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What can be said about w-vectors of finite partially ordered sets ?

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Any partially ordered set (poset for short) to be considered is finite. The cardinality of a finite set X is denoted by $\#(X)$. Let N be the set of non-negative integers and Z the set of integers.

§1. w-vectors

Let P be a poset with elements $x_1, x_2, \ldots, x_p$ labeled so that if $x_i < x_j$ in P then $i < j$ in Z . Given an integer i , $0 \le i < p$, write $w_i = w_i(P)$ for the number of permutations $\pi = \begin{pmatrix} 1 & 2 & \cdots & p \\ a_1 & a_2 & \cdots & a_p \end{pmatrix}$ such that (a) if $x_{a_r} < x_{a_s}$ in P , then $r < s$ (i.e., π is a linear extension of P) and (b) $\#\{ r ; a_r > a_{r+1} \}$, the number of descents of π , is equal to i . We say that the vector $w(P) := (w_0, w_1, \ldots, w_{p-1})$ is the w-vector of P . Consult [Stanley [Sta_2, pp. 211-221] for combinatorial background of w-vectors.

§2. Notation and terminology

A <u>chain</u> is a poset in which any two elements are comparable. The <u>length</u> of a chain C is defined by $\ell(C) := \#(C) - 1$. The <u>rank</u> of a poset P , denoted by $\operatorname{rank}(P)$, is the supremum of lengths of chains contained in P . If $\alpha \le \beta$ in P , we write $\ell(\alpha,\beta)$ for the rank of the subposet $P_\alpha^\beta := \{ x \in P ; \alpha \le x \le \beta \}$ of P . A poset P is called <u>pure</u> if every maximal chain of P has the same length. We say that P satisfies the <u>$\delta^{(n)}$-chain condition</u>, $n \in N$, if (a) for any $\xi \in P$, the subposet $P_\xi :=$ $\{ y \in P ; y \ge \xi \}$ of P is pure and (b) $\operatorname{rank}(P) - \min\{ \ell(C) ; C$ is a maximal chain of $P \} = n$. Thus P satisfies the $\delta^{(0)}$-chain condition if and only if P is pure.

Give a poset P , we write $P^\wedge$ for the poset obtained by adjoining a <u>new</u> pair of elements, $0^\wedge$ and $1^\wedge$, to P such that $0^\wedge < x < 1^\wedge$ for any $x \in P$. A sequence $A = (\alpha_0,\beta_0,\alpha_1,\beta_1,\ldots,\alpha_t,\beta_t)$, which consists of elements of $P^\wedge$, is called <u>rhythmical</u> if (a) $\alpha_0 = 0^\wedge$, $\beta_t = 1^\wedge$, (b) $\alpha_i < \beta_i$ for any i , $0 \le i \le t$, (c) $\alpha_{i+1} < \beta_i$ for any i , $0 \le i < t$ and (d) $\alpha_{i+2} \nleq \beta_i$ for any i , $0 \le i \le t-2$. Let $\ell(A) := \Sigma_{0\le i\le t}\, \ell(\alpha_i,\beta_i) - \Sigma_{0\le i\le t-1}\, \ell(\alpha_{i+1},\beta_i)$. We say that P satisfies the <u>Δ-chain condition</u> if $\ell(A) \le \operatorname{rank}(P^\wedge)$ for any rhythmical sequence A of $P^\wedge$. We easily see that, for any $n \in N$, the $\delta^{(n)}$-chain condition implies the Δ-chain condition.

§3. Results.

Now, what can be said about w-vectors of posets ? In the following, let $w(P) = (w_0, w_1, \dots, w_{p-1})$ be the w-vector of a poset P with $\#(P) = p$ and $s := \max\{ i \,;\, w_i \neq 0 \}$.

THEOREM (Stanley [Sta_2, (4.5.17)]). The sequence w_0 , w_1 , ..., w_s is symmetric, i.e., $w_i = w_{s-i}$ for any i , $0 \le i \le s$, if and only if P is pure.

THEOREM (Stanley). The inequality

$$w_0 + w_1 + \dots + w_i \le w_s + w_{s-1} + \dots + w_{s-i}$$

holds for any i , $0 \le i \le [s/2]$.

THEOREM ([H_2]). Assume that P satisfies the Δ-chain condition. If i and j are non-negative integers with $i + j \le s$, then $w_i \le w_j w_{i+j}$.

THEOREM ([H_2]). Assume that P satisfies the $\delta^{(n)}$-chain condition. Then, the inequality

$$w_s + w_{s-1} + \dots + w_{s-i} \le w_0 + w_1 + \dots + w_i + \dots + w_{i+n}$$

holds for any i , $0 \le i \le [(s-n)/2]$.

Our technique [H_2], which originated in [H_1], is heavily based on commutative algebra, especially the theory of canonical modules [Sta_1] of invariant subrings of tori [Hoc].

It would, of course, be of great interest to find a characterization of w-vectors of posets.

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References

[H_1] T.Hibi, Canonical ideals of Cohen-Macaulay partially ordered sets, to appear.

[H_2] _____, Linear diophantine equations and Stanley's (P,ω)-partitions, in preparation.

[Hoc] M.Hochster, Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes, Ann. of Math. 96 (1972), 318-337.

[Sta_1] R.Stanley, Hilbert functions of graded algebras, Advances in Math. 28 (1978), 57-83.

[Sta_2] _____, "Enumerative Combinatorics, Volume I", Wadsworth, Monterey, Calif., 1986.