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Harmonic functions on Hilbert space and the Lévy Laplacian

Nobuaki Obata (Nagoya University)
尾畑伸明 (名大·理)

Introduction

In his celebrated book [14] P. Lévy introduced an infinite dimensional Laplacian for functions of infinitely many variables $\{\xi_n\}_{n=1}^{\infty}$ by the formula:

$$(1) \qquad \Delta = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{\partial^2}{\partial \xi_n^2} .$$

This is called the *Lévy Laplacian* and has been studied by many authors from various points of view (see, e.g., [1,2,4,6,7,8,11,13] and references quoted there). In this paper the Lévy Laplacian is defined as an operator acting on functions on Hilbert space (for definition, see the formula (2) below) and discussed along with infinite dimensional rotation groups.

The first topic is harmonic functions. Motivated by Lévy's notion of regular functionals and white noise analysis, we propose a notion of regularly analytic functions on Hilbert space which generalize ordinary Brownian functionals. The mean value theorem for regularly analytic functions (Theorem 2.1) then naturally implies the harmonicity of ordinary Brownian functionals (Theorem 3.2).

The second topic, that is, invariance of the Lévy Laplacian under infinite dimensional rotation groups, is discussed with great interest because the Lévy Laplacian depends upon choice and arrangement of complete orthonormal systems of Hilbert space. We determine the maximal rotation group under which the Lévy Laplacian is invariant (Theorem 4.2) and discuss its subgroups.

During the discussion on invariance of the Lévy Laplacian we find a quite interesting permutation group called the Lévy group. In Appendices we shall illustrate a close connection between the Lévy group and certain notions of additive number theory, namely, the density of natural numbers and uniformly distributed sequences.

§1. The Lévy Laplacian and the mean operator

Let H be a real separable Hilbert space with inner product <,> and norm $\|\cdot\|$. We fix a complete orthonormal system (=CONS) of H, say, $\{e_n\}_{n=1}^{\infty}$. For any C^2-function F (in the sense of Fréchet) defined in a neighborhood $\xi \in H$, put

$$(2) \qquad \Delta F(\xi) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{\infty} F''(\xi)(e_n, e_n), \quad \xi \in H,$$

if the limit exists. The operator Δ is called the *Lévy Laplacian.* Evidently the expression (2) coincides with (1) through the Fourier series expansion.

For each $n \geqslant 1$ the unit sphere $S^{n-1} \subset \mathbf{R}^n$ is regarded as a subset of H by means of the map:

$$h = (h_1, \cdots, h_n) \longmapsto \sum_{k=1}^{n} h_k e_k \in H, \qquad h \in S^{n-1}$$

Let $dS_{n-1}(h)$ be the normalized uniform measure on S^{n-1}. The *mean* of a function F over the sphere of radius $\rho \in \mathbf{R}$ with center at $\xi \in H$ is defined by

$$MF(\xi, \rho) = \lim_{n\to\infty} \int_{S^{n-1}} F(\xi+\rho h)\, dS_{n-1}(h),$$

if this limit exists.

The following result, which has been noted in somewhat different forms (e.g., [2, 14]), shows one of the most distinctive features of the Lévy Laplacian. The proof is easy and omitted.

Proposition 1.1. Let F be a C^2-function defined in a neighborhood of $\xi \in H$. If F admits the mean $MF(\xi, \rho)$ for $|\rho| < R$, then

$$\Delta F(\xi) = 2 \lim_{\rho \longrightarrow 0} \frac{MF(\xi, \rho) - F(\xi)}{\rho^2},$$

whenever the limit exists.

§2. Mean value theorem for regularly analytic functions

Assume that a function F admits the expression:

$$F(\xi) = \sum_{n=0}^{\infty} < a_n , (\xi - \xi_0)^{\otimes n} > , \quad a_n \in S^n H, \tag{3}$$

in some neighborhood of ξ_0. Then F is called *regularly analytic at* ξ_0 if the power series $\sum_{n=0}^{\infty} \|a_n\| t^n$ has a non-zero radius of convergence. The expression (3) is called the *power series expansion* of F at ξ_0. A function defined on an open set $\mathcal{O}$ of H is called *regularly analytic on* $\mathcal{O}$ if it is regularly analytic at every point of $\mathcal{O}$. The space of all regularly analytic functions on $\mathcal{O}$ will be denoted by $\mathcal{RA}(\mathcal{O})$.

Example. Consider a quadratic function

$$F(\xi) = < A\xi , \xi > , \quad \xi \in H, \ A \in \mathbf{B}(H),$$

where $\mathbf{B}(H)$ denotes the algebra of all bounded operators on H. Then F is analytic in the usual sense ([15]). However, it is regularly analytic if and only if A is of Hilbert-Schmidt type.

The following is one of the most remarkable properties of regularly analytic functions.

Theorem 2.1 (Mean value theorem). Assume that F is regularly analytic at ξ. Then there exists some $R > 0$ such that

$$MF(\xi, \rho) = F(\xi) \quad \text{whenever } |\rho| < R .$$

Here we only mention a rough idea of the proof. By means of the polar coordinate one can show that the assertion is valid for monomials of the form:

$$F(\eta) = \langle a , (\eta-\xi)^{\otimes n} \rangle , \quad a \in S^n H, \ n \geq 0.$$

For arbitrary regularly analytic functions one may carry out an approximation argument.

The next result is now immediate from Proposition 1.1 and the above theorem.

Corollary 2.2. Every regularly analytic function is harmonic, i.e. $\Delta F = 0$ on $\mathcal{O}$ whenever $F \in \mathcal{RA}(\mathcal{O})$.

§3. Harmonicity of ordinary Brownian functionals

We now start with a Gelfand triple $E \subset H \subset E^*$, where E is a nuclear space contained in H as a dense subspace and E^* denotes the topological dual space of E. The canonical bilinear form on $E \times E^*$ is also denoted by $\langle , \rangle$. The standard Gaussian measure μ on E^* is defined by the characteristic functional:

$$C(\xi) = e^{-\|\xi\|^2/2} = \int_{E^*} e^{i<x,\xi>} d\mu(x) \quad , \quad \xi \in E .$$

We put $(L^2) = L^2(E^*, \mu)$ for simplicity. Any element of (L^2) is called an *ordinary Brownian functional*.

Following [11] we introduce the transformation:

$$Sf(\xi) = \int_{E^*} f(x+\xi) \; d\mu(x) = C(\xi)\int_{E^*} f(x)e^{<x,\xi>} \; d\mu(x)$$

for $f \in (L^2)$ and $\xi \in E$. As is easily shown, Sf is continuously extended to a C^∞-function on H also denoted by Sf. Let S^nH denote the n-th symmetric power of H and put $SH = \sum_{n=0}^{\infty} S^nH$ (usual direct sum of Hilbert spaces). It is known (e.g., [5,11,20]) that the transform S is an analytical expression of the isomorphism $(L^2) \simeq SH$, namely,

Proposition 3.1. For each $f \in (L^2)$ there exists a unique element $a \in SH$ with $\|f\| = \|a\|$ such that

$$Sf(\xi) = < a \; , \; \exp \xi > ,$$

where $\exp \xi = \sum_{n=0}^{\infty} (n!)^{-1/2} \xi^{\otimes n}$. The correspondence $f \longmapsto a$ gives an isometric isomorphism from (L^2) onto SH.

The next result, which has been noted in a weaker forms (e.g., [5,8,13]), is known as harmonicity of ordinary Brownian functionals.

Theorem 3.2. The Lévy Laplacian annihilates (L^2) in the sense that $\Delta(Sf) = 0$ for every $f \in (L^2)$.

Proof. In view of Proposition 3.1 we can show that $Sf \in \mathcal{RA}(H)$ for any $f \in (L^2)$. Hence, by Corollary 2.2 we have $\Delta(Sf) = 0$. Q.E.D.

§4. Invariance of the Lévy Laplacian

In what follows every subset of **B**(H) is assumed to be furnished with the operator-norm topology. It is known that the orthogonal group O(H) is a Banach-Lie group. We denote by $\mathfrak{o}(H)$ the Lie algebra of O(H). The set

$$\mathcal{B}_0 = \left\{ A \in \mathbf{B}(H) \; ; \; \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} < Ae_n, e_n > = 0 \right\}$$

becomes a closed subspace of **B**(H). Set

$$O(H;\mathcal{B}_0) = \left\{ g \in O(H) \; ; \; g\mathcal{B}_0 g^{-1} = \mathcal{B}_0 \right\} .$$

Proposition 4.1. $O(H,\mathcal{B}_0)$ is a closed subgroup of O(H). Moreover, it is a Banach-Lie group with the Lie algebra

$$\mathfrak{o}(H;\mathcal{B}_0) = \{ X \in \mathfrak{o}(H) \; ; \; \mathrm{ad}(X)\mathcal{B}_0 \subset \mathcal{B}_0 \} .$$

Let Dom(Δ) be the space of all C^2-functions on H which admit the limit (2) at every point $\xi \in H$. We take Dom(Δ) to be the domain of the Lévy Laplacian Δ. The orthogonal group O(H) acts on functions on H by means of the map:

$$(U(g)F)(\xi) = F(g^{-1}\xi) \;, \; \xi \in H, \; g \in O(H).$$

We say that the Lévy Laplacian is invariant under a rotation $g \in O(H)$ if $U(g)\Delta = \Delta U(g)$. With these notations,

Theorem 4.2. $O(H;\mathcal{B}_0)$ is the maximal rotation group under which the the Lévy Laplacian is invariant.

Proof. (outline) Assume that F is a C^2-function on H. Then, for each $\xi \in H$, we have

$$F''(\xi)(e_n, e_n) = < A(\xi)e_n \ , \ e_n >$$

for some $A(\xi) \in \mathbf{B}(H)$. Therefore, $F \in \mathrm{Dom}(\Delta)$ if and only if $A(\xi) \in \mathscr{B}$, where $\mathscr{B}$ is the space of all bounded operators $A \in \mathbf{B}(H)$ which admit the limit

$$L(A) = \lim_{N\longrightarrow\infty} \frac{1}{N} \sum_{n=1}^{N} < Ae_n \ , \ e_n > .$$

It is easily shown that $O(H;\mathscr{B}_0)$ is the maximal rotation group which leaves the functional L invariant. On the other hand, we have

$$(U(g)F)''(e_n, e_n) = < A(g^{-1}\xi)g^{-1}e_n \ , \ g^{-1}e_n >$$

for any $g \in O(H)$. Therefore, Δ is invariant under $g \in O(H)$ if and only if $g \in O(H;\mathscr{B}_0)$. Q. E. D.

Remark. The Lévy Laplacian is invariant under any translation, i.e. $\Delta W(\eta) = W(\eta)\Delta$ for any $\eta \in H$, where $W(\eta)$ is defined by

$$(W(\eta)F)(\xi) = F(\xi-\eta).$$

§5. Subgroups of $O(H;\mathscr{B}_0)$

For $g \in \mathbf{B}(H)$ we put

$$\Gamma(g) = \limsup_{N\longrightarrow\infty} \frac{1}{N} \sum_{n=1}^{N} \|(1-g)e_n\|^2$$

$$= \limsup_{N\longrightarrow\infty} \int_{E^*} \frac{1}{N} \sum_{n=1}^{N} <x, \ (1-g)e_n>^2 d\mu(x).$$

Motivated by [5, p. 190] we call this the *integral of average power*. The set

$$O(H;\Gamma) = \left\{ g \in O(H) \ ; \ \Gamma(g) = 0 \right\}$$

becomes a closed subgroup of $O(H)$. Furthermore, we can show the following

Proposition 5.1. $O(H;\Gamma) \subset O(H;\mathscr{B}_0)$. In particular, the Lévy Laplacian is invariant under every orthogonal operator $g \in O(H)$ such that $1-g$ is compact.

Finally we shall devote ourselves to coordinate permutations. Let $\mathbf{N}$ be the set of all natural numbers and $\mathrm{Aut}(\mathbf{N})$ the group of all permutations of $\mathbf{N}$. Then $\mathrm{Aut}(\mathbf{N})$ is regarded as a discrete subgroup of $O(H)$ through the fixed CONS $\{e_n\}_{n=1}^{\infty}$. We set

$$\mathscr{G} = \left\{ g \in \mathrm{Aut}(\mathbf{N}) \ ; \ \lim_{N\to\infty} \frac{1}{N} \left|\{ 1 \leq n \leq N \ ; \ g(n) > N \}\right| = 0 \right\},$$

where $|\cdot|$ denotes the cardinality. It is shown that $\mathscr{G}$ is a subgroup of $\mathrm{Aut}(\mathbf{N})$ and we call it the *Lévy group* after [7].

Proposition 5.2. $\mathscr{G} \subset O(H;\mathscr{B}_0)$, i.e. the Lévy Laplacian is invariant under the Lévy group.

Proof. With the notation introduced in Appendix B, one can show that $O(H;\mathscr{B}_0) \cap \mathrm{Aut}(\mathbf{N}) = \mathscr{G}(L,\mathscr{D})$. The assertion is then follows from Proposition B. 1. Q. E. D.

Proposition 5.3. $O(H;\Gamma) \cap \mathrm{Aut}(\mathbf{N}) = \mathscr{G}_0$, where $\mathscr{G}_0$ is the group of all permutations $g \in \mathrm{Aut}(\mathbf{N})$ whose supports are of null density, i.e. $\delta(\mathrm{supp}\ g) = 0$. Furthermore, we have $\mathscr{G}_0 \subset \mathscr{G}$.

Proof. From the equality

$$\Gamma(g) = 2\,\overline{\delta}(\mathrm{supp}\ g) \ , \ g \in \mathrm{Aut}(\mathbf{N}),$$

which is verified by a direct calculation, the assertion follows.

Q. E. D.

Appendix A. Density of natural numbers and the Lévy group

For any subset $S \subset \mathbf{N}$, $\mathbf{N}$ being the set of all natural numbers, we put

$$\overline{\delta}(S) = \limsup_{N \to \infty} \frac{1}{N} \mid S \cap \{1,2,\cdots,N\} \mid$$

and

$$\underline{\delta}(S) = \liminf_{N \to \infty} \frac{1}{N} \mid S \cap \{1,2,\cdots,N\} \mid,$$

where $|\cdot|$ denotes the cardinality. These are called the *upper* and *lower (asymptotic) density* of S, respectively. If the two are equal, we refer to their common value as the *(asymptotic) density* of S and denote it by $\delta(S)$.

We denote by $\mathcal{F}$ the collection of all subsets of $\mathbf{N}$ which admit the density. The triple $(\mathbf{N}, \mathcal{F}, \delta)$ being regarded as an analogue of a probability space, certain problems of additive number theory were discussed by M. Kac [9,10]. Although $\mathcal{F}$ is not finitely additive, we have the following

Proposition A.1. If $S \in \mathcal{F}$, then $S^c \in \mathcal{F}$ and $\delta(S^c) = 1 - \delta(S)$.

Proposition A.2. Let S_1 and S_2 be members of $\mathcal{F}$. Then the following four conditions are mutually equivalent:

(i) $S_1 \cup S_2 \in \mathcal{F}$; (ii) $S_1 \cap S_2 \in \mathcal{F}$; (iii) $S_1 - S_2 \in \mathcal{F}$; (iv) $S_2 - S_1 \in \mathcal{F}$.

If one of the above conditions is satisfied, we have

$$\delta(S_1 \cup S_2) = \delta(S_1) + \delta(S_2) - \delta(S_1 \cap S_2) .$$

The proofs are immediate from definition. The next result means that the density is *non-atomic*. For the proof, see [19].

Proposition A.3. Let $A \in \mathcal{F}$. For any λ, $0 \leqslant \lambda \leqslant \delta(A)$, there exists a subset $B \subset A$ such that $\delta(B) = \lambda$.

Let Aut(N) be the group of all permutations of N and $\mathcal{G}(\delta)$ the subgroup of all permutations which preserve the density:

$$\mathcal{G}(\delta) = \{ g \in \mathrm{Aut}(N) ;\ g\mathcal{F} = \mathcal{F} \text{ and } \delta(g(S)) = \delta(S) \text{ for any } S \in \mathcal{F} \}.$$

For any $g \in \mathrm{Aut}(N)$ we put

$$\mathrm{supp}\ g = \{ n \in N ;\ g(n) \neq n \}.$$

Then g is a bijection from supp g onto itself. In particular, $\mathrm{supp}\ g = \mathrm{supp}\ g^{-1}$. From the inequality

$$\overline{\delta}(S_1 \cup S_2) \leqslant \delta(S_1) + \delta(S_2), \quad S_1 \in \mathcal{F},\ S_2 \in \mathcal{F},$$

which is verified easily, we see that the set

$$\mathcal{G}_0 = \{ g \in \mathrm{Aut}(N) ;\ \delta(\mathrm{supp}\ g) = 0 \}$$

forms a subgroup of Aut(N). Obviously, the group of all finite permutations, denoted by $\mathfrak{S}_\infty$, is a proper subgroup of $\mathcal{G}_0$.

Here we recall the Lévy group introduced in Section 5. Put

$$F_N^+(g) = \{ 1 \leqslant n \leqslant N ;\ g(n) > N \}, \quad g \in \mathrm{Aut}(N).$$

The set

$$\mathcal{G} = \left\{ g \in \mathrm{Aut}(N) ;\ \lim_{N \to \infty} \frac{1}{N} |F_N^+(g)| = 0 \right\}$$

becomes a subgroup of Aut(N) and called the *Lévy group*. The following characterization is given in [16].

Proposition A.4. The Lévy group $\mathcal{G}$ is the maximal permutation group which keeps $\overline{\delta}$ (or $\underline{\delta}$) invariant.

Proposition A.5. $\mathfrak{S}_\infty \subset \mathcal{G}_0 \subset \mathcal{G} \subset \mathcal{G}(\delta)$.

Proof. The last inclusion relation follows immediately from Proposition A.4. The rest is obvious. Q.E.D.

Example. Let $0 = N_0 < N_1 < \cdots$ be an increasing sequence of integers. Assume that $g \in \mathrm{Aut}(\mathbf{N})$ leaves every *interval* $\{ N_{k-1}+1, \cdots \cdots, N_k \}$ stable, $k = 1,2,\cdots$. Then,

$$\limsup_{N \to \infty} \frac{1}{N} |F_N^+(g)| \leqslant \limsup_{k \to \infty} (N_k/N_{k-1} - 1).$$

In particular, $g \in \mathcal{G}$ whenever $\lim\limits_{k \to \infty} N_k/N_{k-1} = 1$.

Remark. From the above example it follows that $\mathcal{G}_0$ is a proper subgroup of $\mathcal{G}$. Furthermore, it may be shown that $\mathcal{G}$ is a proper subgroup of $\mathcal{G}(\delta)$.

Lemma A.6. Let $A = \{ a_1 < a_2 < \cdots \}$ and $B = \{ b_1 < b_2 < \cdots \}$ be members of $\mathcal{F}$ with the same density. Put $A^c = \{ a'_1 < a'_2 < \cdots \}$ and $B^c = \{ b'_1 < b'_2 < \cdots \}$. Assume that A, A^c, B and B^c are infinite sets. Define a permutation $g \in \mathrm{Aut}(\mathbf{N})$ by

$$g(a_n) = b_n, \quad g(a'_n) = b'_n, \quad n = 1,2,\cdots.$$

Then $g \in \mathcal{G}$.

Proof. For each $N \in \mathbf{N}$ we put

$$\alpha(N) = |A \cap \{1,2,\cdots,N\}| \quad \text{and} \quad \beta(N) = |B \cap \{1,2,\cdots,N\}| .$$

By assumption we have

$$\lim_{N\to\infty} \frac{\alpha(N)}{N} = \delta(A) = \delta(B) = \lim_{N\to\infty} \frac{\beta(N)}{N} .$$

On the other hand, we can show that $|F_N^+(g)| = |\alpha(N)-\beta(N)|$. Therefore

$$\limsup_{N\to\infty} \frac{1}{N} |F_N^+(g)| = \limsup_{N\to\infty} \left| \frac{\alpha(N)}{N} - \frac{\beta(N)}{N} \right| = 0.$$

Hence $g \in \mathcal{G}$. Q. E. D.

Theorem A.7 (Ergodicity). Assume that $A \in \mathcal{F}$ is almost invariant under the Lévy group, i.e.

$$\delta(A \ominus g(A)) = 0 \quad \text{for all } g \in \mathcal{G},$$

where $\ominus$ denotes the symmetric difference. Then $\delta(A) = 0$ or 1.

Proof. Suppose that $0 < \delta(A) < 1$. Replacing A with A^c in case of $1/2 \leqslant \delta(A) < 1$, we may assume that $0 < \delta(A) \leqslant 1/2$. With the help of Proposition A.3 we take a subset $B \subset A^c$ such that $\delta(B) = \delta(A)$. We define a permutation $g \in \mathcal{G}$ according to Lemma A.6. Then

$$\delta(A \ominus g(A)) = \delta(A \ominus B) = \delta(A \cup B) = \delta(A) + \delta(B) > 0 .$$

This contradicts the assumption on A, hence, $\delta(A) = 0$ or 1. Q. E. D.

Appendix B. Uniformly distributed sequences and the Lévy group

We begin with another characterization of the Lévy group. Let ℓ^∞ be the Banach space of all bounded real sequences $a = (a_n)_{n=1}^\infty$ with the norm $\|a\|_\infty = \sup |a_n|$. Put

$$L^+(a) = \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n \ , \quad L^-(a) = \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n \ ,$$

where $a = (a_n)_{n=1}^{\infty} \in \ell^{\infty}$. Clearly, $L^+(-a) = -L^-(a)$. The group Aut(N) acts on ℓ^{∞} as coordinate permutations, i.e. by means of the maps:

$$a = (a_n)_{n=1}^{\infty} \longmapsto ga = (a_{g^{-1}(n)})_{n=1}^{\infty} \ , \quad g \in \mathrm{Aut}(\mathbb{N}).$$

The following result is shown in [16].

Proposition B.1. The Lévy group is the maximal permutation group which keeps L^+ (or L^-) invariant.

Let $\mathcal{D}$ be the space of all $a \in \ell^{\infty}$ such that $L^+(a) = L^-(a)$. Then $\mathcal{D}$ becomes a closed subspace of ℓ^{∞} and the functional:

$$L(a) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n \ , \quad a = (a_n)_{n=1}^{\infty} \in \mathcal{D} \ ,$$

is continuous and linear. We denote by $\mathcal{G}(L,\mathcal{D})$ the group of all permutations which leave L invariant:

$$\mathcal{G}(L,\mathcal{D}) = \{ \ g \in \mathrm{Aut}(\mathbb{N}) \ ; \ g\mathcal{D} = \mathcal{D}, \ L(ga) = L(a) \ \text{for all } a \in \mathcal{D} \ \} \ .$$

Then the following assertion is easy to see.

Proposition B.2. $\mathcal{G} \subset \mathcal{G}(L,\mathcal{D}) \subset \mathcal{G}(\delta)$.

We give notation after [22]. A sequence $(x_n)_{n=1}^{\infty}$, $0 \leqslant x_n < 1$, is called *uniformly distributed* on the interval [0,1) if

$$\lim_{N\to\infty} \frac{1}{N} \mid \{ \ 1 \leqslant n \leqslant N \ ; \ a \leqslant x_n < b \ \} \mid = b - a$$

for any pair a, b of real numbers with $0 \leqslant a < b \leqslant 1$. This property depends upon arrangement of the sequence as J. von Neumann discussed in [21]. Here we mention the following

Proposition B.3. Assume that $x = (x_n)_{n=1}^{\infty}$ is uniformly distributed on $[0,1)$. Then, for any $g \in \mathscr{G}$, the rearranged sequence gx is also uniformly distributed on $[0,1)$.

Proof. Let f be a real-valued continuous function defined on the interval $[0,1]$ and put

$$a = (a_n)_{n=1}^{\infty}, \quad a_n = f(x_n) .$$

By virtue of the Weyl's theorem (see, e.g., [12,22]), we can show that $a \in \mathscr{D}$ and that

$$L(a) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\, dx .$$

For any $g \in \mathscr{G}$, viewing Proposition B.2, we have

$$L(ga) = L(a) = \int_0^1 f(x)\, dx .$$

Consequently, using the Weyl's theorem again, we see that the rearranged sequence gx is uniformly distributed. Q.E.D.

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