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LOCAL SOLVABILITY OF SECOND ORDER FUCHSIAN TYPE EQUATIONS (Collaboration with C.PARENTI)

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§1. DISCUSSION IN A EXAMPLE.

Let $(t,x)\in \mathbf{R}_t\times \mathbf{R}^n_x$ and let us consider

$$P=t\partial_t^2-\Delta_x+a(t,x)\partial_t+\Sigma_{i=1}^n b_i(t,x)\partial_i+c(t,x) \qquad (1.1)$$

near the origin, where Δ_x is the Laplacian, $\partial_t=\partial/\partial t$ and $\partial_i=\partial/\partial x_i$ $(1\leq i\leq n)$. Note that P is hyperbolic on $\{t>0\}$ and elliptic on $\{t<0\}$, that is, P is so-called a mixed type operator.

To illustrate our argument, let us treat here the above operator P and show the local solvability of Pu=f in hyperfunctions & distributions.

1-1. Solvability in hyperfunctions. Let P be the operator in (1.1) and assume that $a(t,x)$, $b_i(t,x)$ and $c(t,x)$ are analytic functions near the origin. Let $\mathcal{B}$ be the stalk of sheaf of hyperfunctions at the origin. Then, we have:

Theorem 1. If $1-a(0,x)\notin\{1,2,\ldots\}$, the map $P:\mathcal{B}\longrightarrow\mathcal{B}$ is surjective.

Sketch of proof. To prove this, it is sufficient to show the following:

(i) $P : \mathcal{A} \longrightarrow \mathcal{A}$ is surjective,

(ii) $P : \mathcal{B}/\mathcal{A} \longrightarrow \mathcal{B}/\mathcal{A}$ is surjective.

(i) is clear from the Cauchy-Kowalewski type theorem for Fuchsian type equations. (ii) follows from the following two facts:

(ii-1) when $p(\pm)=(0,0;\tau=\pm 1,\xi=0)$, $P : \mathcal{C}_{p(\pm)} \longrightarrow \mathcal{C}_{p(\pm)}$ is surjective,

(ii-2) when $q=(0,0;\tau,\xi\neq 0)$, $P : \mathcal{C}_q \longrightarrow \mathcal{C}_q$ is bijective,

where $\mathcal{C}_p$ denotes the stalk of sheaf of microfunctions at p.

Q.E.D.

<u>1-2</u>. Solvability in distributions. Let P be the operator in (1.1) and assume that $a(t,x)$, $b_i(t,x)$ and $c(t,x)$ are C^∞ functions near the origin. Let $\mathcal{D}'$ be the stalk of sheaf of distributions at the origin. Then, we have:

<u>Theorem 2</u>. If $1-a(0,x)\notin \mathbf{Z}$, the map $P : \mathcal{D}' \longrightarrow \mathcal{D}'$ is surjective.

<u>Sketch of proof</u>. Put

$$\mathcal{D}'|_{t>0} = \{ u|_{t>0} \ ; \ u\in\mathcal{D}' \},$$
$$\mathcal{D}'|_{t<0} = \{ u|_{t<0} \ ; \ u\in\mathcal{D}' \}.$$

Then, Theorem 2 follows from the following three facts:

(i) $P : \mathcal{D}'|_{t>0} \longrightarrow \mathcal{D}'|_{t>0}$ is surjective,

(ii) $P : \mathcal{D}'|_{t<0} \longrightarrow \mathcal{D}'|_{t<0}$ is surjective,

(iii) for any $\varphi(x)\in\mathcal{D}'$, there exists a unique solution $u(t,x)\in C^\infty([0,T],\mathcal{D}')$ such that

$$Pu=0 \quad \text{on } t>0, \quad u|_{t=0}=\varphi(x).$$

In fact, we can get Theorem 2 as follows. Let $f\in\mathcal{D}'$. Then, by (i) and (ii) we can choose $v\in\mathcal{D}'$ such that $Pv=f+\delta(t)\otimes\psi(x)$. By (iii) we solve

$$\begin{cases} Pw=0 & \text{on } t>0, \\ w|_{t=0}=(a(0,x)-1)^{-1}\psi(x). \end{cases}$$

Then, we have $P(Y(t)w)=\delta(t)\otimes\psi(x)$. Thus, by putting $u=v-Y(t)w$ we get a solution $u\in\mathcal{D}'$ of $Pu=f$. Q.E.D.

<u>Remark</u>. By the same argument as in the proof of Theorem 1, we can easily get that $P : \mathcal{D}'/C^\infty \longrightarrow \mathcal{D}'/C^\infty$ is surjective. But, unfortunately, we do not know whether $P : C^\infty \longrightarrow C^\infty$ is surjective or not. This is the reason why we proved Theorem 2 in a different way from the proof of Theorem 1.

§2. <u>FURTHER RESULTS</u>.

Let us treat here somewhat more general Fuchsian type operators. Let P_1, P_2 and P_3 be of the form

$$\begin{aligned} P_1 &= t\partial_t^2-t^kA(t,x,\partial_x)+a(t,x)\partial_t \\ &\qquad +t^h\Sigma_{i=1}^n b_i(t,x)\partial_i+c(t,x), \\ P_2 &= t^2\partial_t^2-t^pA(t,x,\partial_x)+a(t,x)t\partial_t \\ &\qquad +t^q\Sigma_{i=1}^n b_i(t,x)\partial_i+c(t,x), \\ P_3 &= t^2\partial_t^2+t^pA(t,x,\partial_x)+a(t,x)t\partial_t \\ &\qquad +t^q\Sigma_{i=1}^n b_i(t,x)\partial_i+c(t,x), \end{aligned}$$

where $\partial_t=\partial/\partial t$, $\partial_i=\partial/\partial x_i$ $(1\leq i\leq n)$, $k,h,p,q\in\mathbf{Z}_+(=\{0,1,2,\ldots\})$,

$$A(t,x,\partial_x) = \Sigma_{i,j=1}^n a_{ij}(t,x)\partial_i\partial_j$$

is a real elliptic differential operator such that

$$A(t,x,\xi/|\xi|) > 0 \quad \text{for } \forall(t,x),\ \forall\xi\in\mathbf{R}^n\setminus\{0\},$$

and $a_{ij}(t,x)$, $a(t,x)$, $b_i(t,x)$ and $c(t,x)$ are C^∞ functions near the origin. Let $\rho(x)=1-a(0,x)$ and let $\rho_1(x),\rho_2(x)$ be the roots of $\rho(\rho-1)+a(0,x)\rho+c(0,x)=0$. Note that $\rho(x)$ is the non-trivial characteristic exponent of P_1 and that $\rho_1(x),\rho_2(x)$ are the characteristic exponents of $P_2$and P_3. Then, we have:

<u>Theorem 3</u>. (1) In case P_1, if $k\geq 0$, $h\geq(k-1)/2$ and $\rho(0)\notin\mathbb{Z}$ hold, the map $P_1 : \mathcal{D}'\longrightarrow\mathcal{D}'$ is surjective.

(2) In case P_2, if $p\geq 1$, $q\geq p/2$ and $\rho_1(0),\rho_2(0)\notin\{-1,-2,\ldots\}$, the map $P_2 : \mathcal{D}'\longrightarrow\mathcal{D}'$ is surjective.

(3) In case P_3, if $p\geq 1$, $q\geq p/2$ and $\rho_1(0),\rho_2(0)\notin\{-1,-2,\ldots\}$, the map $P_3 : \mathcal{D}'\longrightarrow\mathcal{D}'$ is surjective.

The proof of this result is quite similar to that of Theorem 2.

REFERENCES

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要 約

本稿では、2階のフックス型線形偏微分方程式の hyper-function ($\mathcal{B}$) 或いは distribution ($\mathcal{D}'$) での局所可解性が論じられている。

① $(t,x)\in\mathbb{R}_t\times\mathbb{R}^n_x$（原点の近傍）とし、例えば

$$P=t\partial_t^2-\Delta_x+a(t,x)\partial_t+\sum_{i=1}^n b_i(t,x)\partial_{x_i}+c(t,x)$$

という作用素を考えてみる。　Pは$\{t>0\}$では双曲型であり$\{t<0\}$では楕円型となっていて、いわゆる混合型作用素と呼ばれているものになる。　次が成り立つ。

(i)　Pの係数が解析的とする。この時、原点の近傍で

$1-a(0,x)\notin\{1,2,\cdots\}\implies P:\mathcal{B}\to\mathcal{B}$ は全射。

(ii)　Pの係数がC^∞クラスとする。この時、原点の近傍で

$1-a(0,x)\notin\{0,\pm1,\pm2,\cdots\}\implies P:\mathcal{D}'\to\mathcal{D}'$ は全射。

②　(ii)の証明はL^2-評価をベースとした議論による。同様の議論によって　次の様な作用素の$\mathcal{D}'$での局所可解性を扱かうことも出来る。

$$P_1=t\partial_t^2-t^kA(t,x,\partial_x)+a(t,x)\partial_t+t^h\sum_{i=1}^n b_i(t,x)\partial_{x_i}+c(t,x),$$

$$P_2=t^2\partial_t^2-t^pA(t,x,\partial_x)+a(t,x)t\partial_t+t^q\sum_{i=1}^n b_i(t,x)\partial_{x_i}+c(t,x),$$

$$P_3=t^2\partial_t^2+t^pA(t,x,\partial_x)+a(t,x)t\partial_t+t^q\sum_{i=1}^n b_i(t,x)\partial_{x_i}+c(t,x).$$

但し、$A(t,x,\partial_x)$は2階の実係数楕円型作用素、$k,h,p,q\in\mathbb{Z}_+(=\{0,1,2,\cdots\})$で　$h\geq(k-1)/2$, $p\geq1$, $q\geq p/2$　とする。