What can be said about w-vectors of finite partially ordered sets?

(Combinatorial Theory and Related Topics: Mutual Relation among Commutative Algebra, Algebraic Geometry, Representation Theory of Lie Algebras and Partially Ordered Sets)

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What can be said about w-vectors of finite partially ordered sets?

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Any partially ordered set (poset for short) to be considered is finite. The cardinality of a finite set $X$ is denoted by $\#(X)$. Let $N$ be the set of non-negative integers and $Z$ the set of integers.

§1. w-vectors

Let $P$ be a poset with elements $x_1, x_2, \ldots, x_p$ labeled so that if $x_i < x_j$ in $P$ then $i < j$ in $Z$. Given an integer $i$, $0 \leq i < p$, write $w_i = w_i(P)$ for the number of permutations $\pi = (a_1 a_2 \ldots a_p)$ such that (a) if $x_{a_r} < x_{a_s}$ in $P$, then $r < s$ (i.e., $\pi$ is a linear extension of $P$) and (b) $\#(r; a_r > a_{r+1})$, the number of descents of $\pi$, is equal to 1. We say that the vector $w(P) = (w_0, w_1, \ldots, w_{p-1})$ is the w-vector
of $P$. Consult [Stanley [Sta$_2$, pp. 211-221] for combinatorial background of $w$-vectors.

§2. Notation and terminology

A chain is a poset in which any two elements are comparable. The length of a chain $C$ is defined by $\ell(C) := \#(C) - 1$. The rank of a poset $P$, denoted by $\text{rank}(P)$, is the supremum of lengths of chains contained in $P$. If $\alpha \leq \beta$ in $P$, we write $\ell(\alpha, \beta)$ for the rank of the subposet $P^\beta_\alpha := \{ x \in P ; \alpha \leq x \leq \beta \}$ of $P$. A poset $P$ is called pure if every maximal chain of $P$ has the same length. We say that $P$ satisfies the $\delta(n)$-chain condition, $n \in \mathbb{N}$, if (a) for any $\xi \in P$, the subposet $P_{\xi} := \{ y \in P ; y \geq \xi \}$ of $P$ is pure and (b) $\text{rank}(P) - \min\{ \ell(C) ; C$ is a maximal chain of $P \} = n$. Thus $P$ satisfies the $\delta(0)$-chain condition if and only if $P$ is pure.

Give a poset $P$, we write $P^\wedge$ for the poset obtained by adjoining a new pair of elements, $0^\wedge$ and $1^\wedge$, to $P$ such that $0^\wedge < x < 1^\wedge$ for any $x \in P$. A sequence $A = (\alpha_0, \beta_0, \alpha_1, \beta_1, \ldots, \alpha_t, \beta_t)$, which consists of elements of $P^\wedge$, is called rhythmic if (a) $\alpha_0 = 0^\wedge$, $\beta_t = 1^\wedge$, (b) $\alpha_i < \beta_i$ for any $i$, $0 \leq i \leq t$, (c) $\alpha_{i+1} < \beta_i$ for any $i$, $0 \leq i < t$ and (d) $\alpha_{i+2} < \beta_i$ for any $i$, $0 \leq i \leq t-2$. Let $\ell(A) := \sum_{0 \leq i \leq t} \ell(\alpha_i, \beta_i) - \sum_{0 \leq i \leq t-1} \ell(\alpha_i+1, \beta_i)$. We say that $P$ satisfies the $\Delta$-chain condition if $\ell(A) \leq \text{rank}(P^\wedge)$ for any rhythmic sequence $A$ of $P^\wedge$. We easily see that, for any $n \in \mathbb{N}$, the $\delta(n)$-chain condition implies the $\Delta$-chain condition.
§3. Results.

Now, what can be said about \( w \)-vectors of posets? In the following, let \( w(P) = (w_0, w_1, \ldots, w_{p-1}) \) be the \( w \)-vector of a poset \( P \) with \( \#(P) = p \) and \( s := \max\{ i \; ; \; w_i \neq 0 \} \).

**THEOREM (Stanley [Sta\textsubscript{2}, (4.5.17)])**. The sequence \( w_0, w_1, \ldots, w_s \) is symmetric, i.e., \( w_i = w_{s-i} \) for any \( i \), \( 0 \leq i \leq s \), if and only if \( P \) is pure.

**THEOREM (Stanley)**. The inequality

\[
w_0 + w_1 + \ldots + w_i \leq w_s + w_{s-1} + \ldots + w_{s-i}
\]

holds for any \( i \), \( 0 \leq i \leq \lfloor s/2 \rfloor \).

**THEOREM ([H\textsubscript{2}])**. Assume that \( P \) satisfies the \( \Delta \)-chain condition. If \( i \) and \( j \) are non-negative integers with \( i + j \leq s \), then \( w_i \leq w_j w_{i+j} \).

**THEOREM ([H\textsubscript{2}])**. Assume that \( P \) satisfies the \( \delta^{(n)} \)-chain condition. Then, the inequality

\[
w_s + w_{s-1} + \ldots + w_{s-i} \leq w_0 + w_1 + \ldots + w_i + \ldots + w_{i+n}
\]

holds for any \( i \), \( 0 \leq i \leq \lfloor (s-n)/2 \rfloor \).
Our technique \([H_2]\), which originated in \([H_1]\), is heavily based on commutative algebra, especially the theory of canonical modules \([Sta_1]\) of invariant subrings of tori \([Hoc]\).

It would, of course, be of great interest to find a characterization of \(w\)-vectors of posets.

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References


[H_2] _____, Linear diophantine equations and Stanley's \((P, \omega)\)-partitions, in preparation.

