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Relative Recursive Enumerability of Generic Degrees

Dedicated to

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Introduction. Let ω be the set of natural numbers, i.e. $\{0,1,2,3,\cdots\}$. A set $A\subseteq\omega$ is called n-generic if it is Cohen-generic for n-quantifier arithmetic. As characterized by Jockusch[4], this is equivalent to saying that for every Σ_n^0 set of strings S, there is a $\sigma<A$ such that $\sigma\in S$ or $\forall\nu\geq\sigma(\nu\notin S)$. By degree we mean Turing degree (of unsolvability). We call a degree n-generic if it has an n-generic representative. For a degree a, $D(\leq a)$ shows a set of degrees recursive in a.

The relation between n-generic degrees and minimal degrees is widely studied in Chong[1], Chong and Jockusch[2], Haught[3], Jockusch[4], and Kumabe[5]. Jockusch[4] showed that for each $n\geq2$, if a is n-generic and $0<b\leq a$ then there is an n-generic degree c with $c\leq b$. From this and the fact that no n-generic degree is minimal, he showed that any n-generic degree bounds no minimal degree. Chong and Jockusch[2] showed the same result for 1-generic degrees below 0'. Haught[3] showed a stronger result that if a is 1-generic below 0' and $0<b<a$ then b is also 1-generic. On the other hand Chong[1] and Kumabe[5] independently showed that there is a 1-generic degree which bounds a minimal degree. Further Chong[1] showed by a different method that there is a 1-generic degree which bounds a minimal degree below 0'. These results show an interesting downward homoginity property of $D(\leq a)$ for n-generic degrees a with $n\geq2$, but the same result does not hold for all 1-generic degrees.

As 1-generic degrees are not r.e., relative recursive enumerability of n-generic degrees is an interesting problem. Jockusch[4] showed if a is 1-generic there is a $c(<a)$ such that a is r.e. in c. So by the result of Haught[3] above, if a is a 1-generic degree below 0' then there is a 1-generic degree $b(<a)$

such that a is recursive enumerable in b. We show that for all $n \geq 1$ and for any n-generic degree a there is an n-generic degree $c(<a)$ such that a is r.e. in c. This answers to the question in Jockusch[4].

Our notation is standard. A string is a mapping from an initial segment of ω into $\{0,1\}$. Lower case Greek letters other than ω denote strings. For strings σ and ν, $\sigma \geq \nu$ denotes that σ extends ν. and in this case we say that ν is a substring of σ. Further σ and ν are said to be compatible if either extends the other. If σ and ν are incompatible we denote this by $\sigma|\nu$. We identify a set $A \leq \omega$ with its characteristic function. So $\sigma \leq A$ means that the characteristic function of A extends the string σ and in this case we say that σ is a beginning of A. We write $\sigma * \nu$ for the usual concatenation of σ and ν. We identify 0, 1 with the corresponding strings 0, 1 of length 1. We use i only for 0 or 1 and let $[i]=1-i$. Φ denotes the empty string. For each n, $i^{(n)}$ denotes the string σ of length n such that $\sigma(m)=i$ for all $m<n$. For a string σ, $|\sigma|$ denotes the length of σ and σ^{-} is the substring of σ such that $|\sigma^{-}|=|\sigma|-1$. Further for a number $m \leq |\sigma|$, $\sigma[m]$ is the substring of σ of length m. For two strings σ and ν, $\sigma \cap \nu$ is the substring λ of σ and ν such that for all $m<|\lambda|$ $\sigma(m)=\nu(m)$, and $\sigma(|\lambda|) \neq \nu(|\lambda|)$ or at least one of them are not defined. Let Φ_n be n-th partial recursive operator for some fixed recursive enumeration of all the partial recursive operators. Let $\Phi_n(\sigma)(x)=y$ mean that the n-th partial recursive operator with oracle σ and input $x<|\sigma|$, yields output y in at most $|\sigma|$ steps and further that $\Phi_n(\sigma)(u)$ is defined for all $u<x$. Similarly, for an enumeration procedure Ξ, we say that $\Xi(\sigma)(k)=1$ if there is a computation in Ξ with oracle σ enumerating k. Of course B is

recursive in A iff for some e, $\Phi_e(A)=B$. For two partial recursive operators (or enumeration operators) Ψ and Φ, $\Psi\geq\Phi$ denotes that for every string σ and every number n, $\Psi(\sigma)(n)=\Phi(\sigma)(n)$ whenever $\Phi(\sigma)(n)$ is defined. Strings σ and ν are called Φ_n(or n)-split if $\Phi_n(\sigma)$ and $\Phi_n(\nu)$ are incompatible.

The Result.

We first give two definitions and a lemma which will play an important role throughout the proof of the theorem.

Definition 1. Let Ψ be a partial recursive operator.

(1) σ is called Ψ-good if for any $\lambda\geq\Psi(\sigma)$ there is a $\tau\geq\sigma$ with $\Psi(\tau)\geq\lambda$.

(2) σ is called almost Ψ-good if there is a finite set F of strings such that

(2-i) for any $\tau\geq\sigma$ and $\delta\in F$, $\Psi(\tau)\not\geq\delta$,

(2-ii) there is a string $\nu\geq\Psi(\sigma)$ such that $\nu|\delta$ for any $\delta\in F$, and

(2-iii) For any string λ such that $\lambda\geq\Psi(\sigma)$ and $\lambda\not\geq\delta$ for any $\delta\in F$, there is a $\tau\geq\sigma$ with $\Psi(\tau)\geq\lambda$.

Definition 2. (1) A set S of strings is called dense if every string has an extension in S.

(2) A set P of strings is called strongly dense (s-dense) if for any nonrecursive set A and any beginning σ of A there is a beginning ν of A such that $\sigma\leq\nu$ and $\nu\in P$.

Clearly if σ is Ψ-good then σ is almost Ψ-good, and if a set P of strings is s-dense, P is dense. The next lemma corresponds to Lemma 4.6 in Jockusch[4].

Lemma 1. For all $n \geq 1$, if Ψ is a partial recursive operator and there is a dense Σ_n^0 (or s-dense) set P of almost Ψ-good strings, then $\Psi(A)$ is total and n-generic whenever A is n-generic.

Proof. Let F be the finite set of strings as defined in Definition 1-(2). To show that $\Psi(A)$ is total, let for each n, $S_n = \{\sigma: \Psi(\sigma)(n) \text{ is defined}\}$. Then S_n is a dense recursive set of strings. (In fact for any σ let ν be such that $\nu \in P$ and $\nu \geq \sigma$, and let $\nu' \geq \nu$ be such that $|\Psi(\nu')| > n$.) Then by the 1-genericity of A, for each n there is a $\sigma < A$ such that $\sigma \in S_n$. So $\Psi(A)$ is total. Next let S be an arbitrary Σ_n^0 set of strings. Let T be the set of strings ν such that $\Psi(\nu) \geq \lambda$ for some $\lambda \in S$. Then T is a Σ_n^0 set of strings. As A is n-generic, there is a $\nu \leq A$ such that $\nu \in T$ or no extension of ν is in T. If there is a $\nu \leq A$ such that $\nu \in T$ then $\Psi(A)$ extends some string λ in S. If there is a $\nu \leq A$ such that no extension of ν is in T then let $\delta \in P$ be a string such that $\nu \leq \delta < A$. (Such a δ exists because P is a dense Σ_n^0 (s-dense) set.) Since δ is almost Ψ-good, let λ be such that $\delta \leq \lambda < A$ and $\Psi(\lambda) | \tau$ for any $\tau \in F$ (such a λ exists as $\Psi(A)$ is total). As for any $\xi \geq \Psi(\lambda)$ there is a $\mu \geq \delta$ with $\Psi(\mu) \geq \xi$, it follows that no extension of $\Psi(\lambda)$ is in S. Since S was an arbitrary Σ_n^0 set of strings it follows that $\Psi(A)$ is n-generic.□

Theorem. For any $n \geq 1$ and any n-generic degree a, there is an n-generic degree $c (< a)$ such that a is recursive enumerable in c.

Proof. Let A be an n-generic set of degree a. We construct Ψ_s at stage s such that $\Psi_s \geq \Psi_{s-1}$ and $\bigcup_{s=0}^{\infty} \Psi_s = \Psi$ satisfies that $\Psi(A)$ is a set of the desired degree c. Also we construct an enumeration procedure Ξ_s at stage s such that $\Xi_s \geq \Xi_{s-1}$ and $\bigcup_{s=0}^{\infty} \Xi_s = \Xi$ enumerates A relative to $\Psi(A)$ (denote this by $\Xi(\Psi(A)) = A$).

Before constructing Ψ we give the abstract motivation of the construction. To prove the theorem we must construct a partial recursive operator Ψ and an enumeration procedure Ξ which satisfy the following conditions;

(1) $\Xi(\Psi(A)) = A$,

(2) $\Psi(A)$ is n-generic, and

(3) A is not recursive in $\Psi(A)$.

Within the motivation, we use letters α, β, γ to refer to conditions on A, and σ, τ, δ to refer to conditions on $\Psi(A)$. To satisfy (1), it is enough to arrange the following conditions:

(1-i) if $\Psi(\alpha) = \sigma$ and $\Xi(\sigma)(k) = 1$ then $\alpha(k) = 1$, and

(1-ii) if $\alpha(k) = 1$ there is an extension β of α with $\Xi(\Psi(\beta))(k) = 1$.

To satisfy (2), we construct a s-dense set G of Ψ-good strings. (As a matter of fact, we construct a dense recursive set G of almost Ψ-good strings, but here assume as above.) Then by Lemma 1 Ψ preserves n-genericity. To satisfy (3), by the diagonal requirement for each n we must satisfy $\Phi_n(\Psi(A)) \neq A$. In terms of dense sets, it is enough to arrange that for any α, there is a k and a $\beta \geq \alpha$ satisfying

(3-i) $\Phi_n(\Psi(\beta))(k)=0$ and $\beta(k)=1$,

(3-ii) $\Phi_n(\Psi(\beta))(k)=1$ and $\beta(k)=0$, or

(3-iii) there is no extension of $\Psi(\beta)$ that makes Φ_n converge at k.

The construction is organized in terms of strategies. During the course of executing a strategy we may take one of the following actions.

(A) Enumerate axioms into one or both of Ψ and Ξ.

(B) Prohibit such enumeration by strategies of lower priority. We restrain the enumeration of k above σ by prohibiting the enumeration of any axiom to the effect of $\Xi(\tau)(k)=1$ with $\tau \geq \sigma$. Similarly, we restrain Ψ away from σ above α by prohibiting the enumeration of any axiom $\Psi(\beta)=\tau$ with $\tau \geq \sigma$ and $\beta \geq \alpha$.

Note that restraint above α implies restraint above any extension of α.

There are four types of strategies to be considered here. Three of them are designed to satisfy the three types of requirements mentioned above. The remaining strategy is a global constraint imposed on the construction to simplify the analysis of the forcing relation during a typical step. The crux of the problem is, for each α, to understand what axioms enumerated so far imply about the values of Ψ on A or Ξ on $\Psi(A)$ when A extends α. In other words, given the axioms so far, what is the forcing relation for Ψ and Ξ. The analysis can be made very manageable by the following.

(I) For each stage and each condition α maintain the property that α has infinitely many extensions for which there are no axioms in Ψ other than those that already apply to α. Similarly, for each σ maintain the condition that σ has infinitely many extensions for which there are no axioms in Ξ other than those that already apply to σ.

These two property imply that at each stage s the axioms enumerated into Ψ and Ξ do no more than the following.

$\alpha \Vdash$ "$\Psi(A)$ extends σ" $\Longleftrightarrow$ $\Psi(\alpha)$ extends σ.

$\sigma \Vdash F \subsetneq A \Longleftrightarrow \forall_{k \in F}[\Xi(\sigma)(k)=1]$.

(II) To satisfy $\Xi(\Psi(A))=A$ impose:

(II-i) $\Psi(\alpha) \geq \sigma$ and $\Xi(\sigma)(k)=1$ implies $\alpha(k)=1$.

(II-ii) If $\alpha(k)=1$ then the enumeration of k cannot be restrained above α.

Assuming that the construction respects the conditions mentioned so far, for any stage of the construction and for any α, we are free to extend Ψ and Ξ so that there is an extension β of α with $\Psi(\beta)=\sigma$ and $\Xi(\sigma)(k)=1$. We can enumerate the relevant axioms and respect (I) by choosing β and σ to be sufficiently long length. Combining (I),(II) and the possibility of global restraint we obtain the following analysis of the forcing relation.

$\alpha \Vdash$-"$\Psi(A)$ does not extend σ" <===> one of:

(a) $\Psi(\alpha)$ is incompatible with σ.

(b) $\exists k[\alpha(k)=0$ & $\Xi(\sigma)(k)=1]$.

(c) Ψ is restraint away from σ above α.

These combine with earlier observation on the forcing relation to give a complete analysis. Both of the above strategies have a constant effect on the construction. In the case of (II), the strategy impose a global restraint and a stage by stage enumeration of axioms into Ψ and Ξ. However, it does not impose any coherent pattern to the length or distribution of these axioms.

(III) The third strategy is used to produce a Ψ-good condition extending α. First extend α to β and enumerate axioms into Ψ and Ξ so that if $\beta(k)=1$ then $\Xi(\Psi(\beta))(k)=1$. Note that there is no reason that the relevant axioms in Ξ cannot all have the same use, namely the length of $\Psi(\beta)$. We work under the assumption that no higher priority strategy imposes any restraints on the values of Ψ above β to restrain them away from extensions of $\Psi(\beta)$ and also that no higher priority strategy restrains the enumeration of any number greater than the length of β above $\Psi(\beta)$. For each n, impose the restraints that

(III-i) no axiom with use $\beta*1^{(n)}$ may be enumerated into Ψ. However axioms with use extending $\beta*1^{(n)}*0$ may be enumerated, (III-ii) if a strategy of lower priority restrains Ψ away from σ above $\beta*1^{(n)}$ and $\sigma\geq\Psi(\beta)$ then that same strategy provides a mechanism by which the range of Ψ on the conditions extending β is dense below σ, and

(III-iii) similarly, if $k\geq|\beta|$ and a strategy of lower priority restrains the enumeration of k above σ then that strategy provides a mechanism by which the range of Ψ on the conditions extending β is dense below σ.

Suppose that $\sigma\geq\Psi(\beta)$. Providing that the construction respects these conditions, either there is a τ extending σ such that we can enumerate an axiom $\Psi(\beta*1^{(n)}*0)=\tau$ or we can invoke a provided mechanism that enumerates an axiom putting an extension of σ in the rage of Ψ above β. Note that it is always safe to enumerate the axiom mentioned, for large enough n, and respect (II) since every number in $\Xi(\sigma)$ is already in $\beta*1^{(n)}$.

(IV) The final strategy is used to make the conditions forcing $\Phi_n(\Psi(A))\neq A$ dense for all n as in the statement (3-i),(3-ii),(3-iii). Begin with α and move to β as in (III). Let w be the length of β. Enumerate the axiom

$$\Psi(\beta*0)=\Psi(\beta)*0.$$

(IV)-(A) While there is no $\sigma\geq\Psi(\beta*0)$ with $\Phi_n(\sigma)(w)=0$, then

(1) restrain Ψ away from $\Psi(\beta*0)$ above any γ incompatible with $\beta*0$.

(2) restrain the enumeration of w above $\Psi(\beta*0)$,

(3) restrain all Ψ-axioms with use $\beta*1^{(n)}$ beyond those applying to β, and

(4) use the strategy described in (III) to make $\Psi(\beta*0)$ a Ψ-good condition.

(IV)-(B) When the first $\sigma\supseteq\Psi(\beta*0)$ is discovered with $\Phi_n(\sigma)(w)=0$, then drop the above restraints and extend Ψ and Ξ so that there is an n and a τ extending σ such that $\Psi(\beta*1^{(n)}*0)=\tau$.

Interference between strategies occurs when a diagonal strategy of type (IV) moves from condition (A) to (B). For example, the Ψ-good strategies (III) are injured in this case. Namely, when $\Psi(\beta*1^{(n)}*0)=\tau$ is enumetrated as above and a strategy S, of type (III), was attempting to make some γ with $\gamma\supseteq\beta*0$ and $\Psi(\beta*0)\subseteq\Psi(\gamma)\subseteq\tau$, a Ψ-good condition, S cannot be successful. By (II), τ must have some extension τ' with $\Xi(\tau')(w)=1$. But then also by (II), every condition extending $\beta*0$ is prohibited from being mapped by Ψ to such a τ'. Similarly, the restraint imposed by a type (IV) strategy may also be injured by a type (IV) strategy of higher priority. Luckily, a strategy of type (IV) acts at most one time if not itself injured. (Hence γ will be almost Ψ-good.)

During a stage s of the construction, we work to make sure that each condition of length less than s has an extension with an active strategy for each of the first s many requirements. Since the set of actions in the construction is Σ^0_1, any 1-generic set must meet this set infinitely often. By a Friedberg style finite injury argument, for any nonrecursive path (not necessarily generic) every requirement has infinitely many initial segments above which a strategy relevant to that requirement is active and never injured. But there is an important fact. The string γ in the previous paragraph is almost Ψ-good, and so all the strings are almost Ψ-good. To satisfy (2), by Lemma 1, it is enough to construct a dense recursive set of almost Ψ-good strings. So such a finite injury argument does not need. By the notion of "almost Ψ-good", the construction and the proof become extremely simple.

We now give the construction.

Construction.

Stage 0. Let $\Xi_0=\Psi_0=\Phi$. We call 0 maximal string at stage 0.

Stage n+1. For a string ν and a number m with $m<|\nu|$, let $Sub(\nu,m)$ be the substring δ of ν of length m, if any, such that $\delta*0\leq\nu^-$. For a maximal string σ at stage n, we say that σ needs m-attention at stage n+1 if

(1) $Sub(\sigma,m)$ is defined and it is not m-satisfied by the end of stage n, and

(2) $\Phi_m(\Psi_n(\sigma))(m)=0$.

If σ needs m-attention at stage n+1, let m_{n+1} be the least such number m, and let σ_{n+1} be the least such string σ in some fixed recursive enumeration of all the strings. We say $Sub(\sigma_{n+1},m_{n+1})$ is m_{n+1}-satisfied at stage n+1. Let τ_{n+1} be the maximal string at stage n such that $(\tau_{n+1})^-=Sub(\sigma,m)*1^{(k)}$ for some $k\geq0$. Enumerate the axioms:

$\Psi_{n+1}(\sigma_{n+1})=\Psi_n(\sigma_{n+1})*0$, $\Psi_{n+1}(\tau_{n+1}*0)=\Psi_{n+1}(\tau_{n+1})*0$,
$\Psi_{n+1}(\tau_{n+1}^-*1*0*0)=\Psi_n(\sigma_{n+1})*1$,

$\Psi_{n+1}(\tau_{n+1}^{-}*1*1*0)=\Psi_n(\tau_{n+1})*1$,

$\Xi_{n+1}(\Psi_{n+1}(\tau_{n+1}^{-}*1*0*0))(w)=1$ for any w such that

$m_{n+1}\le w\le|\tau_{n+1}^{-}|$.

We call σ_{n+1}, $\tau_{n+1}*0$, $\tau_{n+1}^{-}*1*0*0$ and $\tau_{n+1}^{-}*1*1*0$ maximal strings at stage n+1. For any maximal string δ at stage n such that $\delta\ne\sigma_{n+1},\tau_{n+1}$ if such σ_{n+1} and τ_{n+1} exist, enumerate the axioms:

$\Psi_{n+1}(\delta*0)=\Psi_n(\delta)*0$,

$\Psi_{n+1}(\delta^{-}*1*0)=\Psi_n(\delta)*1$, $\Xi_{n+1}(\Psi_{n+1}(\delta^{-}*1*0))(|\delta^{-}|)=1$.

We call $\delta*0$ and $\delta^{-}*1*0$ maximal strings at stage n+1. For any λ and k let

$\Psi_{n+1}(\lambda)=\cup\{\Psi_m(\lambda')\mid\exists m\le n+1\ [\lambda'\le\lambda\ \&\ \Psi_m(\lambda')$ is explicitly defined at stage m]}

$\Xi_{n+1}(\lambda)(k)=1$ if for some $\lambda'\le\lambda$ and $m\le n+1$, $\Xi_m(\lambda')(k)=1$ is explicitly defined at stage m,

$\Psi(\lambda)=\cup\{\Psi_m(\lambda')\mid\exists m\ [\lambda'\le\lambda\ \&\ \Psi_m(\lambda')$ is explicitly defined at stage m]}, and

$\Xi(\lambda)(k)=1$ if for some $\lambda'\le\lambda$ and m, $\Xi_m(\lambda')(k)=1$ is explicitly defined at stage m}

This completes the construction.

The next lemma follows directly from the construction.

<u>Lemma 2.</u> Let δ be a maximal string at stage n+1.

(1) If σ_{n+2} is defined and $\sigma_{n+2}=\delta$, then δ is a maximal string at stage n+2 and $\Psi_{n+2}(\delta)=\Psi_{n+1}(\delta)*0$. If σ_{n+2} is defined and $\tau_{n+1}=\delta$, then $\delta*0$, $\delta^{-}*1*0*0$ and $\delta^{-}*1*1*0$ are maximal strings at stage n+2. $\Psi_{n+2}(\delta*0)=\Psi_{n+1}(\delta)*0$, $\Psi_{n+2}(\delta^{-}*1*0*0)=\Psi_{n+1}(\sigma_{n+2})*1$, and $\Psi_{n+2}(\delta^{-}*1*1*0)=\Psi_{n+1}(\delta)*1$. Otherwise then $\delta*0$ and $\delta^{-}*1*0$ are maximal strings at stage n+2, $\Psi_{n+2}(\delta*0)=\Psi_{n+1}(\delta)*0$, and $\Psi_{n+2}(\delta^{-}*1*0)=\Psi_{n+1}(\delta)*1$.

(2) $\delta(|\delta|-1)=0$.

(3) $|\Psi_{n+1}(\delta)|=n+1$.

(4) If λ is a maximal string at stage n+1 then $\delta|\lambda$ iff $\delta^{-}|\lambda^{-}$ iff $\delta\neq\lambda$ iff $\Psi_{n+1}(\lambda)\neq\Psi_{n+1}(\delta)$.

(5) If $\lambda<\delta$ then $\Psi_{n+1}(\delta)>\Psi_{n+1}(\lambda)=\Psi_m(\lambda)$ for all $m\geq n$ (so $\Psi_n(\lambda)=\Psi(\lambda)$).

(6) If $\lambda\geq\delta^{-}*1$ then $\Psi_{n+1}(\delta)>\Psi_{n+1}(\lambda)=\Psi_n(\lambda)$.

(7) If $\lambda<\delta$ then there is unique maximal string τ at stage n+1 such that $\tau^{-}=\lambda*1^{(k)}$ for some $k\geq0$.

(8) α is a maximal string at stage n+1 iff $\Psi_{n+1}(\alpha)$ is explicitly defined at stage n+1 iff for any $\beta\geq\alpha$ $\Psi_{n+1}(\beta)=\Psi_{n+1}(\alpha)$ and $\Psi_{n+1}(\alpha^{-})<\Psi_{n+1}(\alpha)$.

(9) $\delta(k)=1$ iff $\Xi_{n+1}(\Psi_{n+1}(\delta))(k)=1$.

(10) If $Sub(\sigma_n,m_n)$ is m_n-satisfied at stage n then for any $s>n$, $Sub(\sigma_s,m_s)\neq Sub(\sigma_n,m_n)$ whenever σ_s is defined.

(11) If σ_{n+2} is not defined or it is defined and $\delta\neq\sigma_{n+2}$ then δ is not a maximal string at stage n+2.

(12) For any string λ and any number n, there is a maximal string τ at stage n such that λ and τ^{-} are compatible.

(13) For each n, if $\Xi_n(\sigma)(k)=1$ is explicitly defined at stage n then $|\sigma|=n$, and there is unique maximal string α at stage

n such that $\alpha(k)=1$ and $\Psi_n(\alpha)=\sigma$. So for each m with $n\geq m$, no axiom of the form $\Xi_m(\sigma)(k)=1$ with $|\sigma|>n$ is enumerated at stage m.

(14) If $\Psi_n(\alpha)\geq\sigma$ and $\Xi(\sigma)(k)=1$ for some α then $\Xi_n(\sigma)(k)=1$.

Proof. (1),$\cdots$,(11) Clear by the construction using induction on stage n.

(12) Clear by (1) and the construction by using induction on stage n.

(13) Clear by (3) and (4).

(14) Clear by (3),(8) and (13).□

By Lemma 2-(10) for each σ let $F(\sigma)$ be the least stage n such that for any stage $s\geq n$, if σ_s is defined then $Sub(\sigma_s,m_s)\not\leq\sigma$.

Clearly Ξ is consistently defined.

Lemma 3. Ψ is consistently defined, i.e. for all n,

(1) if σ_n is defined then τ_n is also defined, and

(2) for any strings λ,τ if $\lambda\geq\tau$ then $\Psi_n(\lambda)\geq\Psi_n(\tau)$.

Proof. (1) is clear by Lemma 2-(7).

(2) We prove (2) by induction on n. Assume the lemma holds for n. Let λ and τ be such that $\lambda\geq\tau$. By Lemma 2-(12) let δ be a maximal string at stage n+1 such that δ^- is compatible with λ. If $\delta^->\lambda$ then by Lemma 2-(5) $\Psi_{n+1}(\lambda)=\Psi_n(\lambda)\geq\Psi_n(\tau)=\Psi_{n+1}(\tau)$. If $\lambda\geq\delta*0(=\delta)>\tau$ then by Lemma 2-(5)(8) $\Psi_{n+1}(\lambda)\geq\Psi_{n+1}(\tau)=\Psi_n(\tau)$. If $\lambda\geq\delta^-*1>\tau$ then by Lemma 2-(5)(6) $\Psi_{n+1}(\lambda)=\Psi_n(\lambda)\geq\Psi_n(\tau)=\Psi_{n+1}(\tau)$. If $\tau\geq\delta^-*0(=\delta)$ then by Lemma 2-(8) $\Psi_{n+1}(\lambda)=\Psi_{n+1}(\tau)$. Finally if $\tau\geq\delta^-*1$ then by lemma 2-(6) $\Psi_{n+1}(\lambda)=\Psi_n(\lambda)\geq\Psi_n(\tau)=\Psi_{n+1}(\tau)$.□

Lemma 4. Let σ be an arbitrary string. If $\sigma \neq \Phi$ and $\sigma(|\sigma|-1)=0$, then

(i) $\sigma*0$ and $\sigma^{-}*1*0$ are maximal strings at some stage, or

(ii) $\sigma*0$, $\sigma^{-}*1*0*0$ and $\sigma^{-}*1*1*0$ are maximal strings at some stage.

and if $\sigma=\Phi$, or $\sigma \neq \Phi$ and $\sigma(|\sigma|-1)=1$ then

(iii) $\sigma*0$ is a maximal string at some stage, or

(iv) $\sigma*0*0$ and $\sigma*1*0$ are maximal strings at some stage.

So there is a maximal string λ at some stage with $\lambda \geq \sigma$.

Proof. We proceed by induction on the length of σ. First by the construction, 0 is a maximal string at stage 0. So the lemma holds for empty string. Let σ be an arbitrary string with $|\sigma| \geq 1$. If $|\sigma| \geq 2$ let $\sigma(|(\sigma^{-})^{-}|)=i$.

If (1) $|\sigma|=1$, (2) $|\sigma| \geq 2$ and $i=0$, or (3) $|\sigma| \geq 2$, $i=1$ and (iii) holds for σ^{-}, then by the inductive hypothesis $\sigma^{-}*0$ is a maximal string at some stage s. Let $\delta=\sigma^{-}*0$. If δ is a maximal string at stage t for any $t \geq s$, then for any $t>s$, σ_t is defined and $\sigma_t=\delta$ by Lemma 2-(10). So $Sub(\sigma_t, m_t) \leq \delta$. But if $t \geq F(\delta)$ this is a contradiction to the assumption on $F(\delta)$. So let $t>s$ be the least stage such that δ is not a maximal string at stage t. Then by Lemma 2-(1),

(A) $\delta=\tau_t$, and $\delta*0$, $\delta^{-}*1*0*0$ and $\delta^{-}*1*1*0$ are maximal strings at stage t, or

(B) $\delta*0$ and $\delta^{-}*1*0$ are maximal strings at stage t.

If (A) is the case and $\sigma=\sigma^{-}*0(=\delta)$ then (ii) holds for σ. If (A) is the case and $\sigma=\sigma^{-}*1(=\delta^{-}*1)$ then (iv) holds for σ. If (B) is the case and $\sigma=\sigma^{-}*0(=\delta)$ then (i) holds for σ. If (B) is the case and $\sigma=\sigma^{-}*1(=\delta^{-}*1)$ then (iii) holds for σ. In all cases, the lemma holds.

Next assume $|\sigma|\geq 2$, $i=1$ and (iv) holds for σ^{-}, i.e. $\sigma^{-}*0*0$ and $\sigma^{-}*1*0$ are maximal strings at some stage s. If $\sigma=\sigma^{-}*0$ then (i) holds for σ. If $\sigma=\sigma^{-}*1$ then (iii) holds for σ. In all cases the lemma holds.□

Definition 3. We say that a string σ is almost Ψ_{s+1}-good at stage s+1 if for any maximal string δ at stage s with $\delta\geq\sigma$, there are maximal strings λ_0,λ_1 at stage s+1 such that $\lambda_i\geq\sigma$ and $\Psi_{s+1}(\lambda_i)=\Psi_s(\delta)*i$ for each i.

Lemma 5. $\Psi(A)$ is total and n-generic.

Proof. Let σ be an arbitrary string. By Lemma 1 it suffices to show that σ is almost Ψ-good. By Lemma 4 let n be such that there is a maximal string λ at stage n with $\lambda>\sigma$. Let $x=\max\{F(\sigma),n\}$.

Let F be the set of strings τ such that $|\tau|=x$ and $\tau|\Psi_x(\lambda)$ for any maximal string λ at stage x with $\lambda\geq\sigma$. Clearly F is finite. By Lemma 2-(3), $|\Psi_x(\lambda)|=x$ for any maximal string λ at stage x. So any string μ of length x is either an element of F or $\mu=\Psi(\lambda)$ for some maximal string λ at stage x. By Lemma 2-(1), for any $v>x$ and any maximal string λ at stage v, $\Psi_v(\lambda)=\Psi_{v-1}(\lambda')*i$ for some i and maximal string λ' at stage v-1.

We first show that $\lambda'>\sigma$ whenever $\lambda>\sigma$.····················(*)

Assume $\lambda>\sigma$. (1) If $\lambda'=\sigma_v$ then $\lambda=\sigma_v$, so $\lambda'>\sigma$. (2) If $\lambda'=\tau_v$ then $\lambda=\lambda'*0$, $(\lambda')^-*1*0*0$, or $(\lambda')^-*1*1*0$. (2-i) If $\lambda=\lambda'*0$ then λ' and σ are compatible. As λ' is a maximal string at stage $v-1\geq x$, $\lambda'>\sigma$ by the assumption on x. (2-ii) If $\lambda=(\lambda')^-*1*0*0$ or $(\lambda')^-*1*1*0$ then assume for a contradiction that $\lambda'\not>\sigma$. By the assumption on x, $\lambda'\neq\sigma$. As $\lambda'=(\lambda')^-*0$ by Lemma 2-(2), $\sigma\geq(\lambda')^-*1$. By Lemma 2-(3)(6), no string extending $(\lambda')^-*1$ is a maximal string at stage $v-1(\geq x)$. This is a contradiction to the assumption on x. (3) Otherwise $\lambda=\lambda'*0$ or $(\lambda')^-*1*0$. If $\lambda=\lambda'*0$ then the proof is exactly same as (2-i). If $\lambda=(\lambda')^-*1*0$ then the proof is exactly same as (2-ii).

If for some $v>x$, $\lambda>\sigma$, and $\tau\in F$, $\Psi_v(\lambda)\geq\tau$ then let v be the least such stage. Further let λ' be the least substring of λ such that $\Psi_v(\lambda)=\Psi_v(\lambda')$. Then by the construction and Lemma 2-(8), $\Psi_v(\lambda')$ is explicitly defined at stage v and λ' is a maximal string at stage v. Then by (*), $\Psi_v(\lambda')=\Psi_{v-1}(\delta)*i$ for some i and maximal string δ at stage $v-1$ with $\delta>\sigma$. By Lemma 2-(3), $|\Psi_{v-1}(\delta)|=v-1\geq x$, so $\Psi_{v-1}(\delta)\geq\tau(\in F)$. By the assumption on v, $v-1=x$. But this is a contradiction to the definition on F. Hence for any $v>x$, $\lambda>\sigma$, and $\tau\in F$, $\Psi_v(\lambda)\not\geq\tau$. So it suffices to show that σ is almost Ψ_s-good at stage s for all $s>x$. Let s be an arbitrary number with $s\geq x$, and δ be an arbitrary maximal string at stage s with $\delta>\sigma$. If σ_{s+1} is defined at stage s+1 then $\mathrm{Sub}(\sigma_{s+1},m_{s+1})\not\geq\sigma$ by the assumption on $F(\sigma)$. (A) If $\delta=\sigma_{s+1}$ then, as $\sigma_{s+1}\geq\mathrm{Sub}(\sigma_{s+1},m_{s+1})$, $(\tau_{s+1})^-\geq\mathrm{Sub}(\sigma_{s+1},m_{s+1})\geq\sigma$. Further by Lemma 2-(1), $\Psi_{s+1}(\delta)=\Psi_s(\delta)*0$, $\Psi_{s+1}((\tau_{s+1})^-*1*0*0)=\Psi_s(\delta)*1$, and δ and $(\tau_{s+1})^-*1*0*0$ are maximal strings at stage s+1. (B) If $\delta=\tau_{s+1}$ then by Lemma 2-(1), $\Psi_{s+1}(\delta*0)=\Psi_s(\delta)*0$, $\Psi_{s+1}(\delta^-*1*1*0)=\Psi_s(\delta)*1$, and $\delta*0$ and $\delta^-*1*1*0$ are maximal strings at stage s+1. (C)

Otherwise by Lemma 2-(1), $\Psi_{s+1}(\delta*0)=\Psi_s(\delta)*0$, $\Psi_{s+1}(\delta^-*1*0)=\Psi_s(\delta)*1$, and $\delta*0$ and $\delta^-*1*0*0$ are maximal strings at stage s+1. In all cases, σ is almost Ψ_{s+1}-good at stage s+1.□

Lemma 6. $\Xi(\Psi(A))=A$.

Proof. It suffices to show that for any numbers s,k and any strings α,σ,

(1) if $\Psi_s(\alpha)\geq\sigma$ and $\Xi_s(\sigma)(k)=1$ then $\alpha(k)=1$, and

(2) if $\alpha(k)=1$ then there is an extension β of $\alpha[k+1]$ such that $\Xi(\Psi(\beta))(k)=1$.

(1) We proceed by induction on stage s. Assume $\Psi_s(\alpha)\geq\sigma$ and $\Xi_s(\sigma)(k)=1$. If $\Psi_{s-1}(\alpha)\geq\sigma$ then by Lemma 2-(14), $\Xi_{s-1}(\sigma)(k)=1$, so by the inductive hypothesis $\alpha(k)=1$. If $\Psi_{s-1}(\alpha)<\sigma$ then $\Psi_{s-1}(\alpha)<\Psi_s(\alpha)$. By Lemma 2-(12) let α_o be a maximal string at stage s such that $(\alpha_o)^-$ and α are compatible. If $(\alpha_o)^-\geq\alpha$ then by Lemma 2-(5) $\Psi_{s-1}(\alpha)=\Psi_s(\alpha)$, which is a contradiction. If $\alpha\geq(\alpha_o)^-*1$ then by Lemma 2-(6) $\Psi_{s-1}(\alpha)=\Psi_s(\alpha)$, also a contradiction. So $\alpha\geq(\alpha_o)^-*0(=\alpha_o$ by Lemma 2-(2)). Hence $|\Psi_s(\alpha)|=s$ and $|\Psi_{s-1}(\alpha)|=s-1$ by Lemma 2-(3)(8). So $\Psi_s(\alpha)=\sigma$ and $\Psi_{s-1}(\alpha)=\sigma^-$. If $\Xi_{s-1}(\sigma)(k)=1$ then $\Xi_{s-1}(\sigma^-)(k)=\Xi_{s-1}(\Psi_{s-1}(\alpha))(k)=1$ by Lemma 2-(13). So by the inductive hypothesis $\alpha(k)=1$. If $\Xi_{s-1}(\sigma)(k)$ is not defined then $\Xi_s(\sigma)(k)=1$ is explicitly defined at stage s and $\alpha_o(k)=1$ by Lemma 2-(13) and the fact that $\Psi_s(\alpha_o)=\sigma$. As $\alpha\geq\alpha_o$, $\alpha(k)=\alpha_o(k)=1$.

(2) Assume $\alpha(k)=1$. By Lemma 4, let n be such that β is a maximal string at stage n for some $\beta\geq\alpha[k+1]$. Then by Lemma 2-(9) $\Xi_n(\Psi_n(\beta))(k)=1$.□

Lemma 7. A is not recursive in $\Psi(A)$.

Proof. It suffices to show that $\Phi_n(\Psi(A))\neq A$ for all n. Let n be an arbitrary number. Let R be a infinite recursive set of numbers such that $\Phi_n=\Phi_m$ whenever $m\in R$. Let m and δ be such that $m\in R$ and $|\delta|=m$. By the 1-genericity of A it suffices to show that

(1) for any $\lambda\geq\delta*0$, $\Phi_m(\Psi(\lambda))(m)$ is not defined, or

(2) for some $\lambda\geq\delta*1$, $\Phi_m(\Psi(\lambda))(m)=0$.

Assume for a contradiction that for some $\lambda'\geq\delta*0$, $\Phi_m(\Psi(\lambda))(m)=0$ and there is no string $\mu\geq\delta*1$ with $\Phi_m(\Psi(\mu))(m)=0$. Let t' be such that $t'\geq\max\{F(\delta):|\delta|=m\}$ and $\Phi_m(\Psi_{t'}(\lambda'))(m)=0$. By Lemma 4 let t and λ be such that $t\geq t'$, $\lambda\geq\lambda'$ and λ is a maximal string at stage t. Then λ needs m-attention at stage t+1, and m is the least such number by the assumption on t. So $Sub(\lambda,m)=\delta$ is m-satisfied at stage t+1 and $\Psi_{t+1}((\tau_{t+1})^{-}*1*0*0)\geq\Psi_t(\lambda)\geq\Psi_t(\lambda')$. Hence $\Phi_m(\Psi_{t+1}((\tau_{t+1})^{-}*1*0*0)(m)=0$. But $(\tau_{n+1})^{-}*1\geq Sub(\lambda',m)(=\delta)*1$. This is a contradiction. □

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