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Kähler-Einstein Metrics on Algebraic Varieties

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This note is more or less a survey on the existence and the degeneration of degenerate Kähler-Einstein Metrics on algebraic varieties. Degenerate Kähler-Einstein metrics naturally arise from algebraic geometry. Analitically such metrics come from degenerate Monge-Ampère equations. Here are two examples:

<u>Example 1.</u> Monge-Ampere equation with degenerate right hand side (See [7]). If one wants to construct a Ricci-flat Kähler metric on the complement of a multi-canonical divisor on an algebraic manifold with ample canonical bundle, one encounters the following Monge-Ampère equation with degenerate right hand side:

$$\text{(E 1)} \qquad \det(g_{i\bar{j}} + u_{i\bar{j}}) = \prod \|\sigma_i\|^{2a_i} e^f \det(g_{i\bar{j}}) \qquad (\mathbb{Q} \ni a_i > 0)$$

In general (with exceptions of Ricci-flat Kähler orbifold metrics), the property of such Ricci-flat metrics is difficult to study although these generalize the Calabi-Yau metrics on K3 surfaces.

<u>Example 2.</u> If one wants to construct a Kähler-Einstein metric on an algebraic manifold X with nef and big canonical bundle, then one must consider the following Monge-Ampère equation with degenerate back ground metric:

(E 2) $$\left(\omega+\sqrt{-1}\,\partial\bar{\partial} u\right)^{n} = e^{u}\,\Omega$$

where $-\mathrm{Ric}\,\Omega = \omega = m^{-1}\,\Phi_{|mK_X|}^{*}$(the Fubini-Study form on the projective space of $H^{0}(X, \mathcal{O}_X(mK_X))$,

$\Phi_{|mK_X|}$ is a m-th pluri-canonical map defined by the complete linear system $|mK_X|$. Note that Ω is a smooth volume form on X uniquely determined up to constant multiple. See [5].

Example 3. If one wants to construct a Kähler-Einstein metric on a complex projective algebraic manifold of general type X, then one have to assume the finite generation of the canonical ring R(X). Let $\mu : X'' \longrightarrow \mathrm{Proj}(R(X))$ be a resolution. Write X' for Proj(R(X)). Then $K_{\mathrm{Proj}(R(X))}$ is a Q-Cartier divisor and so the pull back $\mu^{*}K_{\mathrm{Proj}(R(X))}$ is well-defined:

$$\mu^{*}K_{X'} = K_{X''} - \sum_i a_i E_i$$

where $\sum_i E_i$ is the exceptional divisor for μ and all a_i's are nonnegative rational numbers. Let σ_i be the holomorphic section of $\mathcal{O}_X(E_i)$ with E_i its zero divisor. Then there is a unique smmth volume form(up to constant multiple) Ω on X" such that $\mathrm{Ric}\left(\prod\|\sigma_i\|^{2a_i}\Omega\right) = \mu^{*}$(the Fubini-Study form on X') = ω . The Monge-Ampère equation fulfilled by a Kähler-Einstein metric is the following:

$$\left(\omega+\sqrt{-1}\,\partial\bar{\partial} u\right)^{n} = e^{u}\prod\|\sigma_i\|^{2a_i}\,\Omega$$

In the rest of this note, I explane some ideas used in the study of the Monge-Ampère equations with degenerate background metric which appear in Examples 2 and 3.

1. Tsuji's Maximum Principle (See [5]).
This gives a local a priori C^0- estimate for the solution and is based on the Kawamata's localization lemma.

Kawamata's Localization Lemma (See [5]). Let X be a projective algebraic manifold and D a nef and big divisor on X. Then there exists an integral effective divisor E = $\sum e_i E_i$ such that D - εE is an ample Q-divisor for sufficiently small positive rational numbers

Let me explane this when X is a projective manifold with nef and big canonical bundle although there are other interesting applications. I use the same symbols as in Example 2. Choose $\sigma_i \in H^0(X, \mathcal{O}_X(E_i))$ such that $(\sigma_i = 0) = E_i$. There exists a Hermitian metric for $\mathcal{O}_X(E_i)$ such that

$$\omega + \sqrt{-1} \sum_i \partial\bar\partial \log \|\sigma_i\|^{2a_i\varepsilon} > 0$$

for all sufficiently small positive numbers ε . Rewrite (E2) in the following way:

$$\Big(\big(\underbrace{\omega + \sqrt{-1}\sum_i \partial\bar\partial \log\|\sigma_i\|^{2a_i\varepsilon}}_{\omega_\varepsilon}\big) + \sqrt{-1}\,\partial\bar\partial\big(\underbrace{u - \sum_i \log\|\sigma_i\|^{2a_i\varepsilon}}_{u_\varepsilon}\big)\Big)^n = e^u\Omega$$

Apply usual minimum principle to u_ε! Then we have

$\omega_\varepsilon^n(x_0) \leqq e^{u(x_0)}\Omega(x_0)$ if u_ε takes minimum at $x_0 \in X$.

$$\therefore\ u(x) \geqq \log \frac{\omega_\varepsilon^n}{\Omega}(x_0) + \varepsilon \sum_i \log\|\sigma_i\|^{2a_i}(x) \quad (0 < {}^\forall\varepsilon \ll 1).$$

2. Methods of Curvature Concentration

For simplicity we assume the same situation as above. Let θ be a Kähler from on X and consider

$$(\omega(\varepsilon) + \sqrt{-1}\,\partial\bar{\partial} u(\varepsilon))^n = e^{u(\varepsilon)}\Omega$$

where $\omega(\varepsilon) = \omega + \varepsilon\theta$. Then

$$\begin{aligned} \mathrm{Ric}\,(\underbrace{\omega(\varepsilon) + \sqrt{-1}\,\partial\bar{\partial} u(\varepsilon)}_{\widetilde{\omega(\varepsilon)}}) &= -\sqrt{-1}\,\partial\bar{\partial} u(\varepsilon) - \omega \\ &= -\widetilde{\omega(\varepsilon)} + \varepsilon\theta \end{aligned}$$

If ε is very small then curvature of $\omega(\varepsilon)$ and $\widetilde{\omega(\varepsilon)}$ may concentrates along the exceptional set for the m-pluri canonical map for sufficiently large m. The idea is to take the limit of $\varepsilon \longrightarrow 0$. For applications see [1],[2] and [4].

Combining these ideas, we obtain the following generalzation of Tsuji's Theorem (Tsuji [5] solved the equation (E2)):

<u>Theorem 1.</u> (K.Sugiyama[4], S.Bando-R.Kobayashi[1]) Let X be a projective algebraic manifold of general type and suppose that the canonical ring is finitely generated. Then the regular part of Proj(R(X))=X' admits a Kähler-Einstein metric $\tilde{\omega}$ with negative Ricci curvature. Let μ :X''$\longrightarrow$ X' be a resolution. Then $\mu^*\tilde{\omega}$ defines a closed current on X'' such that

$[\mu^*\tilde{\omega}] = 2\pi c_1(\mu^*(K_{\mathrm{Proj}(R(X))}))$.

Theorem 2. ([1]) Assume that the canonical bundles is nef and big. Then the Kähler-Einstein volume element in Theorem 1 is characterized by

$$\inf \{ \omega^n \mid \mathrm{Ric}(\omega) \geqq -\omega \} \text{ (pointwise).}$$

Using the same ideas, one can study the degeneration of Ricci-negative Kähler-Einstein metrics. For instance,

Theorem 3. Let $\{X_t\}_{|t|\le 1}$ be a projective degeneration of canonical models (i.e., at worst canonical singularities and with ample canonical bundle). If the central fiber has at worst canonical singularities and ball cusp singularities and has ample canonical divisor outside ball cusp singularities, then the Kähler-Einstein metrics ω_t on the general members $\{X_t\}_{t\neq 0}$ converge to a Kähler-Einstein metric on X_0 which is complete toward ball-cusp singularities.

Example. For Persson's example of the degeneration of the Godeaux surfaces (See [3] and [6]), the Kähler-Einstein metrics on the general members converge to the complete Kähler-Einstein metric on the central logarithmic Godeaux surface.

References.

[1] S.Bando-R.Kobayashi.: to appear

[2] R.Kobayashi-A.N.Todorov.: Polarized Period Map for Generalized K3 Surfaces and the Moduli of Einstein Metrics, Tôhoku Math.J.39(1987), pp.341-363.

[3] R.Kobayashi.: Einstein-Kähler Metrics on Open Algebraic Surfaces of General Type, Tôhoku Math.J.37(1985), pp.43-77.

[4] K.Sugiyama.: preprint

[5] H.Tsuji.: Existence and Degeneration of Kähler-Einstein Metrics on Minimal Algebraic Varieties of General Type, preprint.

[6] U.Persson.: degeneration of Algebraic Surfaces, Memoirs of A.M.S.

[7] S.T.Yau.: On the Ricci Curvature of a compact Kähler Manifold and the omplex Monge-Ampère Equations,I, Comm.Pure and Appl.Math. 31 (1978) 339-411.