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<th>$\Sigma_2$ Collection and Maximal Sets</th>
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<tr>
<td>Author(s)</td>
<td>Chong, C.T.</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1988), 644: 4-15</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1988-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/100244">http://hdl.handle.net/2433/100244</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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\(\Sigma_2\) Collection and Maximal Sets

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The subject of reverse recursion theory studies the following basic question (*): What axioms of Peano arithmetic are required, or sufficient, to prove theorems in recursion theory? This question (perhaps first raised by Stephen Simpson) is a natural offshoot of a related, more general question: Which set existence axioms of second order arithmetic are required, or sufficient, to prove theorems in ordinary mathematics (Simpson [1985])? While it was only in recent years that investigations were carried out on (*), some of the answers obtained have nevertheless been very interesting—not only because they provide a better understanding of the fundamental constructions in recursion theory, but also because many of the techniques used to obtain the answers were inspired by those introduced in \(\alpha\) recursion theory. Indeed in many cases the original techniques appear to fit snugly into the new situation, giving the impression of a technical development that is historically correct. Our purpose here is to study the question on the existence of maximal sets, prove a general nonexistence result of these sets for a wide class of models of \(P^- + B\Sigma_2\), and to point out the connection of the proof techniques with those in \(\alpha\) recursion theory.

Let \(P^-\) be the set of axioms of Peano arithmetic minus the induction scheme. These consist of universal closures of the following:

\[
\begin{align*}
  x' &\neq 0 \\
  (x' = y') &\rightarrow (x = y) \\
  x \neq 0 &\rightarrow 0' \leq x \\
  x < y &\leftrightarrow (\exists t)(x + t' = y) \\
  x < y \lor x = y \lor x > y
\end{align*}
\]
\[ x + y = y + x; \quad x \cdot y = y \cdot x \]
\[ x + (y + z) = (x + y) + z \]
\[ x \cdot (y \cdot z) = (x \cdot y) \cdot z \]
\[ x + 0 = x; \quad x \cdot 0 = 0; \quad x^0 = 0' \]
\[ x + y' = (x + y)'; \quad x \cdot y' = (x \cdot y) + x \]
\[ x^v = x^v \cdot x \]
\[ x \cdot (y + z) = (x \cdot y) + (x \cdot z) \]
\[ x + y = x + z \rightarrow y = z \]

The induction scheme is arranged into a hierarchy of increasing complexity strength. For each \( n < \omega \), let \( I\Sigma_n \) be the \( \Sigma_n \) induction scheme which says that for every \( \Sigma_n \) formula \( \varphi \),

\[
[(\varphi(0) \& (\forall x)(\varphi(x) \rightarrow \varphi(x'))) \rightarrow (\forall x)\varphi(x)].
\]

Clearly we have Peano Arithmetic = \( P^- + \{I\Sigma_n | n < \omega\} \).

A scheme which is closely related to the \( \Sigma_n \) induction scheme is the \( \Sigma_n \) least member scheme. This states that if \( \varphi \) is \( \Sigma_n \) and is nonempty, then there is a least member \( a \) satisfying \( \varphi \). And finally, we have the \( \Sigma_n \) collection scheme: if \( \varphi \) is \( \Sigma_n \), then

\[
(\forall y < x)(\exists w)\varphi(y, w)
\]

implies there is a \( b \) such that

\[
(\forall y < x)(\exists w < b)\varphi(y, w).
\]

In other words, on every initial segment of a model of \( P^- \), the existence of a witness for every member in the initial segment implies the existence of a uniform bound where witnesses may be found.

Define \( B\Pi_n \), \( \Pi_n \) and \( L\Pi_n \) similarly for \( \Pi_n \) formulas.

The next theorem provides a classification of the relative strengths of these arithmetical schema:
Proposition (Kirby and Paris [1978]). In every model of $P^- + I\Sigma_0$, we have

\[ I\Sigma_{n+1} \rightarrow B\Sigma_{n+1} \rightarrow I\Sigma_n \]

\[ I\Sigma_n \leftrightarrow \Pi_n \leftrightarrow L\Sigma_n \leftrightarrow \Pi_n \]

\[ B\Pi_n \leftrightarrow B\Sigma_{n+1} \]

Arrows do not reverse except where indicated.

It is not difficult to verify that all the basic notions of recursion theory can be formalized in $P^- + I\Sigma_0$. For example, $n$-tuples can be coded by single elements in models of $P^- + I\Sigma_0$. Indeed, given $\mathcal{M} \models P^- + I\Sigma_0$, one has the following definition:

**DEFINITION.** $H \subset \mathcal{M}$ is $\mathcal{M}$-finite if $H$ has a code in $\mathcal{M}$.

In particular, finite sets are not the only $\mathcal{M}$-finite sets. In any initial segment of $\mathcal{M}$, the $\Delta_0$ sets are all $\mathcal{M}$-finite. Using this notion of $\mathcal{M}$-finiteness, we may define, in a model $\mathcal{M}$ of $P^- + I\Sigma_0$, a set to be recursively enumerable (r.e.) if it is $\Sigma_1(\mathcal{M})$, and is recursive if its complement is r.e. as well. The notion of reduction can also be introduced:

**DEFINITION.** Let $X$ and $Y$ be subsets of $\mathcal{M} \models P^- + I\Sigma_0$. $X$ is pointwise recursive in $Y$ (or weakly recursive in $Y$) if there is an r.e. set $\Phi$ of quadruples such that for all $x$,

\[ x \in X \leftrightarrow (\exists H)(\exists K)[(x, 1, H, K) \in \Phi \& H \subset Y \& K \cap Y = \emptyset], \]

and

\[ x \notin X \leftrightarrow (\exists H)(\exists K)[(x, 0, H, K) \in \Phi \& H \subset Y \& K \cap Y = \emptyset]. \]
(H, K are M-finite sets.)

The notation \( X \leq_{w} Y \) is used to express the relation pointwise recursive in. It is not difficult to see that if \( \mathcal{M} \) is the standard model of arithmetic, then \( \leq_{w} \) is a transitive relation. In general, however, the transitivity of \( \leq_{w} \) is not automatic.

Let \( \mathcal{M} \) be a model of \( P^{-} + I \Sigma_{0} \). Let \( \mathcal{R} \) be denote the collection of all r.e. sets in \( \mathcal{M} \). One may verify that \( \mathcal{R} \) forms a lattice, with \( \emptyset \) and \( \mathcal{M} \) forming respectively the least and greatest element in the lattice. Let \( \mathcal{R}^{*} \) be obtained from \( \mathcal{R} \) by identifying those r.e. sets with \( \mathcal{M} \)-finite difference.

**DEFINITION.** An r.e. set \( M \) is maximal in \( \mathcal{R}^{*} \) if there is no r.e. set lying strictly between \( M \) and \( \mathcal{M} \), modulo \( \mathcal{M} \)-finite sets.

Maximal sets were first constructed by Friedberg [1957] for the standard model \( \mathcal{N} \). It has since become a subject of intense study for recursion theorists (see Soare [1987] for an exposition). Our interest here is to examine the strength of the statement 'there exists a maximal set' vis à vis fragments of the induction scheme. More specifically,

**THEOREM 1.** (a) There is a maximal set in every model of \( P^{-} + I \Sigma_{2} \).
   
   (b) There is a model of \( P^{-} + B \Sigma_{2} + \neg I \Sigma_{2} \) with no maximal set.
   
   (c) There is a model of \( P^{-} + I \Sigma_{0} + \neg I \Sigma_{1} \) with a maximal set.

Our original proof for (a) covered only the case of \( P^{-} + I \Sigma_{3} \). Slaman pointed out that the argument worked for \( I \Sigma_{2} \) as well. We will not discuss the proofs of (a) and (c) (see Chong [to appear]), but will instead take up (b).

To obtain a model as specified by (b), one is reminded of the ordinal \( \aleph_{\alpha}^{L} \) in which Lerman and Simpson [1973] showed that there is no
maximal set. A key property that was used in that paper was that every constructible subset of $\omega$ is $\aleph\omega^L$-finite. Thus the first step towards establishing (b) is to perhaps identify a model $M$ of $P^- + B\Sigma_2 + \neg I\Sigma_2$ with a similar property. This is supplied by a result of Mytilinaios and Slaman [1988]:

**Lemma 1.** There is a model $M_0$ of $P^- + B\Sigma_2 + \neg I\Sigma_2$ such that every set of natural numbers is the standard part of an $M_0$-finite set.

Proof: Starting with $V_{\omega+\omega}$, the collection of all sets of rank less than $\omega + \omega$, form the ultrapower $V^*$ of $V_{\omega+\omega}$ over a nonprincipal ultrafilter. There is an embedding $j$ of $V_{\omega+\omega}$ into $V^*$. The structure $j(N)$ is then a model of full Peano arithmetic with the additional property that every set of natural numbers is the standard part of a $j(N)$-finite set. Now take a nonstandard number $a$ in $j(N)$, and let $M_0$ be the union of the $H_n$'s defined below:

$$H_0 = \{b|b < a\};$$

$$H_{n+1} = \Sigma_1^{n+1}\text{-Hull}\{\{b|\exists c > b)(c \in H_n)\}\}.$$

Here $\Sigma_1^{n+1}(H_n)$ means taking the Skolem hull of $H_n$ in $j(N)$ with respect to the first $n + 1$ $\Sigma_1$ functions. Then $M_0$ is a $\Sigma_1$ elementary substructure of $j(N)$. An argument of Kirby and Paris [1978] shows that $M_0$ is a model of $B\Sigma_2$ but not of $I\Sigma_2$. Furthermore, in $M_0$ every set of natural numbers is the standard part of an $M_0$-finite set.

We say that a set $A$ in a model $M$ is regular if $A|a$ is $M$-finite for every $a$. The next result is well-known:

**Lemma 2.** Every r.e. set $A$ in a model of $P^- + I\Sigma_1$ is regular.

**Lemma 3.** Let $M_0$ be as in Lemma 1. There is a function $f \leq_o \emptyset$ such that $f$ maps $N$ cofinally into $M_0$.

Proof: Define $f(n)$ to be the supremum of $H_n$ in the proof of Lemma 1.
An effective version of Lemma 3 yields the following approximation for the function $f$:

**LEMMA 4.** There is a total recursive function $f'$ such that

(a) For all $n \in \mathbb{N}$, $\lim_{s \to n} f'(s, n) = f(n)$;

(b) For all nonstandard $n$, $\lim_{s \to n} f'(s, n)$ does not exist;

(c) $f'(s, n) \leq f'(t, m)$ for all $s \leq t$ and $n \leq m$.

Thus the model $M_0$ is seen to be endowed with properties reminiscent of the ordinal $\aleph_1^L$: Every set of natural numbers is the standard part of an $M_0$-finite set, and there is a $\Sigma_2$ cofinal function from $\mathbb{N}$ into $M_0$. In Lerman and Simpson [1973], analog of these properties in $\aleph_1^L$ were sufficient to show that no maximal sets exist. The idea was to split $\aleph_1^L$ into the union of $\omega$ many pairwise disjoint simultaneous r.e. sets. By choosing those $n$'s for which $A_n$ has nonempty intersection with a given $\Pi_1$ set $X$, one gets an $\aleph_1^L$-finite subset $K$ of $\omega$, with the property that $X \cap A_n \neq \emptyset$ for each $n \in K$. One can now easily split $K$ into two disjoint infinite $\aleph_1^L$-finite sets $K_1$ and $K_2$, so that the corresponding r.e. sets $\bigcup \{A_n | n \in K_1\}$ and $\bigcup \{A_n | n \in K_2\}$ split $X$ into two non-$\aleph_1^L$-finite pieces.

Now for models of fragments of arithmetic such as $M_0$, a recursive splitting of the universe into $\omega$ pieces is not possible (by the Overspill Lemma), and so a different strategy is required. The intuition remains the same: Given a $\Pi_1$ set $X$, devise a method of recursively guessing (correctly) $\omega$ many elements of $X$, without ‘touching’ $\omega$ many other elements of $X$.

**LEMMA 5.** Let $M_0$ be the model of Lemma 1. If $M$ is r.e. with complement non-$M_0$-finite, then $M$ is contained in an r.e. set $B$ such that neither $B \setminus M$ nor $M_0 \setminus B$ are $M_0$-finite.

Proof: By Lemma 2, $M$ is regular so that $M_0 \setminus M$ is not bounded. Now for each $n \in \mathbb{N}$, there is a standard $m > n$ such that every member of $M | f(n)$ is enumerated by stage $f(m)$. Let $g(n)$ be the the least such $m$. 


The set $K$ of pairs $(n, g(n))$ is the standard part of an $M_0$-finite set $K^*$. Assume without loss of generality that $\bar{M} \cap [f(n), f(n+1)) \neq \emptyset$ for each $n \in \mathcal{N}$. Choose $m_{g(n)}$ to be the least member of $\bar{M}$ greater than or equal to $f(g(n))$. We then have the situation where at any stage $s$, if the value of $m_{g(n+1)}$ is correctly guessed (recursively with the help of the function $f$), then so are all the values of $m_{g(n')}$ for all $n' < n$.

The next step is to ensure that when computing approximations to $m_{g(n)}$, there is no possibility of mistaken identity. In other words, we need a recursive guessing function such that at any stage $s$, if $x$ 'appears' to be $m_{g(n)}$, then $x$ is not $m_{g(n')}$ for any $n' < n$. This is obtained via the function $h$ whose existence is asserted below:

**SUBLEMMA.** There is a recursive function $h$ taking each triple $(s, n', n)$ into 2 such that for standard $n' < n$, $\lim_s h(s, n', n) = h(n', n)$ exists, and such that if $h(s, n', n) = h(n', n)$ then the number which appears to be $m_{g(n)}$ is not equal to $m_{g(n')}$.  

This technical lemma evolves from Chong and Lerman [1976] which studies the existence problem of hyperhypersimple sets in $\aleph_1^L$. The key point is that whilst it is not possible to select recursively from a given $\Pi_1$ set a $\Pi_1$ subset of order type $\omega$, the existence of functions like $h$ allows one to devise a good approximation to this set.

To complete the proof of Lemma 5, one now uses the function $h$ to 'fill up' the complement of $M$ to arrive at the r.e. set $B$ whose complement contains the set of all $m_{g(n)}$'s for $n$ odd. This is done by setting $B$ to be $M$ together with those $x$'s which appear to be $m_{g(n)}$ ($n$ odd) at some stage $s$ where $h(s, n', n) = h(n', n)$ for all $n' < n$. This ensures that $B$ contains all $m_{g(n)}$ for $n$ odd, and excludes all $m_{g(n)}$ for $n$ even.

Lemma 5 implies Theorem 1 (b). A consequence of this theorem is the following result which is of methodological interest:
COROLLARY. There is no finite injury construction of a maximal set.
Proof: Mytilinaios [to appear] showed that every finite injury argument can be carried out in models of $P^- + I\Sigma_1$. Theorem 1 (b) says that $B\Sigma_2$, hence (by the Proposition) $I\Sigma_1$, is not sufficient to do the maximal set construction.

One can generalize Theorem 1 (b) to cover a much wider class of models of $P^- + B\Sigma_2$. To do this we begin with a lemma which is a refinement of Smoryński [1984]:

**Lemma 6.** Let $M \models P^- + I\Sigma_2$. If $K \subset N$ is the standard part of a $\Pi_2$ or $\Sigma_2$ subset of $M$, then $K$ is the standard part of an $M$-finite set.
Proof: Let $M$ be given as in the hypothesis and suppose that $\varphi(x, a)$ is $\Pi_2$ over $M$ with parameter $a$. An analog of Lemma 2 says that in a model of $P^- + I\Sigma_2$ every $\Sigma_2$ (hence $\Pi_2$) set is regular. Let $b$ be a nonstandard number in $M$. Then the initial segment of $b$ intersected with the set of numbers which satisfy $\varphi(x, a)$ is $M$-finite. The standard part of this intersection is $K$. A similar argument applies to $\Sigma_2$ subsets. This proves the lemma.

**Definition.** A function $p$ on a model of $P^- + B\Sigma_2$ is an $N$-function if $p$ is total on $N$ and maps standard numbers to standard numbers.

**Lemma 7.** Let $M \models P^- + I\Sigma_2$. There is an $M' \subset M$ such that $M'$ is a model of $P^- + B\Sigma_2$ but not of $I\Sigma_2$, with the additional property that every standard part of a $N$-function which is $\Sigma_2$ definable is the standard part of an $M'$-finite set.
Proof: In $M$ build the sequence $\{H_n\}$ as in the proof of Lemma 1. Then $M' = \cup_n H_n$ is a $\Sigma_1$ elementary substructure of $M$, with the additional property that there is a function $f \leq^* \emptyset$ mapping $N$ cofinally into $M'$. An analog of Lemma 4 then provides a recursive approximation $f'$ such
that for all \( n \in \mathbb{N} \), \( \lim_{\epsilon} f'(s, n) = f(n) \). Let \( K \) be the standard part of a \( \Sigma_2 \) definable \( \mathcal{N} \)-function \( p \) over \( \mathcal{M}' \), defined by

\[
(i, j) \in p \iff \mathcal{M}' \models (\exists x)(\forall y) \varphi(x, y, a, i, j),
\]

where \( \varphi \) is \( \Delta_0 \) and \( a \) is a parameter. We claim that \( K \) is the standard part of an \( \mathcal{M}' \)-finite set.

Let \( Q \) be a set of triples such that

\[
(i, (m, j)) \in Q \iff \mathcal{M}' \models (\exists x)(\forall y)(\forall t \geq s)(\forall x)(\varphi(x, y, a, i, j) \land f'(t, m) = f'(s, m) \land x \leq f'(s, m)).
\]

Then \( Q \) is \( \Sigma_2 \) definable. Let \( c_0 \in \mathcal{M}' \) be nonstandard, and set \( Q_{c_0} = Q|c_0 \cdot \). Let \( K_{c_0} \) be the standard part of \( K_{c_0} \). Let \( \psi \) be the \( \Sigma_2 \) formula used to define \( Q \). Since \( \mathcal{M}' \) is a \( \Sigma_1 \) elementary substructure of \( \mathcal{M} \), we have for \( (i, (m, j)) \in \mathcal{M}', \mathcal{M}' \models \psi \) implies \( \mathcal{M} \models \psi \). This means that members of \( K_{c_0} \) continue to satisfy the same formula in \( \mathcal{M} \).

Let \( X \) be the set of elements less than \( c_0 \) in \( \mathcal{M} \) satisfying \( \psi \). Then \( X \) is \( \Sigma_2 \) definable over \( \mathcal{M} \) and so by Lemma 6 is \( \mathcal{M} \)-finite. As \( \mathcal{M}' \) is a \( \Sigma_1 \) elementary substructure of \( \mathcal{M} \), \( X \) is also \( \mathcal{M}' \)-finite.

By the definition of \( \psi \), we see that if \( i, m \) and \( j \) are standard such that \( (i, (m, j)) \in X \), then it must be that \( (i, (m, j)) \in K_{c_0} \). Furthermore, by the very nature that \( p \) is an \( \mathcal{N} \)-function, for each standard \( i \) there is a unique standard \( j \) such that \( (i, (m, j)) \) belongs to \( X \) for some \( m \). Hence let \( K^* \) be the set of \( (i, j) \)'s in \( X \) such that \( j \) is the least member of \( \mathcal{M}' \) satisfying \( (i, (m, j)) \in X \) for some \( m < c_0 \). Then \( K \) is the standard part of \( K^* \) and \( K^* \) is \( \mathcal{M}' \)-finite. This proves the lemma.

Now Lemmas 3, 4 and 5 continue to hold for models \( \mathcal{M}' \) satisfying the conclusions of Lemma 7. In particular, the \( \mathcal{N} \)-function \( g \) in the proof of Lemma 5 is \( \Sigma_2 \), and is therefore the standard part of an \( \mathcal{M}' \)-finite set.
The same holds true for the $\mathcal{N}$-function $h$ in the Sublemma. It follows that in $\mathcal{M}'$ there are no maximal sets.

Now in Smoryński [1984], there is a proof of Scott’s Theorem which gives a characterization of subsets of $2^\omega$ which are standard systems of models of Peano arithmetic (i.e. members of $2^\omega$ which are standard parts of ‘finite sets’ of a given model of Peano arithmetic). This is stated as follows:

**Lemma 8.** Let $\mathcal{X}$ be a countable family of sets of natural numbers, then there is a model $\mathcal{M}$ of Peano arithmetic for which $\mathcal{X}$ is the standard system if and only if:

(a) $\mathcal{X}$ is closed under Boolean operations;
(b) $\mathcal{X}$ is closed under Turing reducibility;
(c) $\mathcal{X}$ satisfies a weak form of König’s Lemma: If $X \in \mathcal{X}$ codes an infinite binary tree, then some $Y$ in $\mathcal{X}$ codes an infinite path through $X$.

Thus there exist infinitely many different countable subsets of $2^\omega$ which are standard systems of models of Peano arithmetic. Applying Lemma 7 one finds infinitely many countable models $\mathcal{M}'$ of $P^- + BS_2 + \neg I_2$ with pairwise different standard systems for which all standard parts of $\Pi_2$ or $\Sigma_2$ sets are standard parts of $\mathcal{M}'$-finite sets. Each of these models has no maximal sets. This proves the next result.

**Theorem 2.** There exist infinitely many countable models of $P^- + BS_2 + \neg I_2$ with pairwise different standard systems which have no maximal sets.

Note that in contrast the model $\mathcal{M}_0$ of Theorem 1(b) is uncountable. A theorem of Guaspari allows one to improve the above result to (uncountable) models of $P^- + BS_2 + \neg I_2$ with standard systems of size $\kappa_1$. 
We end this paper with three questions:

(a) Assume $\mathcal{M} \models P^{-} + B\Sigma_{2}$. Is it true that if $\mathcal{M}$ has a maximal set then $\mathcal{M} \models I\Sigma_{2}$? A positive answer to this question would give a complete characterization of the existence of maximal sets over the base theory $P^{-} + B\Sigma_{2}$.

(b) Theorem 1 (c) indicates that the existence of maximal sets does not require any assumption stronger than $P^{-} + I\Sigma_{0}$, provided that the underlying universe is carefully chosen. In the proof of Theorem 1 (c) (Chong [to appear]), the model chosen has the property that there is a $\Sigma_{2}$ map from $\mathcal{N}$ onto the whole universe. Do all models of $P^{-} + I\Sigma_{0} + \neg I\Sigma_{1}$ with maximal sets have this property?

(c) What is the complexity, in the hierarchy of fragments of Peano arithmetic, of various theorems on maximal sets? In particular, is Soare's theorem (Soare [1974]) on the automorphisms of the lattice of r.e. sets sending maximal sets to maximal sets provable in $P^{-} + I\Sigma_{2}$?
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