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A note on Schmidt's built-up systems of fundamental sequences

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Introduction. Let $\langle F_\alpha \rangle_{\alpha\in\Delta}$ be a transfinite sequence of number theoretic functions indexed by an initial segment Δ of the second number class which satisfies the following conditions:

(a) F_0 is strictly increasing,

(b) if F_α is strictly increasing, $F_{\alpha+1}$ is also strictly increasing, $F_\alpha(0) \leq F_{\alpha+1}(0)$ and $F_\alpha(x) < F_{\alpha+1}(x)$ for $0 < x < \omega$,

(c) $F_\alpha(x) = F_{\alpha[x]}(x)$ if α is a limit ordinal, where $\langle \alpha[x] \rangle_{x<\omega}$ is a fundamental sequence for α.

Schmidt[3] introduced the concept of built-up systems of fundamental sequences, and showed that, for the above sequence $\langle F_\alpha \rangle_{\alpha\in\Delta}$, each F_α is strictly increasing if the system of fundamental sequences used is built-up. However there are some standard systems of fundamental sequences in literatures, e.g. Ketonen and Solovay[1], which are not built-up in Schmidt's sense, but which determine a sequence of strictly increasing functions.

The purpose of this note is to extend the concept of built-up systems so that it can be applicable to wider classes of systems of sequences of ordinals.

In §1 we define (n)-built-up systems and quasi-(n)-built-up systems of sequences of ordinals. In §2 we show a theorem on a relation between (n)-built-up systems and $\langle F_\alpha \rangle_{\alpha\in\Delta}$, which corresponds to Theorem 1 in [3], and a theorem on a relation between quasi-(n)-built-up systems and a sequence of number theoretic functions $\langle H_\alpha \rangle_{\alpha \in \Delta}$. In §3, we give an example of (1)-built-up system of fundamental sequences for Γ_0. Finally, in §4, we extend the results in Schmidt[4], by using quasi-(n)-built-up systems of sequences of ordinals.

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§1. Preliminaries

Let Δ be an initial segment of second number class. We will use Greek letters α, β, γ, ... for ordinal numbers in Δ. Let $P:\Delta \longrightarrow \Delta^\omega$ be an assignment of sequences of ordinals for Δ. We shall write $\alpha[i]$ for $(P(\alpha))(i)$ whenever $\alpha \in \Delta$ and $i < \omega$.

If P satisfies the following conditions (A)-(C):

(A) $\alpha[i] = 0$ if $\alpha = 0$ and $i < \omega$,

(B) $\alpha[i] = \beta$ if $\alpha = \beta + 1$ and $i < \omega$,

(C) $\alpha[i] < \alpha$ if α is a limit ordinal and $i < \omega$,

then we call P a <u>system of sequences of ordinals</u> for Δ. Moreover, if a system P of sequences of ordinals satisfies the following conditions $(C)^+$,(D):

$(C)^+$ $\alpha[i] < \alpha[i+1] < \alpha$ if α is a limit ordinal and $i < \omega$,

(D) $\lim_{i<\omega} \alpha[i] = \alpha$ if α is a limit ordinal,

then we call P a <u>system of fundamental sequences</u> for Δ.

In the following, we assume P is a system of sequences of ordinals.

Definition 1.1. Let P be a system of sequences of ordinals for Δ. For each $n < \omega$, $\xrightarrow[n]{1}$, $\xrightarrow[n]{}$, $\overset{}{\underset{n}{\Longrightarrow}}$ are defined as follows:

(1) $\alpha \xrightarrow[n]{1} \beta$ iff $0 < \alpha$ and $\alpha[n] = \beta$,

(2) $\alpha \xrightarrow[n]{} \beta$ iff there is a sequence $\gamma_0, \ldots, \gamma_j$ $(0<j<\omega)$ such that $\gamma_0 = \alpha$, $\gamma_j = \beta$ and $\gamma_i \xrightarrow[n]{1} \gamma_{i+1}$ $(0 \le i < j)$,

(3) $\alpha \underset{n}{\Longrightarrow} \beta$ iff $\alpha \xrightarrow[n]{} \beta$ or $\alpha = \beta$.

For each $n < \omega$, P is (n)-built-up (and quasi-(n)-built-up), if $\alpha[i+1] \xrightarrow[n]{} \alpha[i]$ (and $\alpha[i+1] \underset{n}{\Longrightarrow} \alpha[i]$, respectively) for each limit ordinal and each $i < \omega$.

Built-up systems in Schmidt's sense[3] is the same as (0)-built-up systems of fundamental sequences in our sense. Ketonen and Solovay[1] introduced the relation $\xrightarrow[n]{}$ for studying a standard system of fundamental sequences for ordinals up to ε_0. Their system is (1)-built-up but not (0)-built-up (cf.Theorem 2.4 of [1]).

Proposition 1.1. Let P be quasi-(s)-built-up. If $s \le m$, $n \le m$ and $\alpha \xrightarrow[n]{} \beta$, then $\alpha \xrightarrow[m]{} \beta$.

(Proof) By induction on α. Case 1. $\alpha = 0$. This case is trivial because $\neg(0 \xrightarrow[n]{} \beta)$. Case 2. $\alpha=\gamma+1$. If $\alpha \xrightarrow[n]{} \beta$, then $\gamma \underset{n}{\Longrightarrow} \beta$. So $\alpha=\gamma+1 \xrightarrow[m]{1} \gamma$ and $\gamma \underset{m}{\Longrightarrow} \beta$ by ind. hyp. So $\alpha \xrightarrow[m]{} \beta$. Case 3. α is limit. If $\alpha \xrightarrow[n]{} \beta$ then $\alpha[n] \underset{n}{\Longrightarrow} \beta$. Because P is quasi-(s)-built-up, $\alpha[m] \underset{s}{\Longrightarrow} \alpha[n]$. By ind. hyp., $\alpha[m] \underset{m}{\Longrightarrow} \alpha[n]$ and

$\alpha[n] \underset{m}{\Longrightarrow} \beta$. So, $\alpha \xrightarrow[m]{1} \alpha[m] \underset{m}{\Longrightarrow} \alpha[n] \underset{m}{\Longrightarrow} \beta$.

Corollary 1.2. Let $n \leq m$. If P is (n)-built-up (and quasi-(n)-built-up), then P is (m)-built-up (and quasi-(m)-built-up, respectively).

§2. (n)-built-up systems and hierarchies of number theoretic functions

We say that a function $F:\omega \longrightarrow \omega$ is strictly increasing after n if $F(x) < F(x+1)$ for $n \leq x < \omega$.

Let P be a system of sequences of ordinals for Δ. Suppose that $\langle F_\alpha \rangle_{\alpha \in \Delta}$ is any sequence of number theoretic functions satisfying the following conditions for $n < \omega$.

$(a)_n$ F_0 is strictly increasing after n.

$(b)_n$ If F_α is strictly increasing after n, then
$F_{\alpha+1}$ is also strinctly increasing after n,
$F_\alpha(n) \leq F_{\alpha+1}(n)$, and $F_\alpha(x) < F_{\alpha+1}(x)$ for $n < x < \omega$.

$(c)_n$ $F_\alpha(x) = F_{\alpha[x]}(x)$ for $n \leq x < \omega$, if α is limit.

Remark that conditions $(a)_0$, $(b)_0$ and $(c)_0$ are the same as (a), (b) and (c) in Introduction.

Example 1. Let the fast growing hierarchy $\langle F_\alpha \rangle_{\alpha \in \Delta}$ define by

$F_0(x) = x+1$, $F_{\alpha+1}(x) = F_\alpha^{x+1}(x)$, where F_α^i is defined by
$F_\alpha^0(x)=x$, $F_\alpha^{i+1}(x)=F_\alpha^i(F_\alpha(x))$,
$F_\alpha(x) = F_{\alpha[x]}(x)$ if α is limit.

Then $\langle F_\alpha \rangle_{\alpha \in \Delta}$ satisfies $(a)_n$, $(b)_n$ and $(c)_n$.

Theorem 2.1. If $\langle F_\alpha \rangle_{\alpha \in \Delta}$ satisfies conditions $(a)_n$, $(b)_n$ and $(c)_n$, then the following hold for each α, $\beta \in \Delta$.

(1) $\alpha \xrightarrow[n]{} \beta$ implies $F_\beta(n) \le F_\alpha(n)$.

(2) If P is (n+1)-built-up, then

(2.1) F_α is strictly increasing after n,

(2.2) $\alpha \xrightarrow[m]{} \beta$ implies $F_\beta(s) \le F_\alpha(s)$, $F_\beta(x) < F_\alpha(x)$ for $s < x < \omega$, where $s = \max(n+1, m)$.

Moreover, if P is (n)-built-up and $m \le n$, then $\alpha \xrightarrow[m]{} \beta$ implies $F_\beta(n) \le F_\alpha(n)$, $F_\beta(n+1) < F_\alpha(n+1)$.

(Proof) By induction on α. Case 1. $\alpha = 0$. (1) holds because $\neg(0 \xrightarrow[n]{} \beta)$. (2.1) is $(a)_n$, (2.2) holds because $\neg(0 \xrightarrow[n]{} \beta)$.

Case 2. $\alpha=\gamma+1$. (1) If $\alpha \xrightarrow[n]{} \beta$ then $\gamma \overset{}{\underset{n}{\Longrightarrow}} \beta$. $F_\beta(n) \le F_\gamma(n) \le F_\alpha(n)$ by ind.hyp. and $(b)_n$. (2.1) Since F_γ is strictly increasing after n, by ind. hyp., so is F_α by $(b)_n$. (2.2) Assume $\alpha \xrightarrow[m]{} \beta$. Then $\gamma \underset{m}{\Longrightarrow} \beta$. If $\beta = \gamma$, (2.2) holds by $(b)_n$. Hence, by ind. hyp. (2.2) holds for all $\alpha \xrightarrow[m]{} \beta$.

Case 3. α is limit. (1) If $\alpha \xrightarrow[n]{} \beta$ then $\alpha[n] \underset{n}{\Longrightarrow} \beta$. So

$$F_\beta(n) \le F_{\alpha[n]}(n) = F_\alpha(n) \quad \text{by (1) of ind. hyp.}$$

(2) Let P be (n+1)-built-up. (2.1) For $n \le x < \omega$, $\alpha[x+1] \xrightarrow[n+1]{} \alpha[x]$. So,

$$\begin{aligned} F_\alpha(x+1) = F_{\alpha[x+1]}(x+1) &\ge F_{\alpha[x]}(x+1) && \text{by (2.2) of ind.hyp.} \\ &> F_{\alpha[x]}(x) && \text{by (2.1) of ind.hyp.} \\ &= F_\alpha(x). \end{aligned}$$

(2.2) If $\alpha \xrightarrow[m]{} \beta$ then $\alpha[m] \underset{m}{\Longrightarrow} \beta$. Let $s = \max(n+1,m)$. Then

$$F_\beta(s) \le F_{\alpha[m]}(s) \le F_{\alpha[s]}(s) = F_\alpha(s)$$

by $\alpha[m] \underset{n+1}{\Longrightarrow} \alpha[s]$ and ind.hyp.

$F_\beta(x) \le F_{\alpha[m]}(x) < F_{\alpha[x]}(x) = F_\alpha(x)$

for $s < x < \omega$ by $\alpha[x] \xrightarrow[n+1]{} \alpha[m]$, and ind.hyp. Moreover if P is (n)-built-up and $m \le n$, then $\alpha[n] \overset{}{\underset{n}{\Longrightarrow}} \alpha[m]$ and $\alpha[n+1] \xrightarrow[n]{} \alpha[m]$,

$F_\beta(n) \le F_{\alpha[m]}(n) \le F_{\alpha[n]}(n) = F_\alpha(n)$,

$F_\beta(n+1) \le F_{\alpha[m]}(n+1) < F_{\alpha[n+1]}(n+1) = F_\alpha(n+1)$, by ind.hyp.

<u>Corollary 2.2.</u> If $\langle F_\alpha \rangle_{\alpha\in\Delta}$ satisfies (a),(b) and (c) (i.e. $(a)_0$,$(b)_0$ and $(c)_0$), and P is (1)-built-up, then for each α, F_α is strictly increasing.

Next, we will introduce another hierarchies $\langle H_\alpha \rangle_{\alpha\in\Delta}$ of number theoretic functions which is constructed by the same way as $\langle F_\alpha \rangle_{\alpha\in\Delta}$ except $(c)_n$ (i.e. $F_\alpha(x) = F_{\alpha[x]}(x)$ if α is a limit ordinal).

We fix a system P of sequences of ordinals for Δ, and a function $f:\omega \longrightarrow \omega$ which satisfies $n < f(n)$ and $f(x) < f(x+1)$ for $n \le x < \omega$ (e.g. $f(x) = x + 1$).

Suppose $\langle H_\alpha \rangle_{\alpha\in\Delta}$ is any sequence of number theoretic functions satisfying the following conditions for n.

$(a)_n$ and $(b)_n$ are the same as the case of $\langle F_\alpha \rangle_{\alpha\in\Delta}$,

$(c)^*_n$ $H_\alpha(x) = H_{\alpha[x]}(f(x))$ for $n \le x < \omega$, if α is limit.

<u>Example 2.</u> Let the Hardy hierarchy $\langle H_\alpha \rangle_{\alpha\in\Delta}$ define by

$H_0(x) = x$, $H_\alpha(x) = H_{\alpha[x]}(x+1)$ if $\alpha > 0$.

Then $\langle H_\alpha \rangle_{\alpha\in\Delta}$ satisfies $(a)_n$, $(b)_n$ and $(c)^*_n$ (where we take x+1 as f(x)).

We can prove a theorem which is a relation between quasi-(n)-built-up systems and $\langle H_\alpha \rangle_{\alpha \in \Delta}$.

<u>Theorem 2.3.</u> If $\langle H_\alpha \rangle_{\alpha \in \Delta}$ satisfies conditions $(a)_n$, $(b)_n$ and $(c)^*_n$ and P is quasi-(f(n+1))-built-up, then for each $\alpha \in \Delta$,

(1) H_α is strictly increasing after n,

(2) $\alpha \underset{m}{\longrightarrow} \beta$ implies $H_\beta(x) < H_\alpha(x)$ for $\max(n+1,m) \le x < \omega$.
In addition, $\alpha \underset{n}{\longrightarrow} \beta$ implies $H_\beta(n) \le H_\alpha(n)$.
Moreover, if P is quasi-(f(n))-built-up and $m < n$, then $\alpha \underset{m}{\longrightarrow} \beta$ implies $H_\beta(n) \le H_\alpha(n)$.

(Proof) By induction on α. Assume that P is quasi-(f(n+1))-built-up. <u>Case 1.</u> $\alpha=0$. (1) is $(a)_n$. (2) holds because $\neg(0 \underset{m}{\longrightarrow} \beta)$.
<u>Case 2.</u> $\alpha=\gamma+1$. (1) Since H_γ is strictly increasing after n by ind.hyp., so is H_α by $(b)_n$. (2) Assume $\alpha \underset{m}{\longrightarrow} \beta$. Then $\gamma \underset{m}{\Longrightarrow} \beta$. If $\beta = \gamma$, (2) holds by $(b)_n$. Hence, by ind. hyp., (2) holds for all $\alpha \underset{m}{\longrightarrow} \beta$.
<u>Case 3.</u> α is limit. (1) For $n \le x < \omega$, $\alpha[x+1] \underset{f(n+1)}{\Longrightarrow} \alpha[x]$, $f(n+1) \le f(x+1)$,

$$\begin{aligned} H_\alpha(x+1) = H_{\alpha[x+1]}(f(x+1)) &\ge H_{\alpha[x]}(f(x+1)) && \text{by (2) of ind.hyp.} \\ &> H_{\alpha[x]}(f(x)) && \text{by (1) of ind.hyp.} \\ &= H_\alpha(x). \end{aligned}$$

(2) If $\alpha \underset{m}{\longrightarrow} \beta$ then $\alpha[m] \underset{m}{\Longrightarrow} \beta$. For $\max(n+1,m) \le x < \omega$, by ind. hyp. and $\alpha[x] \underset{f(n+1)}{\Longrightarrow} \alpha[m]$, $f(n+1) \le f(x)$, so

$$H_\beta(x) \le H_{\alpha[m]}(x) < H_{\alpha[m]}(f(x)) \le H_{\alpha[x]}(f(x)) = H_\alpha(x).$$

If $\alpha \underset{n}{\longrightarrow} \beta$ then $\alpha[n] \underset{n}{\Longrightarrow} \beta$. By ind. hyp.,

$$H_\beta(n) \le H_{\alpha[n]}(n) < H_{\alpha[n]}(f(n)) = H_\alpha(n).$$

Moreover if P is quasi-(f(n))-built-up, $\alpha \underset{m}{\longrightarrow} \beta$ and and $m < n$ then $\alpha[m] \underset{m}{\Longrightarrow} \beta$ and $\alpha[n] \underset{f(n)}{\Longrightarrow} \alpha[m]$, by ind. hyp.,

$$H_\beta(n) \le H_{\alpha[m]}(n) < H_{\alpha[m]}(f(n)) \le H_{\alpha[n]}(f(n)) = H_\alpha(n).$$

§3. A (1)-built-up system of fundamental sequences for Γ_0.

We give a system P of fundamental sequences for Γ_0 (for details about Γ_0 see Schütte[5]), by modifying the system in §3 of Schmidt[3]. If we restrict P to ordinals below ε_0, P corresponds to a standard system of fundamental sequences below ε_0 (e.g. Ketonen and Solovay[1]). Then we can prove that P is (1)-built-up (cf.Theorem 2.4 in [1]).

All ordinals below Γ_0 can be generated from 0 by the two functions ν and κ defined by

$\nu(\alpha, \beta) = \omega^\alpha + \beta$,

$\kappa(0, \beta) = \varepsilon_\beta$ = the β-th inaccessible of ν,

for $\gamma > 0$, $\kappa(\gamma, \beta)$ = the β-th ordinal which is inaccessible for all $\lambda\delta.\kappa(\alpha, \delta)$ such that $\alpha < \gamma$.

If $\gamma < \Gamma_0$, γ is a limit ordinal, then there is exactly one pair $(\alpha, \beta) \in {\Gamma_0}^2$ and one $\rho \in \{\kappa, \nu\}$ such that $\alpha, \beta < \gamma$ and $\gamma = \rho(\alpha, \beta)$. We will write $\gamma =_{nf} \rho(\alpha,\beta)$ ($\rho(\alpha,\beta)$ is the normal form of γ). We define a system P of fundamental sequences for all limit ordinals $\gamma < \Gamma_0$ by induction on γ.

(1) $\gamma =_{nf} \nu(\alpha, \beta) = \omega^\alpha + \beta$. β is not a successor, α is not 0.

(1.1) If $\gamma = \omega^{\delta+1}$, then $\gamma[i] = \begin{cases} \omega^\delta\cdot(i+1) & \text{if } \delta \ge \varepsilon_0, \\ \omega^\delta\cdot i & \text{if } \delta < \varepsilon_0. \end{cases}$

(1.2) If $\gamma = \omega^\alpha$, α is limit, then $\gamma[i] = \omega^{\alpha[i]}$.

(1.3) If $\gamma = \omega^\alpha + \beta$, β is limit, then $\gamma[i] = \omega^\alpha + \beta[i]$.

(2) $\gamma =_{nf} \kappa(\alpha,\beta)$.

(2.1) If $\gamma = \kappa(\alpha,\beta)$, β is limit, then $\gamma[i] = \kappa(\alpha,\beta[i])$.

(2.2) If $\gamma = \kappa(0,0) = \varepsilon_0$, then $\gamma[0] = \omega$, $\gamma[i+1] = \omega^{\gamma[i]}$.

(2.3) If $\gamma = \kappa(0,\eta+1) = \varepsilon_{\eta+1}$, then $\gamma[0] = \varepsilon_\eta+1$, $\gamma[i+1] = \omega^{\gamma[i]}$.

(2.4) If $\gamma = \kappa(\delta+1,0)$, then $\gamma[0] = \kappa(\delta,0)$, $\gamma[i+1] = \kappa(\delta,\gamma[i])$.

(2.5) If $\gamma = \kappa(\delta+1,\eta+1)$, then

$$\gamma[0] = \kappa(\delta+1,\eta)+1,\ \gamma[i+1] = \kappa(\delta,\gamma[i]).$$

(2.6) If $\gamma = \kappa(\alpha,0)$, α is limit, then $\gamma[i] = \kappa(\alpha[i],0)$.

(2.7) If $\gamma = \kappa(\alpha,\eta+1)$, α is limit, then $\gamma[i] = \kappa(\alpha[i],\kappa(\alpha,\eta)+1)$.

<u>Theorem 3.1.</u> P is a (1)-built-up system of fundamental sequences for Γ_0.

<u>Lemma 3.2.</u> Let $m < \omega$ and $\alpha, \beta, \gamma, \sigma, \tau < \Gamma_0$.

(1) If $\omega^\alpha + \beta > \beta$ and $\beta \xrightarrow[1]{} \gamma$, then $\omega^\alpha\cdot m + \beta \xrightarrow[1]{} \omega^\alpha\cdot m + \gamma$.

(2) For each α such that $1 < \alpha$, $\alpha \xrightarrow[1]{} 1$.

(3) If $\alpha \xrightarrow[1]{} \beta$ and there is no γ such that $\alpha \underset{1}{\Longrightarrow} \varepsilon_\gamma \xrightarrow[1]{} \beta$, then $\omega^\alpha \xrightarrow[1]{} \omega^\beta$.

(4) If $\beta \xrightarrow[1]{} \gamma$ and there is no δ such that $\beta \underset{1}{\Longrightarrow} \kappa(\alpha+1,\delta) \xrightarrow[1]{} \gamma$, then $\kappa(\alpha,\beta) \xrightarrow[1]{} \kappa(\alpha,\gamma)$.

(5) If $\alpha \xrightarrow[1]{} \beta$, then $\kappa(\alpha,0) \xrightarrow[1]{} \kappa(\beta,0)$.

(6) If $\alpha \xrightarrow[1]{} \beta$, $\alpha < \sigma$ and $\gamma = \kappa(\sigma,\tau)$ for some τ, then $\kappa(\alpha,\gamma+1) \xrightarrow[1]{} \kappa(\beta,\gamma+1)$.

(7) Let α be limit, $\alpha =_{nf} \rho(\sigma,\tau)$, $\eta < \Gamma_0$. Then either the following condition (a) or (b) holds.

(a) there is an m such that all of the $\alpha[i]$ for $m < i$ are in the range of $\lambda\xi.\kappa(\eta,\xi)$.

(b) there is a $\xi < \Gamma_0$ such that $\kappa(\eta,\xi) \le \alpha[i] < \kappa(\eta,\xi+1)$ for all i or $\alpha[i] < \kappa(\eta,0)$ for all $i < \omega$.

(Proof) (1) For m = 1, by induction on β. For m > 1, by induction on m. (2),(3) By induction on α. (4) By induction on α with subsidiary induction on β. (5),(6) By induction on α. (7) By induction on α; if $\rho = \nu$ then (b) holds.

(Proof of Theorem 3.1) By induction on γ, we will show that

$\gamma[i+1] \xrightarrow[1]{} \gamma[i]$.

(1.1) $\gamma=\omega^{\delta+1}$. By Lemma 3.2(2), $\gamma[i+1]=\gamma[i]+\omega^{\delta} \overset{}{\underset{1}{\Longrightarrow}} \gamma[i]+1 \xrightarrow[1]{} \gamma[i]$.

(1.2) $\gamma=\omega^{\alpha}$, α is limit. By Lemma 3.2(3)(7) and ind.hyp.,

$$\gamma[i+1] = \omega^{\alpha[i+1]} \xrightarrow[1]{} \omega^{\alpha[i]} = \gamma[i].$$

(1.3) $\gamma=\omega^{\alpha}+\beta$, β is limit. By Lemma 3.2(1) and ind.hyp.,

$$\gamma[i+1] = \omega^{\alpha}+\beta[i+1] \xrightarrow[1]{} \omega^{\alpha}+\beta[i] = \gamma[i].$$

(2.1) $\gamma=\kappa(\alpha,\beta)$, β is limit. $\gamma[i]=\kappa(\alpha,\beta[i])$. Now (a) of Lemma 3.2(7) cannot hold for $\beta =_{nf} \kappa(\alpha+1,\tau)$. (For then $\gamma = \lim_{i<\omega} \gamma[i] = \lim_{i<\omega} \kappa(\alpha,\beta[i]) = \lim_{i<\omega} \beta[i] = \beta$, it is contradiction.) Hence (b) must hold. Therefore by Lemma 3.2(4) and ind.hyp.,

$$\gamma[i+1] = \kappa(\alpha,\beta[i+1]) \xrightarrow[1]{} \kappa(\alpha,\beta[i]) = \gamma[i].$$

(2.2) $\gamma=\varepsilon_0$. By Lemma 3.2(3) and induction on i,

$$\gamma[1] = \omega^{\omega} \xrightarrow[1]{} \omega = \gamma[0],\ \gamma[i+2] = \omega^{\gamma[i+1]} \xrightarrow[1]{} \omega^{\gamma[i]} = \gamma[i+1].$$

(2.3) $\gamma=\varepsilon_{\eta+1}$. By Lemma 3.2(2), $\gamma[1] = \omega^{\varepsilon_{\eta}+1} \xrightarrow[1]{} \omega^{\varepsilon_{\eta}}\cdot 2 \xrightarrow[1]{} \varepsilon_{\eta}+1 = \gamma[0]$. By Lemma 3.2(3) and induction on i,

$$\gamma[i+2] = \omega^{\gamma[i+1]} \xrightarrow[1]{} \omega^{\gamma[i]} = \gamma[i+1].$$

(2.4) $\gamma=\kappa(\delta+1,0)$. $\gamma[0]=\kappa(\delta,0)$, $\gamma[i+1]=\kappa(\delta,\gamma[i])$. Now $\gamma[i] <$

$\kappa(\delta+1,0)$. By Lemma 3.2(4) and induction on i,

$$\gamma[1] = \kappa(\delta,\gamma[0]) \xrightarrow[1]{} \kappa(\delta,0) = \gamma[0],$$

$$\gamma[i+2] = \kappa(\delta,\gamma[i+1]) \xrightarrow[1]{} \kappa(\delta,\gamma[i]) = \gamma[i+1].$$

(2.5) $\gamma=\kappa(\delta+1,\eta+1)$. By Lemma 3.2(6), $\gamma[1] = \kappa(\delta,\kappa(\delta+1,\eta)+1) \overset{}{\underset{1}{\Longrightarrow}}$ $\kappa(0,\kappa(\delta+1,\eta)+1) \xrightarrow[1]{} \omega^{\kappa(0,\kappa(\delta+1,\eta))+1} = \omega^{\kappa(\delta+1,\eta)+1} \xrightarrow[1]{}$ $\omega^{\kappa(\delta+1,\eta)}\cdot 2 \xrightarrow[1]{} \kappa(\delta+1,\eta)+1 = \gamma[0]$. Now $\kappa(\delta+1,\eta) < \gamma[i] <$ $\kappa(\delta+1,\eta+1)$. By Lemma 3.2(4) and induction on i,

$$\gamma[i+2] = \kappa(\delta,\gamma[i+1]) \xrightarrow[1]{} \kappa(\delta,\gamma[i]) = \gamma[i+1].$$

(2.6) $\gamma=\kappa(\alpha,0)$, α is limit. By Lemma 3.2(5),

$$\gamma[i+1] = \kappa(\alpha[i+1],0) \xrightarrow[1]{} \kappa(\alpha[i],0) = \gamma[i].$$

(2.7) $\gamma=\kappa(\alpha,\eta+1)$, α is limit. By Lemma 3.2(6) and ind.hyp.,

$$\gamma[i+1] = \kappa(\alpha[i+1],\kappa(\alpha,\eta)+1) \xrightarrow[1]{} \kappa(\alpha[i],\kappa(\alpha,\eta)+1) = \gamma[i].$$

§3. (n)-built-upness and Bachmann's property 𝔹[n]

In this section, we extend the theorems in Schmidt[4] by using (n)-built-up system. In the following, we assume that P is a system of fundamental sequences for Δ.

Definition 4.1. P has property 𝔹[n] iff if $\alpha[i] < \mu \le \alpha[i+1]$ then $\alpha[i] \le \mu[n]$ for each limit $\alpha \in \Delta$, $i < \omega$ and $\mu \in \Delta$.

Theorem 4.1. P has property 𝔹[n] iff P is (n)-built-up.

(Proof) Let P have 𝔹[n], $\alpha \in \Delta$ be limit. Then for $\alpha[i] < \mu \le$ $\alpha[i+1]$, $\mu \xrightarrow[n]{} \alpha[i]$ by ind. on μ, in particular, $\alpha[i+1] \xrightarrow[n]{} \alpha[i]$. Let P be (n)-built-up, $\alpha \in \Delta$ be limit and $\alpha[i] < \mu \le \alpha[i+1]$. We define $\langle\alpha_k\rangle_{k<\omega}$ inductively as follows: $\alpha_0 = \alpha[i+1]$,

$$\alpha_{k+1} = \begin{cases} \mu & \text{if } \alpha_k = \mu, \\ (\alpha_k)[j] & \text{if } \alpha_k \text{ is limit} > \mu, \text{ where j is the least} \\ & \quad \text{such that } (\alpha_k)[j] > \mu, \\ \beta & \text{if } \alpha_k = \beta + 1. \end{cases}$$

Because P is (n)-built-up, we can prove $\alpha \geq \mu$ and $\alpha_k \xrightarrow[n]{} \alpha[i]$ by ind. on k. Then $\langle\alpha_k\rangle_{k<\omega}$ is non-ascending sequence, hence there is an $m < \omega$ such that $\alpha_k = \mu$ for $k \geq m$. Hence $\mu \xrightarrow[n]{} \alpha[i]$, in particular, $\alpha[i] \leq \mu[n]$.

<u>Theorem 4.2.</u> For any $n < \omega$, there is no assignment of fundamental sequences with property $\mathbb{B}[n]$ for the whole of the second number class.

(Proof) Let Ω be the first uncountable ordinal. Assume there is such a system for fixed $n < \omega$. Then $\lambda\alpha.\alpha[n]$ is a regressive function (i.e. $\alpha[n] < \alpha$ for $0 < \alpha < \Omega$). and therefore there is an $A \subset \Omega$ of order type Ω and a $\beta \in \Omega$ such that $\alpha[n] = \beta$ for all $\alpha \in A$ (cf. Levy[2]). Let $\{\alpha_i : i < \omega\}$ be the first ω elements of A, $\alpha = \lim_{n<\omega} \alpha_i$ and $\langle\alpha[i]\rangle_{i<\omega}$ be the fundamental sequence for α. Since $\alpha > \beta$, there is an $i < \omega$ such that $\beta < \alpha[i] < \alpha$. Let $m < \omega$ be the number such that $\alpha[i] < \alpha_m < \alpha$, and p be the greatest number such that $\alpha[p] < \alpha_m$. Then $\alpha[p] < \alpha_m \leq \alpha[p+1]$, so $\beta < \alpha[p] \leq \alpha_m[n]$. But $\alpha_m[n] = \beta$ because $\alpha_m \in A$. Contradiction.

Finally, we define properties $\mathbb{A}$ and $\mathbb{C}[n]$ to extend Remark of [4] as follows:

P has property $\mathbb{A}$ iff if $\beta < \alpha < \Delta$, then $\alpha \xrightarrow[n]{} \beta$ for some $n < \omega$.

P has property $\mathbb{C}[n]$ iff if $\langle F_\alpha \rangle_{\alpha<\Delta}$ is a sequence of number theoretic functions satisfies the properties $(a)_n$, $(b)_n$ and (c), and $\beta < \gamma \in \Delta$, then F_β is dominated by F_γ (i.e. there is an m such that for all $n \geq m$, $F_\beta(n) < F_\gamma(n)$).

We can prove the following theorem which is an extension of Remark of [4].

<u>Theorem 4.3.</u> If P is (n)-built-up for some $n < \omega$, then P has properties $\mathbb{A}$ and $\mathbb{C}[n]$.

(Proof) Assume P is (n)-built-up. First we prove that P has $\mathbb{A}$ by ind. on α. <u>Case 1.</u> $\alpha = 0$. Trivial. <u>Case 2.</u> $\alpha = \gamma + 1$. $\gamma \overset{}{\underset{m}{\Longrightarrow}} \beta$ for some $m < \omega$ by ind. hyp. Then $\alpha \xrightarrow[m]{} \gamma \underset{m}{\Longrightarrow} \beta$. <u>Case 3.</u> α is limit. $\alpha[i] > \beta$ for some i. Then $\alpha[i] \xrightarrow[k]{} \beta$ for some k by ind. hyp. Let $r = \max(n,k,i)$. By using Proposition 1.1, $\alpha \xrightarrow[r]{} \alpha[r] \underset{r}{\Longrightarrow} \alpha[i] \xrightarrow[r]{} \beta$. Next we prove that P has $\mathbb{C}[n]$. Because P has $\mathbb{A}$, $\gamma \xrightarrow[r]{} \beta$ for some r. If we put $m = \max(n,r)+1$, then by Theorem 1.3, $F_\beta(x) < F_\alpha(x)$ for $m \leq x < \omega$.

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