i)\}$$

$$+ \frac{S(\mu_i)C(\mu_{i+2})S(\lambda_{i+2})(S(\lambda_i)-C(\lambda_i))}{S(\mu_{i+2})} -C(\mu_i)C(\lambda_{i-2})$$

$-S(\lambda_{i-2})S(\mu_i)C(\mu_{i-2})/S(\mu_{i-2})$

ここで　$C(\lambda_i)-S(\lambda_i)=1/\lambda_i$　であるから

$$a_1 = \frac{S(\mu_{i+1})S(\mu_{i+2})\lambda_i}{S(\lambda_{i+1})S(\lambda_{i+2})S(\lambda_i)} \{C(\mu_i)C(\lambda_{i+2})/\lambda_i$$

$$- \frac{S(\mu_i)C(\mu_{i+2})S(\lambda_{i+2})}{S(\mu_{i+2})\lambda_i} - C(\mu_i)C(\lambda_{i-2})$$

$$- \frac{S(\lambda_{i-2})S(\mu_i)C(\mu_{i-2})}{S(\mu_{i-2})} \}$$

$$= \frac{S(\mu_{i+1})S(\mu_{i+2})}{S(\lambda_{i+1})S(\lambda_{i+2})S(\lambda_i)} \{C(\mu_i)C(\lambda_{i+2}) -$$

$$- \frac{S(\mu_i)C(\mu_{i+2})S(\lambda_{i+2})}{S(\mu_{i+2})} - \lambda_i C(\mu_i)C(\lambda_{i-2})$$

$$- \frac{\lambda_i S(\lambda_{i-2})S(\mu_i)C(\mu_{i-2})}{S(\mu_{i-2})} \}$$

となり

$$a_3 = \frac{1}{S(\lambda_i)} \{C(\mu_i)C(\lambda_{i+2}) - S(\mu_i)C(\mu_{i+2})S(\lambda_{i+2})/S(\mu_{i+2})$$

$$-\lambda_i C(\mu_i)C(\lambda_{i-2}) - \lambda_i S(\lambda_{i-2})S(\mu_i)C(\mu_{i-2})/S(\mu_{i-2})\}$$

と置けば

$$a_1 = \frac{S(\mu_{i+1})S(\mu_{i+2})}{S(\lambda_{i+1})S(\lambda_{i+2})} a_3 \qquad (9)$$

となる。また

$$a_2 = v_{i1} v_{i4} \mu_{i-1} - v_{i2} v_{i3} / \mu_{i-1}$$

$$= (v_{i1} v_{i4} + v_{i2} v_{i3}) S(\mu_{i-1}) + (v_{i1} v_{i4} - v_{i2} v_{i3}) C(\mu_{i-1})$$

$$= (v_{i1} v_{i4} + v_{i2} v_{i3}) S(\mu_{i-1}) + C(\mu_{i-1})$$

$$= S(\mu_{i-1}) \{ (v_{i1} v_{i4} + v_{i2} v_{i3}) + C(\mu_{i-1}) / S(\mu_{i-1}) \}$$

$$= S(\mu_{i-1}) \{ \frac{C(\mu_i) S(\mu_{i-2}) \{ C(\lambda_{i+2}) - C(\lambda_i) C(\lambda_{i-2}) \}}{S(\lambda_{i-1}) S(\lambda_i) S(\lambda_{i-2})}$$

$$- \frac{S(\mu_i) C(\mu_{i-1})}{S(\lambda_{i-1})} + \frac{C(\mu_{i-1})}{S(\mu_{i-1})} \}$$

$$= \frac{S(\mu_{i-1})}{S(\lambda_{i-1})} \{ \frac{C(\mu_i) S(\mu_{i-2}) \{ C(\lambda_{i+2}) - C(\lambda_i) C(\lambda_{i-2}) \}}{S(\lambda_i) S(\lambda_{i-2})}$$

$$- S(\mu_i) C(\mu_{i-2}) + C(\mu_{i-1}) S(\lambda_{i-1}) / S(\mu_{i-1}) \}$$

ここで、定理の仮定より

$$\frac{C(\mu_{i-1}) S(\lambda_{i-1})}{S(\mu_{i-1})} = - \frac{S(\mu_{i-2})}{S(\lambda_{i-2})} C(\lambda_{i-2}) C(\mu_i)$$

$$- \frac{S(\mu_i)}{S(\lambda_i)} C(\mu_{i-2})C(\lambda_i) - \frac{S(\lambda_{i+2})S(\mu_{i-2})S(\mu_i)}{S(\mu_{i+2})S(\lambda_{i-2})S(\lambda_i)} C(\mu_{i+2})$$

であるから

$$a_2 = \frac{S(\mu_{i-1})}{S(\lambda_{i-1})} \{ \frac{C(\mu_i)S(\mu_{i-2})\{C(\lambda_{i+2})-C(\lambda_i)C(\lambda_{i-2})\}}{S(\lambda_i)S(\lambda_{i-2})}$$

$$-S(\mu_i)C(\mu_{i-2}) - \frac{S(\mu_{i-2})}{S(\lambda_{i-2})} C(\lambda_{i-2})C(\mu_i)$$

$$- \frac{S(\mu_i)}{S(\lambda_i)} C(\mu_{i-2})C(\lambda_i) - \frac{S(\lambda_{i+2})S(\mu_{i-2})S(\mu_i)}{S(\mu_{i+2})S(\lambda_{i-2})S(\lambda_i)} C(\mu_{i+2})\}$$

$$= \frac{S(\mu_{i-1})S(\mu_{i-2})}{S(\lambda_{i-1})S(\lambda_{i-2})S(\lambda_i)} \{C(\mu_i)C(\lambda_{i+2}) - C(\mu_i)C(\lambda_i)C(\lambda_{i-2})$$

$$- \frac{S(\mu_i)C(\mu_{i-2})S(\lambda_{i-2})S(\lambda_i)}{S(\mu_{i-2})} - S(\lambda_i)C(\lambda_{i-2})C(\mu_i)$$

$$-\frac{S(\mu_i)S(\lambda_{i-2})}{S(\mu_{i-2})}C(\mu_{i-2})C(\lambda_i)-\frac{S(\lambda_{i+2})S(\mu_i)C(\mu_{i+2})}{S(\mu_{i+2})}$$

$$=\frac{S(\mu_{i-1})S(\mu_{i-2})}{S(\lambda_{i-1})S(\lambda_{i-2})S(\lambda_i)}\{C(\mu_i)C(\lambda_{i+2})-C(\mu_i)C(\lambda_{i-2})(S(\lambda_i)$$

$$+C(\lambda_i))-\frac{C(\mu_{i-2})S(\lambda_{i-2})S(\mu_i)\{S(\lambda_i)+C(\lambda_i)\}}{S(\mu_{i-2})}$$

$$-S(\lambda_i)C(\lambda_{i-2})C(\mu_i)-\frac{S(\lambda_{i+2})S(\mu_i)C(\mu_{i+2})}{S(\mu_{i+2})}\}$$

ここで $S(\lambda_i)+C(\lambda_i)=\lambda_i$ であるから

$$a_2=\frac{S(\mu_{i-1})S(\mu_{i-2})}{S(\lambda_{i-1})S(\lambda_{i-2})S(\lambda_i)}\{C(\mu_i)C(\lambda_{i+2})-S(\mu_i)C(\mu_{i+2})$$

$$S(\lambda_{i+2})/S(\mu_{i+2})-\lambda_i C(\mu_i)C(\lambda_{i-2})-\lambda_i S(\lambda_{i-2})S(\mu_i)C(\mu_{i-2})/S(\mu_{i-2})\}$$

$$=\frac{S(\mu_{i-1})S(\mu_{i-2})}{S(\lambda_{i-1})S(\lambda_{i-2})}a_3 \qquad (10)$$

となる。同様な計算により

$$d_3 = -\frac{C(\mu_i)C(\lambda_{i+2})S(\mu_{i+2})+S(\mu_i)S(\lambda_{i+2})C(\mu_{i+2})}{S(\lambda_i)S(\mu_{i+2})}$$

$$+\frac{C(\mu_i)C(\lambda_{i-2})S(\mu_{i-2})-S(\mu_i)S(\lambda_{i-2})C(\mu_{i-2})}{\lambda_i S(\lambda_i)S(\mu_{i-2})}$$

とおけば

$$d_1 = \frac{S(\mu_{i+1})S(\mu_{i+2})}{S(\lambda_{i+1})S(\lambda_{i+2})} d_3, \quad d_2 = \frac{S(\mu_{i-1})S(\mu_{i-2})}{S(\lambda_{i-1})S(\lambda_{i-2})} d_3 \quad (11)$$

となる。ゆえに、(7), (8), (9), (10), (11) により

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \frac{S(\mu_{i+1})S(\mu_{i+2})}{S(\lambda_{i+1})S(\lambda_{i+2})} \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix},$$

$$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \frac{S(\mu_{i-1})S(\mu_{i-2})}{S(\lambda_{i-1})S(\lambda_{i-2})} \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}$$

となり

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \approx \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

が示された。　　　　　　　　　　　　　　　　　　　　（証明終わり）