数理解析研究所講究録
643巻 1988年 154-169

An explicit sixth-order pseudo-Runge-Kutta formula

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§0 はじめに.

常微分方程式の初期値問題:

(1) $$y'=f(x,y),\ y(x_0)=y_0$$

の近似式について述べる. 論文[1]である種の近似式で位数6の公式を導いたが近似式の係数が特別の値をとる($w_2=0$)場合について議論している. ここでは係数が一般的な場合について調べる。

§1 近似式の導出.

(1) 式の近似式を次のように与える。

(2) $$y_{n+1}=y_n-v(y_n-y_{n-1})+h\sum_{i=0}^{p}w_i k_i,$$

$$k_0=f(x_{n-1},y_{n-1}),\quad k_1=f(x_n,y_n),$$

$$k_i=f\Big(x_n+a_i h,\ y_n+b_i(y_n-y_{n-1})+h\sum_{j=0}^{i-1}b_{ij}k_j\Big)\quad (i=2,3,\dots,p)$$

$$a_i = \sum_{j=0}^{i-1} b_{ij}\,.$$

(2)式が order 6 になるための係数条件を tree で表現すると次のようになる。

(3)  $$(-1)^i \frac{v}{i!} + \sum_{l=0}^{4} \frac{a_l^{i-1}}{(l-1)!} w_l = \frac{1}{i!}$$
$(i=1,2,\ldots,6)$,

 $$\frac{v}{6!} - \frac{w_0}{5!} + P_{24} w_2 + (P_{34} + P_{23} b_{32}) w_3 + \left(P_{44} + P_{23} b_{43} + \left(P_{33} + \frac{1}{3!} a_2^3 b_{32}\right) b_{44}\right) w_4 = \frac{1}{6!},$$

 $$\frac{v}{6!} - \frac{w_0}{5!} + P_{24} w_2 + \left(P_{34} + \frac{1}{4!} a_2^4 b_{32}\right) w_3 + \left(P_{44} + \frac{1}{4!} a_2^4 b_{43} + \frac{1}{4!} a_3^4 b_{44}\right) w_4 = \frac{1}{6!},$$

 $$\frac{6}{6!} v - \frac{5}{5!} w_0 + a_2 P_{23} w_2 + a_3 \left(P_{33} + \frac{1}{3!} a_2^3 b_{32}\right) w_3 + a_4 \left(P_{44} + \frac{1}{3!} a_3^3 b_{43} + \frac{1}{3!} a_4^3 b_{44}\right) w_4 = \frac{5}{6!},$$

$$P_{ij} = (-1)^{i+1} \left\{ \frac{1}{(i+2)!} b_j + \frac{1}{(i+1)} b_{j0} \right\},$$
$(i=2,3,4,\ j=1,2,3,4)$.

$$a_j^{i+2} = (i+2)! \left\{ P_{ij} + \frac{1}{(i+1)!} \sum_{l=2}^{j-1} a_l^{i+1} b_{jl} \right\} \quad (i=0,1,\ j=2,3,4),$$

ただし　$a_0 = -1$, $a_1 = 0$　とする。

(3)式を $w_2 = 0$ と $w_2 \neq 0$ の二通りの場合に区けて解くと、

[1]　$w_2 = 0$ のとき、

(4)　$a_3 = \frac{1}{\sqrt{3}}$, $v = \frac{139 - 240 a_3}{11}$, $w_0 = \frac{18 - 31 a_3}{11}$,
$a_4 = 1$.

$$w_1 = \frac{40-64a_3}{11}, \quad w_3 = \frac{-18(8a_3-5)}{11}, \quad w_4 = \frac{2-a_3}{11},$$

$$b_2 = -(2a_2^3+3a_2^2), \quad b_{20} = a_2^2 + a_2^3,$$

$$b_3 = \frac{2(2+\sqrt{3})}{3(2a_2+1)} - \frac{3\sqrt{3}+2}{3\sqrt{3}}, \quad b_{30} = -\frac{(2+\sqrt{3})(3a_2+2)}{9(2a_2+1)(a_2+1)} + \frac{\sqrt{3}+1}{3\sqrt{3}},$$

$$b_{32} = \frac{2+\sqrt{3}}{9a_2(2a_2+1)(a_2+1)},$$

$$b_4 = -\frac{12}{2a_2+1} + 12\frac{6a_3+4}{7a_3+4} - 5,$$

$$b_{40} = \frac{2(3a_2+2)}{(a_2+1)(2a_2+1)} - 6\frac{5a_3+3}{7a_3+4} + 2,$$

$$b_{42} = \frac{2}{a_2(a_2+1)(2a_3+1)}, \quad b_{43} = 6\frac{3a_3+1}{7a_3+4},$$

$$b_{i1} = a_i - \sum_{\substack{j=0 \\ j\neq 1}}^{i-1} b_{ij}, \quad (i=2,3,4).$$

[II]　$w_2 \neq 0$ のとき.

次の二つの場合に分けて考える.

(イ)　$a_4 \neq 1$ のとき.

(5)
$$v = \frac{q_1 - q_2 q_3}{q_4 - q_3 q_5}, \quad w_3 = \frac{q_1 - q_4 v}{a_3(1+a_3)(a_2-a_3)(a_4-a_3)q_3},$$

$$w_4 = \frac{\frac{9}{20}a_2 - \frac{11}{30} - \left\{\left(\frac{1}{20}a_2 + \frac{11}{30}\right)v + a_3^3(a_2-a_3)(1+a_3)w_3\right\}}{a_4^3(a_2-a_4)(a_4+1)},$$

$$w_2 = \frac{\frac{9}{20} - \left\{\frac{1}{20}v + a_3^3(a_3+1)w_3 + a_4^3(a_4+1)w_4\right\}}{a_2^3(a_2+1)},$$

$$w_0 = \frac{1}{2}(v-1) + \sum_{i=2}^{4} a_i w_i,$$

$$w_1 = 1 + v - (w_0 + w_2 + w_3 + w_4).$$

$$b_{32} = \frac{62a_4 - 49 + (2a_4+1)v}{120(a_4-a_3)a_2(2a_2+1)(a_2+1)w_3},$$

$$b_{43} = \frac{160a_2 + 49 - v}{120\{5(2a_2+1)^2 a_2(a_2+1)b_{32} + a_3(a_3+1)(a_2-a_3)\}w_4},$$

$$b_{42} = \frac{1}{a_2(2a_2+1)(a_2+1)}\left\{\frac{49 - 62a_3 - (2a_3+1)v}{120(a_4-a_3)w_4} - a_3(2a_3+1)(a_3+1)b_{43}\right\},$$

$$b_2 = -(2a_2^3 + 3a_2^2), \quad b_{20} = a_2^2 + a_2^3,$$

$$b_4 = 6\left\{(a_2+a_2^2)b_{42} + (a_3+a_3^2)b_{43} - \frac{5}{6}\right\},$$

$$b_{40} = -\frac{1}{2}b_4 + a_2 b_{42} + a_3 b_{43} - \frac{1}{2}a_4^2,$$

$$b_3 = 6a_2(a_2+1)b_{32} - a_3^2(2a_3+3),$$

$$b_{30} = -\frac{1}{2}b_3 + a_2 b_{32} - \frac{1}{2}a_3^2,$$

$$b_{i1} = a_i - \sum_{\substack{j=0 \\ j\neq 1}}^{i-1} b_{ij} \quad (i=2,3,4).$$

ただし

$$g_1 = a_4^2\left(\frac{5}{6}a_2^2 - \frac{9}{20}\right) - \left(\frac{9}{20}a_2 - \frac{11}{30}\right)(a_2+a_4),$$

$$g_2 = \frac{1}{12}\left\{10a_2a_4 - 7(a_2+a_4) + \frac{27}{5}\right\},$$

$$g_3 = a_2a_3 + a_3a_4 + a_4a_2,$$

$$g_4 = a_4^2\left(\frac{1}{6}a_2^2 - \frac{1}{20}\right) - \left(\frac{1}{20}a_2 + \frac{1}{30}\right)(a_2+a_4),$$

$$g_5 = \frac{1}{12}\left(2a_2a_4 + a_2 + a_4 + \frac{3}{5}\right),$$

である．また定数 a_3 は次の二次方程式：

$$z_1 a_3^2 + z_2 a_3 + z_3 = 0,$$

の根である．ここで z_1, z_2, z_3 は次のように与えられる．

$$z_1 = u_{11}\{2u_2u_7u_8 - (u_2u_{12} - u_1)(u_5u_8 + u_6u_7) - 2u_1u_2u_5u_6$$
$$+ u_9(u_3u_6u_8 + u_4u_8) - u_{10}(u_3u_6^2 + u_4u_6u_8)\},$$

$$z_2 = u_{13}\{2u_2u_7u_8 - (u_2u_{12} - u_1)(u_5u_8 + u_6u_7)$$
$$- 2u_1u_5u_6u_{12}\} + (u_3u_9 - u_4u_{10})(u_5u_8 + u_6u_7)$$
$$+ 2(u_4u_7u_8u_9 - u_3u_5u_6u_{10})$$
$$+ u_{11}\{u_2u_7^2 - (u_2u_{12} - u_1)u_5u_7 - u_1u_5^2u_{12}\},$$

$$z_3 = u_4u_7^2u_9 + u_5u_7(u_3u_9 - u_4u_{10}) - u_3u_5^2u_{10}$$
$$+ u_{13}\{u_2u_7^2 - (u_2u_{12} - u_1)u_5u_7 - u_1u_5^2u_{12}\},$$

$$u_1 = 2\{10a_2a_4 - 7(a_2 + a_4) + \frac{27}{5}\},$$

$$u_2 = -2\{2a_2a_4 + a_2 + a_4 + \frac{3}{5}\},$$

$$u_3 = (2a_2 + 1)(62a_4 - 49) + u_1, \quad u_4 = -\frac{1}{5},$$

$$u_5 = 24\{a_4^2(\frac{1}{6}a_2^2 - \frac{1}{20}) - (\frac{1}{20}a_2 + \frac{1}{30})(a_2 + a_4) + u_2a_2a_4\},$$

$$u_6 = (a_2 + a_4)u_2,$$

$$u_7 = 24\{a_4^2(\frac{5}{6}a_2^2 - \frac{9}{20}) - (\frac{9}{20}a_2 - \frac{11}{30})(a_2 + a_4)\}$$
$$- u_1a_2a_4,$$

$$u_8 = -(a_2 + a_4)u_1, \quad u_9 = 10a_2^2 + 10a_2 + 2,$$

$$u_{10} = -310a_2^2 + 10a_2 + 98, \quad u_{11} = 10a_2 + 5,$$

$u_{12} = 160a_2 + 49$, $u_{13} = 5a_2 + 3$.

(6) (ロ) $a_4 = 1$ のとき.

(5) 式において

$z_1 = z_2 = z_3 = 0$,

となり，定数 v, w_i, b_i, b_{ij} は (5) 式で与えられる．この場合．a_2, a_3 は free parameter となる．

§2 局所打切り誤差.

(2) 式において4段6位のときの打切り誤差を調べることにする．

まず

$$d_{ij} = (-1)^j \left\{ \frac{b_i}{(j+1)!} + \frac{b_{i0}}{j!} \right\} \quad (i = 2, 3, 4, j = 1, 2, \ldots 5),$$

$$e_{30} = 3d_{23} + \frac{1}{6}a_2^4,$$

$$e_{31} = d_{33} + \frac{1}{6}a_3^3 b_{32},$$

$$e_{41} = d_{34} + \frac{1}{4!}a_2^4 b_{32},$$

$$e_{42} = d_{34} + d_{22} b_{32},$$

$$e_{43} = 4\left(d_{34} + \frac{1}{4!}a_2^4 b_{32}\right) + 3\left(d_{34} + d_{22} b_{32}\right),$$

$$e_{51} = d_{35} + \frac{1}{5!}a_2^5 b_{32},$$

$$e_{52} = d_{35} + \tfrac{1}{5} a_2 d_{23} b_{32},$$

$$e_{53} = 13 d_{35} + (3 d_{24} + \tfrac{1}{12} a_2^5) b_{32},$$

$$e_{54} = 16 d_{35} + (6 d_{24} + \tfrac{1}{12} a_2^5) b_{32},$$

$$e_{55} = 12 d_{35} + (7 d_{24} + a_2 d_{23}) b_{32},$$

$$e_{56} = 9 d_{35} + (4 d_{24} + a_2 d_{23}) b_{32},$$

$$e_{57} = d_{35} + d_{24} b_{32},$$

$$p_{31} = d_{43} + \tfrac{1}{6} a_2^3 b_{42} + \tfrac{1}{6} a_3^3 b_{43},$$

$$p_{41} = d_{44} + \tfrac{1}{24} a_2^4 b_{42} + \tfrac{1}{24} a_3^4 b_{43},$$

$$p_{43} = d_{44} + d_{23} b_{42} + e_{32} b_{43},$$

$$p_{51} = d_{45} + \tfrac{1}{5!} a_2^5 b_{42} + \tfrac{1}{5!} a_3^5 b_{43},$$

$$p_{52} = d_{45} + \tfrac{1}{5} a_2 d_{23} b_{42} + \tfrac{1}{5} a_3 (d_{33} + \tfrac{1}{3!} a_2^3 b_{32}) b_{43},$$

$$p_{53} = 13 d_{45} + (3 d_{24} + \tfrac{1}{12} a_2^5) b_{42}$$
$$+ (3 d_{34} + \tfrac{1}{8} a_2^4 b_{32} + \tfrac{1}{12} a_3^5) b_{43},$$

$$p_{54} = 16 d_{45} + (6 d_{24} + \tfrac{1}{12} a_2^5) b_{42}$$
$$+ (6 d_{34} + \tfrac{1}{4} a_2^4 b_{32} + \tfrac{1}{12} a_3^5) b_{43},$$

$$p_{55} = 12 d_{45} + (7 d_{24} + a_2 d_{23}) b_{32}$$
$$+ (7 d_{34} + \tfrac{1}{6} a_2^4 b_{32} + 2 d_{23} b_{32} + a_3 (d_{33} + \tfrac{1}{3!} a_2^3 b_{32}) b_{43})$$

$$p_{56} = 9 d_{45} + (4 d_{24} + a_2 d_{23}) b_{42}$$
$$+ (4 d_{34} + \tfrac{1}{3!} a_2^4 b_{32} + a_3 (d_{33} + \tfrac{1}{3!} a_2^3 b_{32})) b_{43},$$

$$p_{57} = d_{45} + d_{24} b_{42} + (d_{34} + d_{23} b_{32}) b_{43},$$

$$p_{58} = d_{45} + d_{24} b_{42} + (d_{34} + \tfrac{1}{4!} a_2^4 b_{32}) b_{43},$$

とおき. θ_i ($i=0\sim28$) を次の様に定める.

$$\theta_0 = \frac{1}{720}w_0 - \frac{1}{5040} - \frac{v}{5040},$$

$$\theta_1 = (3a_2d_{24} + \tfrac{3}{2}a_2^2d_{23})w_2 + (3a_3e_{41} + \tfrac{3}{2}a_3^2e_{31})w_3 + (3a_4p_{41} + \tfrac{3}{2}a_4^2p_{31})w_4,$$

$$\theta_2 = (15d_{25} + \tfrac{1}{12}a_2^6)w_2 + (15e_{51} + \tfrac{1}{12}a_3^6)w_3 + (15p_{51} + \tfrac{1}{12}a_4^6)w_4,$$

$$\theta_3 = d_{25}w_2 + e_{51}w_3 + p_{51}w_4,$$

$$\theta_4 = d_{25}w_2 + e_{57}w_3 + p_{57}w_4,$$

$$\theta_5 = \frac{1}{48}(a_2^6w_2 + a_3^6w_3 + a_4^6w_4),$$

$$\theta_6 = \frac{1}{2}(a_2^2d_{23}w_2 + a_3^2e_{31}w_3 + a_4^2p_{31}w_4),$$

$$\theta_7 = (\tfrac{1}{36}a_2^6 + 10d_5)w_2 + (\tfrac{1}{36}a_3^6 + 10e_{51})w_3 + (\tfrac{1}{36}a_4^6 + 10p_{51})w_4,$$

$$\theta_8 = \frac{1}{36}\sum_{i=2}^{4}a_i^6w_i,\quad \theta_9 = \frac{1}{16}\sum_{i=2}^{4}a_i^6w_i,\quad \theta_{10} = \frac{1}{12}\sum_{i=2}^{4}a_i^6w_i,$$

$$\theta_{11} = \frac{1}{48}\sum_{i=2}^{4}a_i^6w_i,\quad \theta_{12} = \frac{1}{72}\sum_{i=2}^{4}a_i^6w_i,\quad \theta_{13} = \frac{1}{720}\sum_{i=2}^{4}a_i^6w_i,$$

$$\theta_{14} = (\tfrac{1}{2}a_2^2d_{23} + \tfrac{1}{36}a_2^6 + 10d_{25})w_2 + (\tfrac{1}{2}a_3^2e_{31} + \tfrac{1}{36}a_3^6 + 10e_{51})w_3 + (\tfrac{1}{2}a_4^2p_{31} + \tfrac{1}{36}a_4^6 + 10p_{51})w_4,$$

$$\theta_{15} = (7a_2d_{24} + 15d_{25})w_2 + (a_3e_{43} + 15e_{52})w_3 + (a_4p_{42} + 15p_{52})w_4,$$

$$\theta_{16} = (16d_{25} + \tfrac{1}{2}a_2^2d_{23})w_2 + (e_{54} + \tfrac{1}{2}a_3^2e_{31})w_3 + (p_{54} + \tfrac{1}{2}a_4^2p_{31})w_4,$$

$$Q17 = \left(\frac{3}{2}a_2^2 d_{23} + 6a_2 d_{24}\right)w_2 + \left(\frac{3}{2}a_3^2 e_{31} + 6a_3 e_{41}\right)w_3$$
$$+ \left(\frac{3}{2}a_4^2 P_{31} + 6a_4 P_{41}\right)w_4,$$

$$Q18 = (12d_{25} + a_2 d_{24})w_2 + (e_{55} + a_3 e_{42})w_3$$
$$+ (P_{55} + a_4 P_{43})w_4,$$

$$Q19 = d_{25}w_2 + e_{57}w_3 + P_{58}w_4,$$

$$Q20 = (9d_{25} + a_2 d_{24})w_2 + (e_{56} + a_3 e_{42})w_3$$
$$+ (P_{56} + a_4 P_{43})w_4,$$

$$Q21 = 4a_2 d_{24}w_2 + 4a_3 e_{41}w_3 + 4a_4 P_{41}w_4,$$

$$Q22 = \left(\frac{1}{2}a_2^2 d_{23} + 10d_{25}\right)w_2 + \left(\frac{1}{2}a_3^2 e_{31} + 10e_{51}\right)w_3$$
$$+ \left(\frac{1}{2}a_4^2 P_{31} + 10P_{51}\right)w_4,$$

$$Q23 = d_{25}w_2 + e_{57}w_3 + P_{57}w_4,$$

$$Q24 = (a_2 d_{24} + 5d_{25})w_2 + (a_3 e_{41} + 5e_{52})w_3$$
$$+ (a_4 P_{43} + 5P_{52})w_4,$$

$$Q25 = Q23,$$

$$Q26 = \left(\frac{1}{72}a_2^6 + 13d_{35} + \frac{1}{2}a_2^2 d_{23}\right)w_2$$
$$+ \left(\frac{1}{72}a_3^6 + e_{53} + \frac{1}{2}a_3^2 e_{31}\right)w_3$$
$$+ \left(\frac{1}{72}a_4^6 + P_{53} + \frac{1}{2}a_4^2 P_{31}\right)w_4,$$

$$Q27 = \frac{1}{2}a_2^2 d_{23}w_2 + \frac{1}{2}a_3^2 e_{31}w_3 + \frac{1}{2}a_4^2 P_{31}w_4,$$

$$Q28 = a_2 d_{24}w_2 + a_3 e_{41}w_3 + a_4 P_{41}w_4$$

このとき (2) 式の打切り誤差は次のように表わせる.

(7)

$$
\begin{aligned}
T(x_n, y_n; h) = {} & \Theta_1 f_{yy} \cdot S1 \cdot G2 + \Theta_2 f_y \cdot G2 \cdot R1 \\
& + \Theta 3 f_y \cdot T5 + \Theta 4 f_y^3 \cdot T3 + \Theta 5\, T1 \cdot S4 + \Theta 6 f_{yy} \cdot T1 \cdot T2 \\
& + \Theta 7 f_y \cdot T1 \cdot S3 + \Theta 8\, T2 \cdot S3 + \Theta 9 \cdot G2 \cdot D^2 f_{yy} + \\
& \Theta 10\, T1 \cdot T2 \cdot D f_{yy} + \Theta 11 f_{yy} G3 + \Theta 12 f_{yy} (T2)^2 + \Theta 13\, T6 \\
& + \Theta 14 f_y \cdot f_{yy}\, T1 \cdot T2 + \Theta 15 f_y\, T1 \cdot (S1)^2 + \Theta 16 f_y^2 \cdot T1 \cdot S2 \\
& + \Theta 17\, T1 \cdot S1 \cdot S2 + \Theta 18 f_y^3\, T1 \cdot S1 + \Theta 19 f_y^5 \cdot T1 \\
& + \Theta 20 f_y^2 \cdot T2 \cdot S1 + \Theta 21 \cdot T2 \cdot (S1)^2 + \Theta 22 f_y \cdot T2 \cdot S2 \\
& + \Theta 23 f_y^4 \cdot T2 + \Theta 24 f_y \cdot T3 \cdot S1 + \Theta 25 f_y^2 \cdot T4 \\
& + \Theta 26 f_y^2 \cdot f_{yy} \cdot \Theta 2 + \Theta 27\, T3 \cdot S2 + \Theta 28\, S1 \cdot T4.
\end{aligned}
$$

ここで $Sj = D^j f_y$, $Gj = (Df)^j$, $R1 = D f_{yy}$,

$Tj = D^j f$. $D = \left(\frac{\partial}{\partial x} + f \frac{\partial}{\partial y}\right)$, $f_y = \frac{\partial f}{\partial y}$

を表わす.

関数 f の偏導関数に次のような評価を仮定する.

$$|f(x, y)| \leqq L_1, \quad \left| \frac{\partial^{i+j} f}{\partial x^i \partial y^j} \right| \leqq L_2^{i+j} / L_1^{j-1} \quad (i+j \leqq 7).$$

$(L_1, L_2 : 定数)$.

このとき打切り誤差式(7)は

(8)

$$|T(x_n, y_n; h)| \leqq \sum_{i=1}^{118} \Theta\Theta i\, L_1 L_2^6 h^7 \left(= T_u L_1 L_2^6 h^7\right)$$

のようになる.

定数 $\Theta\Theta i$ $(= \Theta\Theta i(\Theta_1, \Theta_2, \ldots \Theta 28))$ は次式で与えられる.

```
QQ1=ABS(2*Q1+2*Q14):QQ2=ABS(Q2):QQ3=ABS(3*Q7+2*Q9)
QQ4=ABS(5*Q3+2*Q5):QQ5=ABS(Q6)
QQ6=ABS(Q2):QQ7=ABS(Q7):QQ8=ABS(Q5)
QQ9=ABS(Q3):QQ10=ABS(3*Q6)
QQ11=ABS(3*Q7+4*Q9):QQ12=ABS(10*Q3+4*Q5)
QQ13=ABS(2*Q1+Q15):QQ14=ABS(3*Q6+Q10+2*Q17+3*Q24)
QQ15=ABS(Q7+2*Q9):QQ16=ABS(10*Q3+6*Q5)
QQ17=ABS(Q6+Q10+Q24+Q17):QQ18=ABS(5*Q3+4*Q5)
QQ19=ABS(Q3+Q5):QQ20=ABS(2*Q8+4*Q28)
 QQ21=ABS(Q1):QQ22=ABS(6*Q8+6*Q28)
 QQ23=ABS(6*Q8+4*Q28):QQ24=ABS(2*Q8+Q28)
 QQ25=ABS(3*Q6):QQ26=ABS(2*Q9)
 QQ27=ABS(3*Q8):QQ28=ABS(4*Q5)
 QQ29=ABS(Q1):QQ30=ABS(Q9):QQ31=ABS(Q8)
 QQ32=ABS(Q6):QQ33=ABS(Q1+2*Q14+2*Q15+3*Q20)
 QQ34=ABS(2*Q2+2*Q16):QQ35=ABS(Q7+4*Q25)
 QQ36=ABS(Q5):QQ37=ABS(2*Q2+3*Q11+Q16)
 QQ38=ABS(3*Q7+Q9+6*Q25):QQ39=ABS(3*Q7+2*Q9+4*Q25)
 QQ40=ABS(Q1+Q14+Q15+Q20):QQ41=ABS(Q7+Q9+Q25)
 QQ42=ABS(3*Q4+Q16):QQ43=ABS(Q4)
 QQ44=ABS(Q2+3*Q4+2*Q16):QQ45=ABS(Q2+Q4+Q11+Q16)
 QQ46=ABS(3*Q6+Q10):QQ47=ABS(3*Q8)
 QQ48=ABS(Q8+4*Q28):QQ49=ABS(Q9):QQ50=ABS(6*Q5)
 QQ51=ABS(Q6+Q17):QQ52=ABS(3*Q8+6*Q28)
 QQ53=ABS(Q8):QQ54=ABS(4*Q5):QQ55=ABS(3*Q8+4*Q28)
 QQ56=ABS(Q5):QQ57=ABS(Q8+Q28):QQ58=ABS(2*Q15)
 QQ59=ABS(2*Q10+3*Q24+2*Q17):QQ60=ABS(2*Q10+Q24+Q17)
 QQ61=ABS(Q10)QQ62=ABS(3*Q11):QQ63=ABS(Q14)
 QQ64=ABS(Q15):QQ65=ABS(Q10):QQ66=ABS(Q14)
 QQ67=ABS(2*Q10+2*Q17):QQ68=ABS(4*Q12+2*Q21)
 QQ69=ABS(2*Q10+Q17):QQ70=ABS(4*Q12+4*Q21)
 QQ71=ABS(Q10):QQ72=ABS(6*Q13):QQ73=ABS(Q10)
 QQ74=ABS(Q11):QQ75=ABS(Q12):QQ76=ABS(Q13)
 QQ77=ABS(Q14+Q20):QQ78=ABS(Q15+2*Q20)
 QQ79=ABS(4*Q12):QQ80=ABS(15*Q13)
 QQ81=ABS(2*Q12+Q21):QQ82=ABS(20*Q13):QQ83=ABS(15*Q13)
 QQ84=ABS(Q12+Q21):QQ85=ABS(6*Q13):QQ86=ABS(Q13)
 QQ87=ABS(3*Q24+Q17):QQ88=ABS(Q24):QQ89=ABS(3*Q24+Q17)
 QQ90=ABS(Q25):QQ91=ABS(2*Q22+2*Q23):QQ92=ABS(Q22+Q23)
 QQ93=ABS(Q22+Q23+Q26+Q18):QQ94=ABS(Q21)
 QQ95=ABS(5*Q21):QQ96=ABS(2*Q21):QQ97=ABS(Q17)
 QQ98=ABS(Q17):QQ99=ABS(Q20):QQ100=ABS(Q16)
 QQ101=ABS(Q18):QQ102=ABS(Q18+2*Q26)
 QQ103=ABS(Q18):QQ104=ABS(Q19):QQ105=ABS(Q19)
 QQ106=ABS(2*Q17):QQ107=ABS(Q28):QQ108=ABS(2*Q27)
 QQ109=ABS(Q28):QQ110=ABS(3*Q27)
 QQ111=ABS(Q27):QQ112=ABS(Q26):QQ113=ABS(9*Q27)
 QQ114=ABS(Q27):QQ115=ABS(4*Q27):QQ116=ABS(6*Q27)
 QQ117=ABS(5*Q27):QQ118=ABS(Q27)
```

局所打切り誤差 T_u を最小にするように free parameters を決定しよう.

(4)式のとき a_2 が free parameter で最適値は

$$a_2 = 0.1$$

でこのとき

$$|T(x_n, y_n; h)| \leqq 1.31\, h^7 L_1 L_2^6,$$

(5)式での free parameter a_2, a_3 の最適値は

$$a_2 = 0.05 \quad , \quad a_4 = 0.285.$$

またこのとき

$$|T(x_n, y_n; h)| \leqq 4.36\, h^7 L_1 L_2^6.$$

(6)式においては同様に free parameter a_2, a_4 は

$$a_2 = 0.6 \quad , \quad a_3 = 0.3$$

のとき最適で打切り誤差は

$$|T(x_n, y_n; h)| \leqq 1.06\, h^7 L_1 L_2^6.$$

となる.

§3 安定性.

微分方程式 $y' = \lambda y$, $y(0) = 1$, $(\lambda \in C,\ Re(\lambda) < 0)$ を(2)式で近似すると次の特性多項式が得られる.

(7) $$y_{n+1} = (l_1 + l_2 i)\, y_n + (l_3 + l_4 i)\, y_{n-1},$$

l_1, l_2, l_3, l_4 は定数で次式で与えられる.

$$l_1 = \alpha (v_1 w_2 + v_2 w_3 + v_3 w_4)$$
$$\quad + \beta (w_1 + u_1 w_2 + u_2 w_3 + u_3 w_4),$$
$$l_2 = 1 - v + \alpha (w_1 + u_1 w_2 + u_2 w_3 + u_3 w_4)$$
$$\quad - \beta (v_1 w_2 + v_2 w_3 + v_3 w_4),$$
$$l_3 = v + \alpha (w_0 + c_1 w_2 + c_2 w_3 + c_3 w_4)$$
$$\quad - \beta (d_1 w_2 + d_2 w_3 + d_3 w_4),$$
$$l_4 = \alpha (d_1 w_2 + d_2 w_3 + d_3 w_4)$$
$$\quad + \beta (w_0 + c_1 w_2 + c_2 w_3 + c_3 w_4),$$
$$c_1 = -b_2 + \alpha b_{20},$$
$$c_2 = -b_3 + \alpha (b_{30} - b_{32} b_2) + (\alpha^2 - \beta^2) b_{32} b_{20},$$
$$c_3 = -b_4 + \alpha (b_{40} - b_2 b_{42} - b_3 b_{43})$$
$$\quad + (\alpha^2 - \beta^2) \{ b_{42} b_{20} + b_{43} (b_{30} - b_{32} b_2) \}$$
$$\quad + \{ (\alpha^2 - \beta^2) \alpha - 2\alpha\beta^2 \} b_{43} b_{32} b_{20},$$
$$d_1 = \beta b_{20},$$
$$d_2 = \beta (b_{30} - b_{32} b_2) + 2\alpha\beta b_{32} b_{20},$$
$$d_3 = \beta (b_{40} - b_2 b_{42} - b_3 b_{43})$$
$$\quad + 2\alpha\beta \{ b_{42} b_{20} + b_{43} (b_{30} - b_{32} b_2) \}$$
$$\quad + \{ 2\alpha^2 \beta + \beta (\alpha^2 - \beta^2) \} b_{43} b_{32} b_{20},$$
$$u_1 = 1 + b_2 + \alpha b_{21},$$
$$u_2 = 1 + b_3 + \alpha \{ b_{31} + b_{32} (1 + b_2) \} + (\alpha^2 - \beta^2) b_{32} b_{21},$$

$$u_3 = 1 + b_4 + \alpha\{b_{41} + b_{42}(1+b_2) + b_{43}(1+b_3)\}$$
$$+ (\alpha^2-\beta^2)\{b_{42}b_{21} + b_{43}(b_{31} + b_{32}(1+b_2))\}$$
$$+ \alpha(\alpha^2 - 3\beta^2)b_{43}b_{32}b_{21},$$

$$v_1 = \beta b_{21},$$

$$v_2 = \beta\{b_{31} + b_{32}(1+b_2)\} + 2\alpha\beta b_{32}b_{21},$$

$$v_3 = \beta\{b_{41} + b_{42}(1+b_2) + b_{43}(1+b_3)\}$$
$$+ 2\alpha\beta\{b_{42}b_{21} + b_{43}(b_{31} + b_{32}(1+b_2))\}$$
$$+ \beta(3\alpha^2 - \beta^2)b_{43}b_{32}b_{21},$$
$$(h\lambda = \alpha + \beta i).$$

これより (7)式の特性多項式の根 (ξ_1, ξ_2)

$$|\xi_i| \leqq 1, \quad (i=1,2)$$
$$|\xi_1\xi_2| \neq 1,$$

を満たすための条件(絶対安定条件)は

(1) $\quad l_3^2 + l_4^2 < 1,$

(2) $\quad \{l_1 + (l_1l_3 + l_2l_4)\}^2 + \{l_2 + (l_1l_4 - l_2l_3)\}^2$
$\quad \leqq 1 - (l_3^2 + l_4^2),$

となる.

(4), (5), (6) 式の安定領域((1), (2) を満足する $(\alpha+\beta i)$ の領域)は図のようになる.

§4 数値実験

(4),(5),(6)式により次の微分方程式：

$$y' = -y + x^2$$

$$y(0) = 3 \quad , \quad y(x) = e^{-x} + 2 - 2x + x^2.$$

の近似解を求めよう。

（進み幅 $h = 1/2^4$, $x = 0 \sim 5.00$）

絶対誤差

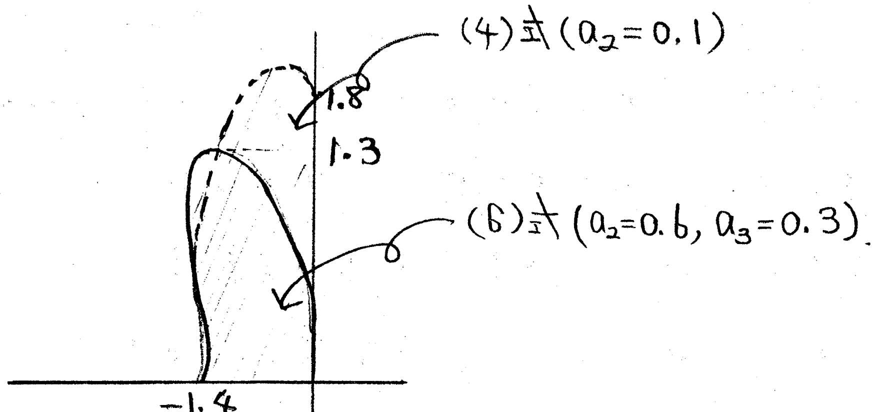
x	(4)式 $a_2=0.1$	(5)式 $a_2=0.05, a_4=0.285$	(6)式 $a_2=0.6, a_3=0.3$
0.125	-4.67E-12	-1.546E-10	-5.87E-12
0.1875	-8.60E-12	-1.00E-9	-1.08E-11
0.3125	-1.50E-11	-2.36E-8	-1.90E-11
0.5	-2.17E-11	-2.33E-6	-2.74E-11
1.00	-2.81E-11	-4.73E-1	-3.56E-11
1.500	-2.61E-11	-9.62E+4	-3.31E-11
3.00	-1.19E-11	-8.06E+20	-1.50E-11
5.00	-2.71E-12		-3.43E-12

(7)式で $h\lambda = \alpha + \beta i = 0$ とおいたとき、特性多項式の根は $\xi_1 = 1$, $\xi_2 = -v$ となり、

A_0-安定条件は

$$|\xi_2| = |-v| < 1$$

となる．しかるに(5)式ではこの条件を満足するような parameter a_2, a_4 $(0<a_2, a_4 \leqq 1)$ は存在しない．よって係数(5)は A_0-不安定である．



絶対安定領域

参考論文

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