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<td>TAHARA, Hidetoshi</td>
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Kyoto University
LOCAL SOLVABILITY OF SECOND ORDER FUCHSIAN TYPE EQUATIONS (Collaboration with C. PARENTI)

上智大 工科 田原秀敏
(Hidetoshi TAHARA)

§1. DISCUSSION IN A EXAMPLE.

Let \((t, x) \in \mathbb{R}_t \times \mathbb{R}_x^n\) and let us consider

\[
P = t^{2} \Delta_x + a(t, x) \partial_t + \sum_{i=1}^{n} b_i(t, x) \partial_i + c(t, x)
\]  \hspace{1cm} (1.1)

near the origin, where \(\Delta_x\) is the Laplacian, \(\partial_t = \partial/\partial t\) and \(\partial_i = \partial/\partial x_i\) \((1 \leq i \leq n)\). Note that \(P\) is hyperbolic on \((t > 0)\) and elliptic on \((t < 0)\), that is, \(P\) is so-called a mixed type operator.

To illustrate our argument, let us treat here the above operator \(P\) and show the local solvability of \(P u = f\) in hyperfunctions \& distributions.

1-1. Solvability in hyperfunctions. Let \(P\) be the operator in (1.1) and assume that \(a(t, x), b_i(t, x)\) and \(c(t, x)\) are analytic functions near the origin. Let \(\mathcal{B}\) be the stalk of sheaf of hyperfunctions at the origin. Then, we have:

Theorem 1. If \(1 - a(0, x) \notin \{1, 2, \ldots\}\), the map \(P : \mathcal{B} \to \mathcal{B}\) is surjective.

Sketch of proof. To prove this, it is sufficient to show the following:
(1) $P : A \rightarrow A$ is surjective.

(ii) $P : B/Q \rightarrow B/A$ is surjective.

(i) is clear from the Cauchy-Kowalewski type theorem for Fuchsian type equations. (ii) follows from the following two facts:

(ii-1) when $p(\pm)=(0,0; \tau=\pm1, \xi=0)$, $P : \mathcal{C}_p(\pm) \rightarrow \mathcal{C}_p(\pm)$ is surjective,

(ii-2) when $q=(0,0; \tau,\xi\neq0)$, $P : \mathcal{C}_q \rightarrow \mathcal{C}_q$ is bijective, where $\mathcal{C}_p$ denotes the stalk of sheaf of microfunctions at $p$.

Q.E.D.

1-2. Solvability in distributions. Let $P$ be the operator in (1.1) and assume that $a(t,x)$, $b_1(t,x)$ and $c(t,x)$ are $C^\infty$ functions near the origin. Let $\mathcal{D}'$ be the stalk of sheaf of distributions at the origin. Then, we have:

**Theorem 2.** If $1-a(0,x)\notin Z$, the map $P : \mathcal{D}' \rightarrow \mathcal{D}'$ is surjective.

**Sketch of proof.** Put

$$\mathcal{D}'|_{t>0} = \{ u|_{t>0} : u \in \mathcal{D}' \},$$

$$\mathcal{D}'|_{t<0} = \{ u|_{t<0} : u \in \mathcal{D}' \}.$$

Then, Theorem 2 follows from the following three facts:

(i) $P : \mathcal{D}'|_{t>0} \rightarrow \mathcal{D}'|_{t>0}$ is surjective,

(ii) $P : \mathcal{D}'|_{t<0} \rightarrow \mathcal{D}'|_{t<0}$ is surjective,

(iii) for any $\phi(x)\in\mathcal{D}'$, there exists a unique solution $u(t,x)\in C^\infty([0,T], \mathcal{D}')$ such that

$Pu=0$ on $t>0$, $u|_{t=0} = \phi(x)$.
In fact, we can get Theorem 2 as follows. Let \( f \in \mathcal{D} \). Then, by (i) and (ii) we can choose \( v \in \mathcal{D} \) such that \( P v = f + \delta(t) \psi(x) \). By (iii) we solve

\[
\begin{cases}
Pw = 0 \quad \text{on } t > 0, \\
w|_{t=0} = (a(0,x) - 1)^{-1} \psi(x).
\end{cases}
\]

Then, we have \( P(Y(t)w) = \delta(t) \psi(x) \). Thus, by putting \( u = v - Y(t)w \) we get a solution \( u \in \mathcal{D} \) of \( Pu = f \). Q.E.D.

Remark. By the same argument as in the proof of Theorem 1, we can easily get that \( P : \mathcal{D}^C \to \mathcal{D}^C \) is surjective. But, unfortunately, we do not know whether \( P : C^\infty \to C^\infty \) is surjective or not. This is the reason why we proved Theorem 2 in a different way from the proof of Theorem 1.

§2. FURTHER RESULTS.

Let us treat here somewhat more general Fuchsian type operators. Let \( P_1, P_2, P_3 \) be of the form

\[
P_1 = t^2 a_t - t^k A(t,x,\partial_x) + a(t,x) \partial_t \\
+ t^{h_n} \sum_{i=1}^{n} b_i(t,x) \partial_i + c(t,x),
\]

\[
P_2 = t^2 a_t - t^k P A(t,x,\partial_x) + a(t,x) \partial_t \\
+ t^{q_n} \sum_{i=1}^{n} b_i(t,x) \partial_i + c(t,x),
\]

\[
P_3 = t^2 a_t + t^k A(t,x,\partial_x) + a(t,x) \partial_t \\
+ t^{q_n} \sum_{i=1}^{n} b_i(t,x) \partial_i + c(t,x),
\]

where \( a_t = a/\partial t, \quad a_i = a/\partial x_i \quad (1 \leq i \leq n) \), \( k, h, p, q \in \mathbb{Z}^+ (= \{0, 1, 2, \ldots\}) \),

\[
A(t,x,\partial_x) = \sum_{i,j=1}^{n} a_{ij}(t,x) \partial_i \partial_j
\]

is a real elliptic differential operator such that

\[
A(t,x,\xi/|\xi|) > 0 \quad \text{for } \forall (t,x), \quad \forall \xi \in \mathbb{R}^n \setminus \{0\},
\]

- 3 -
and $a_{ij}(t,x)$, $a(t,x)$, $b_1(t,x)$ and $c(t,x)$ are $C^\infty$ functions near the origin. Let $\rho(x) = 1 - a(0,x)$ and let $\rho_1(x), \rho_2(x)$ be the roots of $\rho(\rho-1)+a(0,x)\rho+c(0,x)=0$. Note that $\rho(x)$ is the non-trivial characteristic exponent of $P_1$ and that $\rho_1(x), \rho_2(x)$ are the characteristic exponents of $P_2$ and $P_3$. Then, we have:

**Theorem 3.**

1. In case $P_1$, if $k \geq 0$, $h \geq (k-1)/2$ and $\rho(0) \notin \mathbb{Z}$ hold, the map $P_1 : \mathcal{D}' \to \mathcal{D}'$ is surjective.

2. In case $P_2$, if $p \geq 1$, $q \geq p/2$ and $\rho_1(0), \rho_2(0) \notin \{-1, -2, \ldots\}$, the map $P_2 : \mathcal{D} \to \mathcal{D}'$ is surjective.

3. In case $P_3$, if $p \geq 1$, $q \geq p/2$ and $\rho_1(0), \rho_2(0) \notin \{-1, -2, \ldots\}$, the map $P_3 : \mathcal{D}' \to \mathcal{D}'$ is surjective.

The proof of this result is quite similar to that of Theorem 2.

**REFERENCES**


**要約**

本稿では、2階のフックス型線形偏微分方程式の hyper-function ($\mathcal{E}$) やは distribution ($\mathcal{D}'$) での局所可解性が論じられている。

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1. \((t, x) \in \mathbb{R}_t \times \mathbb{R}^n_{x}\)（原点の近傍）とし、例えば

\[ P = t \partial_t^2 - \Delta_x + a(t, x) \partial_x + \sum_{i=1}^n b_i(t, x) \partial_{x_i} + c(t, x) \]

という作用素を考えている。\(P\)は\(|t| \geq \alpha\)では双曲型であり\(|t| < \alpha\)では楕円型となっていて、いわゆる混合型作用素と呼ばれているものになる。次が成り立つ。

(i) \(P\)の係数が解析的とする。この時、原点の近傍で

\[ 1 - a(t, x) \notin \{ 1, 2, \ldots \} \Rightarrow P: \mathcal{B} \rightarrow \mathcal{B} \text{は全射。} \]

(ii) \(P\)の係数が\(C^\infty\)クラスとする。この時、原点の近傍で

\[ 1 - a(t, x) \notin \{ 0, \pm 1, \pm 2, \ldots \} \Rightarrow P: \mathcal{B} \rightarrow \mathcal{B} \text{は全射。} \]

2. (ii) の証明は \(L^2\) 評価をベースとした議論による。

同様の議論によって次の様に作用素の \(\varphi'\)での局所可解性を扱うことも出来る。

\[ P_1 = t \partial_t^2 - t^k A(t, x, \partial_x) \]

\[ + a(t, x) \partial_x + t^h \sum_{i=1}^n b_i(t, x) \partial_{x_i} + c(t, x), \]

\[ P_2 = t^2 \partial_t^2 - t^p A(t, x, \partial_x) \]

\[ + a(t, x) t \partial_x + t^q \sum_{i=1}^n b_i(t, x) \partial_{x_i} + c(t, x), \]

\[ P_3 = t^2 \partial_t^2 + t^p A(t, x, \partial_x) \]

\[ + a(t, x) t \partial_x + t^q \sum_{i=1}^n b_i(t, x) \partial_{x_i} + c(t, x). \]

但し、\(A(t, x, \partial_x)\)は 2階の実係数楕円型作用素、\(k, h, p, q \in \mathbb{Z}_f (=\{0, 1, 2, \ldots \})\) で

\[ h \geq (k - 1)/2, \quad p \geq 1, \quad q \geq p/2 \]

とする。