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Combinatorics on $P_\kappa\lambda$

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It is known that if $P_\kappa\lambda$ has the partition property, then κ is λ-ineffable. (Magidor [2]) We shall discuss the converse direction under the existence of stationary codings.

Definition. Let $X \subset P_\kappa\lambda = \{x \subset \lambda : |x| < \kappa\}$.

(i) We call a function $F:\{(x,y): x,\ y \in X \text{ and } x \subsetneq y\} \longrightarrow 2$ a partition of X. $H \subset X$ is homogeneous for F if there is a $k<2$ so that $F((x,y)) = k$ for every x, y in H with $x \subsetneq y$.

(ii) X has the partition property (Part*(X)) if every partition of X has a stationary homogeneous set.

(iii) X is λ-ineffable if for any $f:X \longrightarrow P_\kappa\lambda$ with $f(x) \subset x$ for every $x \in X$ there is an $A \subset \lambda$ so that $\{x \in X: f(x) = x \cap A\}$ is stationary.

(iv) Let $t:\lambda \longrightarrow P(X)$. t' is a flip of t if $t'(\alpha) = t(\alpha)$ or $X - t(\alpha)$ for all $\alpha<\lambda$.

(v) X has the flipping property if every $t:\lambda \longrightarrow P(X)$ has a flip t' so that $\underset{\alpha<\lambda}{\Delta} t'(\alpha) = \{x \in X: x \in t'(\alpha) \text{ for all } \alpha \in x\}$ is stationary.

(vi) X has a stationary coding if X is stationary and there is a injective $c:X \rightarrow \lambda$ such that $c(x) \in y$ whenever $x \subsetneq y$.

Lemma (DiPrisco, Zwicker [1]). (iii) and (v) are equivalent.

Combining the flipping property and SC(stationary coding), we get;

Proposition. If X is λ-ineffable with SC, then Part*(X).

Proof. Suppose that X is λ-ineffable and $c:X \rightarrow \lambda$ is a stationary coding. For a partition F of X, we define $t:\lambda \longrightarrow P(X)$ as follows.

$$t(\alpha) = \begin{cases} \{x: F(c^{-1}(\alpha), x) = 1\} & \text{if } \alpha \in \text{range}(c) \\ X & \text{otherwise} \end{cases}$$

Since X has the flipping property, there is a flip t' of t so that $\underset{\alpha<\lambda}{\Delta} t'(\alpha) = S$ is stationary.

Let $S_1 = \{x \in S: t'(c(x)) = t(c(x))\}$ and $S_2 = S - S_1$. Either S_1 or S_2 is stationary.

We shall show that both of them are homogeneous for F.

Suppose that $x \subsetneq y \in S_1$. Since $c(x) \in y$ and $y \in S$, $y \in t'(c(x)) = t(c(x))$. Hence $F(x,y) = 1$.

If $x \subsetneq y \in S_2$, $y \in t'(c(x)) = X - t(c(x))$. Thus $F(x,y) = 0$. $\square$

But it it is open whether a λ-ineffable set with SC exists whenever κ is λ-ineffable. If κ is λ-supercompact, the answer is of course " Yes ". In fact, every X with normal measure one is λ-ineffable and there is a Y with normal measure one on which $\{\langle x, \sup(x)\rangle : x \in Y\}$ is a stationary coding. ([2] and [4])

Shelah proved the following.

Proposition (Shelah [3]). If κ is ineffable, $\lambda \overset{w}{\not\rightarrow} (\omega)^{<\omega}_{\kappa}$ and $\lambda^{<\kappa} = \lambda$, then there is a stationary coding set.

Moreover it is well known that;

Proposition. If Part*(X), there is a $Y \subset X$ with SC.

Proof. Define F by

$$F(x,y) = \begin{cases} 1 \text{ if } c(x) \in y \\ 0 \text{ otherwise.} \end{cases}$$

(c is any injective map from X to λ.) Let Y be a stationary homo-geneous set for F. Pick an $x \in Y$. Since Y is unbounded, there is a $y \in Y$ such that $c(x) \in y$. Hence $F(x,y) = 1$. This means $F(x,y) = 1$ for all $x \subsetneq y$ in Y. Now we have shown that c is a stationary coding for Y.

The above proposition shows that every injection of $P_\kappa\lambda$ into λ can be a stationary coding if $P_\kappa\lambda$ has the partition property.

References.

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[4] Solovay, Reinhardt, Kanamori, Strong axioms of infinity and elementary embeddings, Ann. Math. Logic 13 (1978), 73-116.

[5] Zwicker, $P_\kappa\lambda$ combinatorics I: Stationary coding sets rationalize the club filter, in Axiomatic set theory, Contemporary Math. 31 (1984), 243-259.