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A note on Schmidt's built-up systems of fundamental sequences

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Introduction. Let $\langle F_\alpha \rangle_{\alpha \in \Delta}$ be a transfinite sequence of number theoretic functions indexed by an initial segment $\Delta$ of the second number class which satisfies the following conditions:

(a) $F_0$ is strictly increasing,

(b) if $F_\alpha$ is strictly increasing, $F_{\alpha+1}$ is also strictly increasing, $F_\alpha(0) < F_{\alpha+1}(0)$ and $F_\alpha(x) < F_{\alpha+1}(x)$ for $0 < x < \omega$,

(c) $F_\alpha(x) = F_{\alpha[x]}(x)$ if $\alpha$ is a limit ordinal, where $\langle \alpha[x] \rangle_{x<\omega}$ is a fundamental sequence for $\alpha$.

Schmidt[3] introduced the concept of built-up systems of fundamental sequences, and showed that, for the above sequence $\langle F_\alpha \rangle_{\alpha \in \Delta}$, each $F_\alpha$ is strictly increasing if the system of fundamental sequences used is built-up. However there are some standard systems of fundamental sequences in literatures, e.g. Ketonen and Solovay[1], which are not built-up in Schmidt's sense, but which determine a sequence of strictly increasing functions.

The purpose of this note is to extend the concept of built-up systems so that it can be applicable to wider classes of systems of sequences of ordinals.

In §1 we define $(n)$-built-up systems and quasi-$(n)$-built-up systems of sequences of ordinals. In §2 we show a theorem on a relation between $(n)$-built-up systems and $\langle F_\alpha \rangle_{\alpha \in \Delta}$, which corresponds to Theorem 1 in [3], and a theorem on a relation...
between quasi-(n)-built-up systems and a sequence of number
theoretic functions $\langle H_\alpha \rangle_{\alpha \in \Delta}$. In §3, we give an example of
(1)-built-up system of fundamental sequences for $\Gamma_0$. Finally, in
§4, we extend the results in Schmidt[4], by using quasi-(n)-
built-up systems of sequences of ordinals.

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§1. Preliminaries

Let $\Delta$ be an initial segment of second number class. We will
use Greek letters $\alpha, \beta, \gamma, \ldots$ for ordinal numbers in $\Delta$. Let
$P: \Delta \to \Delta^\omega$ be an assignment of sequences of ordinals for $\Delta$. We
shall write $\alpha[i]$ for $(P(\alpha))(i)$ whenever $\alpha \in \Delta$ and $i < \omega$.

If $P$ satisfies the following conditions (A)-(C):

(A) $\alpha[i] = 0$ if $\alpha = 0$ and $i < \omega$,

(B) $\alpha[i] = \beta$ if $\alpha = \beta + 1$ and $i < \omega$,

(C) $\alpha[i] < \alpha$ if $\alpha$ is a limit ordinal and $i < \omega$,

then we call $P$ a system of sequences of ordinals for $\Delta$.

Moreover, if a system $P$ of sequences of ordinals satisfies the
following conditions (C)$^+, (D)$:

(C)$^+$ $\alpha[i] < \alpha[i+1] < \alpha$ if $\alpha$ is a limit ordinal and $i < \omega$,

(D) $\lim_{i<\omega} \alpha[i] = \alpha$ if $\alpha$ is a limit ordinal,

then we call $P$ a system of fundamental sequences for $\Delta$.

In the following, we assume $P$ is a system of sequences of
Definition 1.1. Let $P$ be a system of sequences of ordinals for $\Delta$. For each $n < \omega$, $\frac{1}{n}$, $\frac{n}{\alpha}$, $\frac{\alpha}{n}$ are defined as follows:

1. $\alpha \xrightarrow{\frac{1}{n}} \beta$ iff $0 < \alpha$ and $\alpha[n] = \beta$.

2. $\alpha \xrightarrow{n} \beta$ iff there is a sequence $\gamma_0, \ldots, \gamma_j$ ($0 < j < \omega$) such that $\gamma_0 = \alpha$, $\gamma_j = \beta$ and $\gamma_i \xrightarrow{\frac{1}{n}} \gamma_{i+1}$ ($0 \leq i < j$).

3. $\alpha \xrightarrow{n} \beta$ iff $\alpha \xrightarrow{n} \beta$ or $\alpha = \beta$.

For each $n < \omega$, $P$ is $(n)$-built-up (and quasi-$(n)$-built-up), if $\alpha[i+1] \xrightarrow{n} \alpha[i]$ (and $\alpha[i+1] \xrightarrow{n} \alpha[i]$, respectively) for each limit ordinal and each $i < \omega$.

Built-up systems in Schmidt's sense[3] is the same as (0)-built-up systems of fundamental sequences in our sense.

Ketonen and Solovay[1] introduced the relation $\xrightarrow{n}$ for studying a standard system of fundamental sequences for ordinals up to $\varepsilon_0$. Their system is (1)-built-up but not (0)-built-up (cf. Theorem 2.4 of [1]).

Proposition 1.1. Let $P$ be quasi-$(s)$-built-up. If $s \leq m$, $n \leq m$ and $\alpha \xrightarrow{n} \beta$, then $\alpha \xrightarrow{m} \beta$.

(Proof) By induction on $\alpha$. Case 1. $\alpha = 0$. This case is trivial because $\{0 \xrightarrow{n} \beta\}$. Case 2. $\alpha = \gamma + 1$. If $\alpha \xrightarrow{n} \beta$, then $\gamma \xrightarrow{n} \beta$. So $\alpha = \gamma + 1 \xrightarrow{\frac{1}{m}} \gamma$ and $\gamma \xrightarrow{m} \beta$ by ind. hyp. So $\alpha \xrightarrow{m} \beta$. Case 3. $\alpha$ is limit. If $\alpha \xrightarrow{n} \beta$ then $\alpha[n] \xrightarrow{\frac{n}{\alpha}} \beta$. Because $P$ is quasi-$(s)$-built-up, $\alpha[m] \xrightarrow{s} \alpha[n]$. By ind. hyp., $\alpha[m] \xrightarrow{m} \alpha[n]$ and
\[\alpha[n] \overset{m}{\longrightarrow} \beta. \text{ So, } \alpha \overset{1}{\longrightarrow} \alpha[m] \overset{m}{\longrightarrow} \alpha[n] \overset{m}{\longrightarrow} \beta.\]

**Corollary 1.2.** Let \( n \leq m \). If \( P \) is \((n)\)-built-up (and quasi-(n)-built-up), then \( P \) is \((m)\)-built-up (and quasi-(m)-built-up, respectively).

**§2. \((n)\)-built-up systems and hierarchies of number theoretic functions**

We say that a function \( F: \omega \rightarrow \omega \) is **strictly increasing after \( n \)** if \( F(x) < F(x+1) \) for \( n \leq x < \omega \).

Let \( P \) be a system of sequences of ordinals for \( \Delta \). Suppose that \( \langle F_{\alpha} \rangle_{\alpha \in \Delta} \) is any sequence of number theoretic functions satisfying the following conditions for \( n < \omega \).

- (a) \( F_0 \) is strictly increasing after \( n \).
- (b) \( F_\alpha \) is strictly increasing after \( n \), then \( F_{\alpha+1} \) is also strictly increasing after \( n \).
- (c) \( F_\alpha(n) \leq F_{\alpha+1}(n) \), and \( F_\alpha(x) < F_{\alpha+1}(x) \) for \( n < x < \omega \).

Let \( F_\alpha(x) = F_{\alpha[x]}(x) \) for \( n \leq x < \omega \), if \( \alpha \) is limit.

Remark that conditions \((a)_0\), \((b)_0\) and \((c)_0\) are the same as \((a)\), \((b)\) and \((c)\) in Introduction.

**Example 1.** Let the fast growing hierarchy \( \langle F_{\alpha} \rangle_{\alpha \in \Delta} \) define by

\[
F_0(x) = x+1, \quad F_{\alpha+1}(x) = F_{\alpha}^{x+1}(x), \text{ where } F_{\alpha}^1 \text{ is defined by } F_\alpha^0(x) = x, \quad F_{\alpha+1}^1(x) = F_{\alpha}^1(F_\alpha(x)),
\]

\[F_\alpha(x) = F_{\alpha[x]}(x) \text{ if } \alpha \text{ is limit.}\]

Then \( \langle F_{\alpha} \rangle_{\alpha \in \Delta} \) satisfies \((a)_n\), \((b)_n\) and \((c)_n\).
Theorem 2.1. If \( \langle F_\alpha \rangle_{\alpha \in \Delta} \) satisfies conditions (a), (b), and (c), then the following hold for each \( \alpha, \beta \in \Delta \).

1. \( \alpha \xrightarrow{n} \beta \) implies \( F_\beta(n) \leq F_\alpha(n) \).

2. If \( P \) is \((n+1)\)-built-up, then

   (2.1) \( F_\alpha \) is strictly increasing after \( n \).

   (2.2) \( \alpha \xrightarrow{m} \beta \) implies \( F_\beta(s) \leq F_\alpha(s) \), \( F_\beta(x) < F_\alpha(x) \)

       for \( s < x < \omega \), where \( s = \max(n+1, m) \).

       Moreover, if \( P \) is \((n)\)-built-up and \( m \leq n \), then \( \alpha \xrightarrow{m} \beta \)

       implies \( F_\beta(n) \leq F_\alpha(n) \), \( F_\beta(n+1) < F_\alpha(n+1) \).

(Proof) By induction on \( \alpha \).

   Case 1. \( \alpha = 0 \). (1) holds because \( \langle 0 \xrightarrow{n} \beta \rangle \). (2.1) is \((a)\), (2.2) holds because \( \langle 0 \xrightarrow{n} \beta \rangle \).

   Case 2. \( \alpha = \gamma + 1 \). (1) If \( \alpha \xrightarrow{n} \beta \) then \( \gamma \xrightarrow{n} \beta \). \( F_\beta(n) \leq F_\gamma(n) \leq F_\alpha(n) \) by ind. hyp. and \((b)\). (2.1) Since \( F_\gamma \) is strictly

       increasing after \( n \), by ind. hyp., so is \( F_\alpha \) by \((b)\). (2.2) Assume \( \alpha \xrightarrow{m} \beta \). Then \( \gamma \xrightarrow{m} \beta \). If \( \beta = \gamma \), (2.2) holds by \((b)\). Hence, by

       ind. hyp. (2.2) holds for all \( \alpha \xrightarrow{m} \beta \).

   Case 3. \( \alpha \) is limit. (1) If \( \alpha \xrightarrow{n} \beta \) then \( \alpha[n] \xrightarrow{n} \beta \). So

       \( F_\beta(n) \leq F_{\alpha[n]}(n) = F_\alpha(n) \) by (1) of ind. hyp.

   (2) Let \( P \) be \((n+1)\)-built-up. (2.1) For \( n \leq x < \omega \), \( \alpha[x+1] \xrightarrow{n+1} \alpha[x] \). So,

       \( F_\alpha(x+1) = F_{\alpha[x+1]}(x+1) \geq F_{\alpha[x]}(x+1) \) by (2.2) of ind. hyp.

       > \( F_{\alpha[x]}(x) \) by (2.1) of ind. hyp.

       = \( F_\alpha(x) \).

   (2.2) If \( \alpha \xrightarrow{m} \beta \) then \( \alpha[m] \xrightarrow{m} \beta \). Let \( s = \max(n+1, m) \). Then

       \( F_\beta(s) \leq F_{\alpha[m]}(s) \leq F_{\alpha[s]}(s) = F_\alpha(s) \)

       by \( \alpha[m] \xrightarrow{n+1} \alpha[s] \) and ind. hyp.

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\[ F_{\beta}(x) \leq F_{\alpha[m]}(x) < F_{\alpha[x]}(x) = F_{\alpha}(x) \]
for \( s < x < \omega \) by \( \alpha[x] \xrightarrow{\alpha[m]} \), and \( \text{ind.hyp}. \) Moreover if \( P \) is \( (n)\)-built-up and \( m \leq n \), then \( \alpha[n] \xrightarrow{n} \alpha[m] \) and \( \alpha[n+1] \xrightarrow{n} \alpha[m] \),
\[ F_{\beta}(n) \leq F_{\alpha[m]}(n) \leq F_{\alpha[n]}(n) = F_{\alpha}(n), \]
\[ F_{\beta}(n+1) \leq F_{\alpha[m]}(n+1) < F_{\alpha[n+1]}(n+1) = F_{\alpha}(n+1) , \text{ by ind.hyp.} \]

**Corollary 2.2.** If \( \langle F_{\alpha}\rangle_{\alpha \in \Delta} \) satisfies (a), (b) and (c) ( i.e. (a)_0, (b)_0 and (c)_0 ), and \( P \) is (1)-built-up, then for each \( \alpha \), \( F_{\alpha} \) is strictly increasing.

Next, we will introduce another hierarchies \( \langle H_{\alpha}\rangle_{\alpha \in \Delta} \) of number theoretic functions which is constructed by the same way as \( \langle F_{\alpha}\rangle_{\alpha \in \Delta} \) except (c) \( n \) ( i.e. \( F_{\alpha}(x) = F_{\alpha[x]}(x) \) if \( \alpha \) is a limit ordinal ).

We fix a system \( P \) of sequences of ordinals for \( \Delta \), and a function \( f: \omega \rightarrow \omega \) which satisfies \( n < f(n) \) and \( f(x) < f(x+1) \) for \( n \leq x < \omega \) ( e.g. \( f(x) = x + 1 \) ).

Suppose \( \langle H_{\alpha}\rangle_{\alpha \in \Delta} \) is any sequence of number theoretic functions satisfying the following conditions for \( n \).
\[ \text{(a)}_n \text{ and (b)}_n \text{ are the same as the case of } \langle F_{\alpha}\rangle_{\alpha \in \Delta}, \]
\[ \text{(c)* } H_{\alpha}(x) = H_{\alpha[x]}(f(x)) \text{ for } n \leq x < \omega, \text{ if } \alpha \text{ is limit.} \]

**Example 2.** Let the Hardy hierarchy \( \langle H_{\alpha}\rangle_{\alpha \in \Delta} \) define by
\[ H_0(x) = x, \ H_{\alpha}(x) = H_{\alpha[x]}(x+1) \text{ if } \alpha > 0. \]
Then \( \langle H_{\alpha}\rangle_{\alpha \in \Delta} \) satisfies (a)_n, (b)_n and (c)* ( where we take \( x+1 \) as \( f(x) \) ).
We can prove a theorem which is a relation between quasi-(n)-built-up systems and \( \langle H_\alpha \rangle_{\alpha \in \Delta} \).

**Theorem 2.3.** If \( \langle H_\alpha \rangle_{\alpha \in \Delta} \) satisfies conditions (a) \( n \), (b) \( n \) and (c) \( n \) and \( P \) is quasi-(f(n+1))-built-up, then for each \( \alpha \in \Delta \),

1. \( H_\alpha \) is strictly increasing after \( n \),
2. \( \alpha \xrightarrow{m} \beta \) implies \( H_\beta(x) < H_\alpha(x) \) for \( \max(n+1,m) \leq x < \omega \).
   In addition, \( \alpha \xrightarrow{n} \beta \) implies \( H_\beta(n) \leq H_\alpha(n) \).
   Moreover, if \( P \) is quasi-(f(n))-built-up and \( m < n \), then \( \alpha \xrightarrow{m} \beta \) implies \( H_\beta(n) \leq H_\alpha(n) \).

**Proof.** By induction on \( \alpha \). Assume that \( P \) is quasi-(f(n+1))-built-up. **Case 1.** \( \alpha = 0 \). (1) is (a) \( n \). (2) holds because \( I(0 \xrightarrow{m} \beta) \).
**Case 2.** \( \alpha = \gamma + 1 \). (1) Since \( H_\gamma \) is strictly increasing after \( n \) by ind. hyp., so is \( H_\alpha \) by (b) \( n \). (2) Assume \( \alpha \xrightarrow{m} \beta \). Then \( \gamma \xrightarrow{m} \beta \). If \( \beta = \gamma \), (2) holds by (b) \( n \). Hence, by ind. hyp., (2) holds for all \( \alpha \xrightarrow{m} \beta \).
**Case 3.** \( \alpha \) is limit. (1) For \( n \leq x < \omega \), \( \alpha[x+1] \xrightarrow{f(n+1)} \alpha[x] \),

\[
H_\alpha(x+1) = H_\alpha[x+1](f(x+1)) \geq H_\alpha[x](f(x+1)) \text{ by (2) of ind. hyp.}
\]

\[
> H_\alpha[x](f(x)) \text{ by (1) of ind. hyp.}
\]

\[
= H_\alpha(x).
\]

(2) If \( \alpha \xrightarrow{m} \beta \) then \( \alpha[m] \xrightarrow{m} \beta \). For \( \max(n+1,m) \leq x < \omega \), by ind. hyp. and \( \alpha[x] \xrightarrow{f(n+1)} \alpha[m] \), \( f(n+1) \leq f(x) \), so

\[
H_\beta(x) \leq H_\alpha[m](x) < H_\alpha[m](f(x)) \leq H_\alpha[x](f(x)) = H_\alpha(x).
\]

If \( \alpha \xrightarrow{n} \beta \) then \( \alpha[n] \xrightarrow{n} \beta \). By ind. hyp.,

\[
H_\beta(n) \leq H_\alpha[n](n) < H_\alpha[n](f(n)) = H_\alpha(n).
\]
Moreover if $P$ is quasi-$(f(n))$-built-up, $\alpha \xrightarrow{m} \beta$ and and $m < n$ then $\alpha[m] \xrightarrow{m} \beta$ and $\alpha[n] \xrightarrow{f(n)} \alpha[m]$, by ind. hyp.,

$$H_\beta(n) < H_{\alpha[m]}(n) < H_{\alpha[m]}(f(n)) < H_{\alpha[n]}(f(n)) = H_\alpha(n).$$

§3. A $(1)$-built-up system of fundamental sequences for $\Gamma_0$.

We give a system $P$ of fundamental sequences for $\Gamma_0$ (for details about $\Gamma_0$ see Schütte[5]), by modifying the system in §3 of Schmidt[3]. If we restrict $P$ to ordinals below $\varepsilon_0$, $P$ corresponds to a standard system of fundamental sequences below $\varepsilon_0$ (e.g. Ketonen and Solovay[1]). Then we can prove that $P$ is $(1)$-built-up (cf. Theorem 2.4 in [1]).

All ordinals below $\Gamma_0$ can be generated from 0 by the two functions $\nu$ and $\kappa$ defined by

$$\nu(\alpha, \beta) = \omega^\alpha + \beta,$$

$$\kappa(0, \beta) = \varepsilon_\beta = \text{the } \beta\text{-th inaccessible of } \nu,$$

for $\gamma > 0$, $\kappa(\gamma, \beta) = \text{the } \beta\text{-th ordinal which is inaccessible for all } \lambda \delta. \kappa(\alpha, \delta) \text{ such that } \alpha < \gamma$.

If $\gamma < \Gamma_0$, $\gamma$ is a limit ordinal, then there is exactly one pair $(\alpha, \beta) \in \Gamma_0^2$ and one $\rho \in (\kappa, \nu)$ such that $\alpha, \beta < \gamma$ and $\gamma = \rho(\alpha, \beta)$. We will write $\gamma = \nu_{nf}(\alpha, \beta)$ ( $\rho(\alpha, \beta)$ is the normal form of $\gamma$ ). We define a system $P$ of fundamental sequences for all limit ordinals $\gamma < \Gamma_0$ by induction on $\gamma$.

1. $\gamma = \nu_{nf}(\alpha, \beta) = \omega^\alpha + \beta$. $\beta$ is not a successor, $\alpha$ is not 0.

1.1. If $\gamma = \omega^\delta + 1$, then $\gamma[i] = \begin{cases} \omega^\delta \cdot (i+1) & \text{if } \delta \geq \varepsilon_0, \\ \omega^\delta \cdot i & \text{if } \delta < \varepsilon_0. \end{cases}$

1.2. If $\gamma = \omega^\alpha$, $\alpha$ is limit, then $\gamma[i] = \omega^{\alpha[i]}$.

1.3. If $\gamma = \omega^\alpha + \beta$, $\beta$ is limit, then $\gamma[i] = \omega^\alpha + \beta[i]$. 

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(2) \( \gamma = \text{nf} \ k(\alpha, \beta) \).

(2.1) If \( \gamma = k(\alpha, \beta) \), \( \beta \) is limit, then \( \gamma[1] = k(\alpha, \beta[1]) \).

(2.2) If \( \gamma = k(0, 0) = \varepsilon_0 \), then \( \gamma[0] = \omega \), \( \gamma[1+1] = \omega \gamma[1] \).

(2.3) If \( \gamma = k(0, \eta+1) = \varepsilon_{\eta+1} \), then \( \gamma[0] = \varepsilon_{\eta+1} \), \( \gamma[1+1] = \omega \gamma[1] \).

(2.4) If \( \gamma = k(\delta+1, 0) \), then \( \gamma[0] = k(\delta, 0) \), \( \gamma[1+1] = k(\delta, \gamma[1]) \).

(2.5) If \( \gamma = k(\delta+1, \eta+1) \), then

\[ \gamma[0] = k(\delta+1, \eta)+1 \), \( \gamma[1+1] = k(\delta, \gamma[1]) \).

(2.6) If \( \gamma = k(\alpha, 0) \), \( \alpha \) is limit, then \( \gamma[1] = k(\alpha[1], 0) \).

(2.7) If \( \gamma = k(\alpha, \eta+1) \), \( \alpha \) is limit, then \( \gamma[1] = k(\alpha[1], k(\alpha, \eta)+1) \).

**Theorem 3.1.** \( P \) is a \((1)\)-built-up system of fundamental sequences for \( \Gamma_0 \).

**Lemma 3.2.** Let \( m < \omega \) and \( \alpha, \beta, \gamma, \sigma, \tau < \Gamma_0 \).

(1) If \( \omega^\alpha + \beta > \beta \) and \( \beta \xrightarrow{1} \gamma \), then \( \omega^\alpha \cdot m + \beta \xrightarrow{1} \omega^\alpha \cdot m + \gamma \).

(2) For each \( \alpha \) such that \( 1 < \alpha \), \( \alpha \xrightarrow{1} 1 \).

(3) If \( \alpha \xrightarrow{1} \beta \) and there is no \( \gamma \) such that \( \alpha \xrightarrow{1} \varepsilon_{\gamma} \xrightarrow{1} \beta \), then \( \omega^\alpha \xrightarrow{1} \omega^\beta \).

(4) If \( \beta \xrightarrow{1} \gamma \) and there is no \( \delta \) such that \( \beta \xrightarrow{1} k(\alpha+1, \delta) \xrightarrow{1} \gamma \), then \( k(\alpha, \beta) \xrightarrow{1} k(\alpha, \gamma) \).

(5) If \( \alpha \xrightarrow{1} \beta \), then \( k(\alpha, 0) \xrightarrow{1} k(\beta, 0) \).

(6) If \( \alpha \xrightarrow{1} \beta \), \( \alpha < \sigma \) and \( \gamma = k(\sigma, \tau) \) for some \( \tau \), then \( k(\alpha, \gamma+1) \xrightarrow{1} k(\beta, \gamma+1) \).

(7) Let \( \alpha \) be limit, \( \alpha = \text{nf} \ \rho(\sigma, \tau) \), \( \eta < \Gamma_0 \). Then either the following condition (a) or (b) holds.

(a) there is an \( m \) such that all of the \( \alpha[i] \) for \( m < i \) are in the range of \( \lambda \xi. k(\eta, \xi) \).
(b) there is a $\xi < \Gamma_0$ such that $\kappa(\eta, \xi) \leq \alpha[i] < \kappa(\eta, \xi + 1)$ for all $i$ or $\alpha[i] < \kappa(\eta, 0)$ for all $i < \omega$.

(Proof) (1) For $m = 1$, by induction on $\beta$. For $m > 1$, by induction on $m$. (2), (3) By induction on $\alpha$. (4) By induction on $\alpha$ with subsidiary induction on $\beta$. (5), (6) By induction on $\alpha$. (7) By induction on $\alpha$; if $\rho = \nu$ then (b) holds.

(Proof of Theorem 3.1) By induction on $\gamma$, we will show that $\gamma[i + 1] \rightarrow \gamma[i]$.

(1.1) $\gamma = \omega^{\delta + 1}$. By Lemma 3.2(2), $\gamma[i + 1] = \gamma[i] + \omega^\delta \rightarrow \gamma[i] + 1 \rightarrow \gamma[i]$.

(1.2) $\gamma = \omega^\alpha$, $\alpha$ is limit. By Lemma 3.2(3)(7) and ind.hyp.,

$\gamma[i + 1] = \omega^\alpha[i + 1] \rightarrow \omega^\alpha[i] = \gamma[i]$.

(1.3) $\gamma = \omega^\alpha + \beta$, $\beta$ is limit. By Lemma 3.2(1) and ind.hyp.,

$\gamma[i + 1] = \omega^\alpha + \beta[i + 1] \rightarrow \omega^\alpha + \beta[i] = \gamma[i]$.

(2.1) $\gamma = \kappa(\alpha, \beta), \beta$ is limit. $\gamma[i] = \kappa(\alpha, \beta[i])$. Now (a) of Lemma 3.2(7) cannot hold for $\beta = \eta_\text{nf} \kappa(\alpha + 1, \tau)$. (For then $\gamma = \lim_{i < \omega} \gamma[i]$ and it is contradiction.) Hence (b) must hold. Therefore by Lemma 3.2(4) and ind.hyp.,

$\gamma[i + 1] = \kappa(\alpha, \beta[i + 1]) \rightarrow \kappa(\alpha, \beta[i]) = \gamma[i]$.

(2.2) $\gamma = \varepsilon_0$. By Lemma 3.2(3) and induction on $i$,

$\gamma[i] = \omega^i \rightarrow \omega = \gamma[0], \gamma[i + 2] = \omega^{\gamma[i + 1]} \rightarrow \omega^\gamma[i] = \gamma[i + 1]$.

(2.3) $\gamma = \varepsilon_{\eta + 1}$. By Lemma 3.2(2), $\gamma[i] = \omega^{\eta + 1} \rightarrow \varepsilon_{\eta + 2} \rightarrow \varepsilon_{\eta + 1} = \gamma[0]$. By Lemma 3.2(3) and induction on $i$,

$\gamma[i + 2] = \omega^{\gamma[i + 1]} \rightarrow \omega^\gamma[i] = \gamma[i + 1]$.

(2.4) $\gamma = \kappa(\delta + 1, 0), \gamma[0] = \kappa(\delta, 0), \gamma[i + 1] = \kappa(\delta, \gamma[i])$. Now $\gamma[i] <
κ(δ + 1, 0). By Lemma 3.2(4) and induction on i,
\[ γ[1] = κ(δ, γ[0]) \rightarrow κ(δ, 0) = γ[0], \]
\[ γ[i+2] = κ(δ, γ[i+1]) \rightarrow κ(δ, γ[i]) = γ[i+1]. \]

(2.5) γ = κ(δ + 1, η + 1). By Lemma 3.2(6), \[ γ[1] = κ(δ, κ(δ + 1, η) + 1) \rightarrow κ(0, κ(δ + 1, η) + 1) = \omega κ(δ + 1, η) + 1 \rightarrow \omega κ(δ + 1, η) \cdot 2 \rightarrow κ(δ + 1, η) + 1 = γ[0]. \] Now κ(δ + 1, η) < γ[1] < κ(δ + 1, η + 1). By Lemma 3.2(4) and induction on i,
\[ γ[i+2] = κ(δ, γ[i+1]) \rightarrow κ(δ, γ[i]) = γ[i+1]. \]

(2.6) γ = κ(α, 0), α is limit. By Lemma 3.2(5),
\[ γ[i+1] = κ(α[i+1], 0) \rightarrow κ(α[i], 0) = γ[i]. \]

(2.7) γ = κ(α, η + 1), α is limit. By Lemma 3.2(6) and ind. hyp.,
\[ γ[i+1] = κ(α[i+1], κ(α, η) + 1) \rightarrow κ(α[i], κ(α, η) + 1) = γ[i]. \]

§3. (n)-built-upness and Bachmann's property B[n]

In this section, we extend the theorems in Schmidt[4] by using (n)-built-up system. In the following, we assume that P is a system of fundamental sequences for Δ.

Definition 4.1. P has property B[n] iff if α[i] < μ ≤ α[i+1] then α[i] ≤ μ[n] for each limit α ∈ Δ, i < ω and μ ∈ Δ.

Theorem 4.1. P has property B[n] iff P is (n)-built-up.

(Proof) Let P have B[n], α ∈ Δ be limit. Then for α[i] < μ ≤ α[i+1], μ \rightarrow α[i] by ind. on μ, in particular, α[i+1] \rightarrow α[i].

Let P be (n)-built-up, α ∈ Δ be limit and α[i] < μ ≤ α[i+1]. We define \( \langle α_k \rangle_{k<ω} \) inductively as follows:
\[ α_0 = α[i+1], \]
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\[ \alpha_{k+1} = \begin{cases} 
\mu & \text{if } \alpha_k = \mu, \\
(\alpha_k)[j] & \text{if } \alpha_k \text{ is limit }> \mu, \text{ where } j \text{ is the least such that } (\alpha_k)[j] > \mu, \\
\beta & \text{if } \alpha_k = \beta + 1. 
\end{cases} \]

Because \( P \) is \((n)\)-built-up, we can prove \( \alpha \geq \mu \) and \( \alpha_k \xrightarrow{n} \alpha[i] \) by ind. on \( k \). Then \( \langle \alpha_k \rangle_{k<\omega} \) is non-ascending sequence, hence there is an \( m < \omega \) such that \( \alpha_k = \mu \) for \( k \geq m \). Hence \( \mu \xrightarrow{n} \alpha[i] \), in particular, \( \alpha[i] \leq \mu[n] \).

**Theorem 4.2.** For any \( n < \omega \), there is no assignment of fundamental sequences with property \( B[n] \) for the whole of the second number class.

(Proof) Let \( \Omega \) be the first uncountable ordinal. Assume there is such a system for fixed \( n < \omega \). Then \( \lambda \alpha.\alpha[n] \) is a regressive function (i.e. \( \alpha[n] < \alpha \) for \( 0 < \alpha < \Omega \)), and therefore there is an \( A \subseteq \Omega \) of order type \( \Omega \) and a \( \beta \in \Omega \) such that \( \alpha[n] = \beta \) for all \( \alpha \in A \) (cf. Levy[2]). Let \( \langle \alpha_i \rangle_{i < \omega} \) be the first \( \omega \) elements of \( A \), \( \alpha = \lim_{n<\omega} \alpha_i \) and \( \langle \alpha[i] \rangle_{i<\omega} \) be the fundamental sequence for \( \alpha \).

Since \( \alpha > \beta \), there is an \( i < \omega \) such that \( \beta < \alpha[i] < \alpha \). Let \( m < \omega \) be the number such that \( \alpha[i] < \alpha_m < \alpha \), and \( p \) be the greatest number such that \( \alpha[p] < \alpha_m \). Then \( \alpha[p] < \alpha_m \leq \alpha[p+1] \), so \( \beta < \alpha[p] \leq \alpha_m[n] \). But \( \alpha_m[n] = \beta \) because \( \alpha_m \in A \). Contradiction.

Finally, we define properties \( A \) and \( C[n] \) to extend Remark of [4] as follows:
P has property A iff if $\beta < \alpha < \Delta$, then $\alpha \rightarrow_n \beta$ for some $n < \omega$.

P has property C[n] iff if $\langle F_\alpha \rangle_{\alpha < \Delta}$ is a sequence of number theoretic functions satisfies the properties (a)$_n$, (b)$_n$ and (c), and $\beta < \gamma \in \Delta$, then $F_\beta$ is dominated by $F_\gamma$ (i.e. there is an $m$ such that for all $n \geq m$, $F_\beta(n) < F_\gamma(n)$).

We can prove the following theorem which is an extension of Remark of [4].

**Theorem 4.3.** If P is (n)-built-up for some $n < \omega$, then P has properties A and C[n].

(Proof) Assume P is (n)-built-up. First we prove that P has A by ind. on $\alpha$. Case 1. $\alpha = 0$. Trivial. Case 2. $\alpha = \gamma + 1$. $\gamma \rightarrow_m \beta$ for some $m < \omega$ by ind. hyp. Then $\alpha \rightarrow_m \gamma \rightarrow_m \beta$. Case 3. $\alpha$ is limit. $\alpha[i] > \beta$ for some $i$. Then $\alpha[i] \rightarrow_k \beta$ for some $k$ by ind. hyp. Let $r = \max(n, k, i)$. By using Proposition 1.1, $\alpha \rightarrow_r \alpha[r] \rightarrow_r \alpha[i] \rightarrow_r \beta$. Next we prove that P has C[n]. Because P has A, $\gamma \rightarrow_r \beta$ for some $r$. If we put $m = \max(n, r) + 1$, then by Theorem 1.3, $F_\beta(x) < F_\gamma(x)$ for $m \leq x < \omega$.

**References.**


[3] D.Schmidt, Built-up systems of fundamental sequences and hierarchies of number theoretic functions, Arch. math. Logik


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