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ASYMPTOTIC THEORY IN CHANGE POINT PROBLEM

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1. Introduction.

Let Π_k be a normal population whose mean and variance are μ_k and σ_k^2, respectively, $k=1,2,\cdots,n$. Many authors have examined the problem of testing the null hypothesis

$$H_1: \mu_1=\mu_2=\cdots=\mu_n$$

against the two sided alternative

$$K_1: \bigcup_{1\leq k\leq n-1} K_{1k}, \text{ where } K_{1k}: \mu_1=\mu_2=\cdots=\mu_k\neq\mu_{k+1}=\cdots=\mu_n,$$

or against the one-sided alternative

$$K_2: \bigcup_{1\leq k\leq n-1} K_{2k}, \text{ where } K_{2k}: \mu_1=\mu_2=\cdots=\mu_k>\mu_{k+1}=\cdots=\mu_n.$$

This problem is known as the problem of detecting a change in means, (see, e.g., Sen and Srivastava (1975), or Srivastava and Worsley (1986)). Apart from normal means, Hsu (1979) treated the problem of detecting a change of means in a sequence of gamma variables. Worsley (1986) treated the problem somewhat generally, that is, in the framework of a sequence of exponential family random variables. He derived the likelihood ratio test for the two sided alternative and obtained the null distribution of the test, conditional on the total sum of observations, in the terms of iterated integrals. More recently, Yao and Davis (1986) has determined the asymptotic null distribution of the likelihood ratio test as the number of the populations n tends to infinity for the case of normal means. Siegmund (1986) also has treated the approximation of the asymptotic null distribution of the modified likelihood ratio test.

In this paper we will proceed as follows. In Section 2 some mathematical preliminaries are provided. Section 3 is expository and the asymptotic distribution of the likelihood ratio test for a shift in normal means given by Yao and Davis (1986) reviewed briefly. In Section 4 we treat the asymptotic null distribution of a modified likelihood ratio test for a shift in normal variances.

2. Mathematical preliminaries

Definition 2.1. A process $W(t)$, $t\geq 0$, satifying the following conditions is called the standard Brownian motion.

(1) Gaussian

(2) $EW(t)=0$, $Cov(W(s),W(t))=s\wedge t$

(3) continuous

Proposition 2.1 (Law of the Iterated Logarithm). Let $W(t)$, $t\geq 0$, be the standard Brownian motion. Then

$$\limsup_{t\to 0} |W(t)|/\sqrt{t\log|\log t|} = 1 \quad \text{a.s.}$$

$$\limsup_{t\to \infty} |W(t)|/\sqrt{t\log|\log t|} = 1 \quad \text{a.s.}$$

Definition 2.2. A Gaussian process $W^0(t)$ defined by

$$W^0(t)=W(t)-tW(1),\ 0\le t\le 1,$$

is called the standard Brownian bridge.

Proposition 2.2. Let $W^0(t)$ be the standard Brownian bridge. Then

$$W^0(0)=W^0(1)=0,\ EW^0(t)=0,\ \text{and } \mathrm{Cov}(W^0(s),W^0(t))=s(1-t),\ s\le t.$$

Definition 2.3. A stationary Gaussian process $X(t)$, $-\infty<t<\infty$, satisfying

$EX(t)=0$ and $\mathrm{Cov}(X(s),X(t))=e^{-(t-s)}$, $s\le t$, is called the Ornstein-Uhlenbeck (O-U)process.

Note that we use the symbol $W(t)$, $W^0(t)$ and $X(t)$ for these three particular process respectively.

In this paper some transformations among these three process play an essential role, and we summarize them here.

Proposition 2.4. Let $Y((1/2)\log t)=W(t)/\sqrt{t}$, $t>0$. Then $Y(t)$, $-\infty<t<\infty$, is the O-U process $X(t)$.

Remark: Proposition 2.4 implies that a square root boundary for $W(t)$ is transformed to a constant boundary for $X(t)$.

Proposition 2.5. (DeLong (1981), p. 2212). Let $Y(t)$, $t\ge 0$, be defined by

$$Y(t) = (1+t)W^0(t/(1+t)).$$

Then $Y(t)$ is the standard Brownian motion $W(t)$ and

$$P\{\max_{t_1<t<t_2} W^0(t)/\sqrt{t(1-t)} < c\} = P\{\max_{1<t<f(t_1,t_2)} W(t)/\sqrt{t} < c\},$$

where $f(t_1,t_2)=t_2(1-t_1)/t_1(1-t_2)$.

Remark: Proposition 2.5 implies that a $\sqrt{t(1-t)}$ boundary for $W^0(t)$ is transformed to a $\sqrt{t}$ boundary for $W(t)$ and therefore to a constant boundary for $X(t)$.

Proposition 2.6 (DeLong(1981), p. 2203).

$$P\{\max_{1<t<T} |W(t)|/\sqrt{t} < c\} = \sum_{i=1}^{\infty} \alpha_i(c,q)T^{-\beta_i(c,q)},$$

where $q=1/2$, β_i is the i-th root of $M(\beta_i+q,q,-c^2/2)=0$, M is the confluent hypergeometric function, and α_i is equal to

$$\frac{e^{-c^2/2}(c^2/2)^q\ M(\beta_i+q,q+1,-c^2/2)}{\Gamma(q+1)\beta_i(d/dx)M(x+q,q,-c^2/2)}$$

with the derivative of M evaluated at $x=\beta_i$.

Proposition 2.7 (DeLong (1981), p. 2205). For T and c large,

$$P\{\max_{1\le t\le T} |W(t)|/\sqrt{t} \ge c\} \sim \frac{c}{\sqrt{2\pi}} e^{-c^2/2}\{(\log T)(1-1/c^2)+2/c^2+O(c^{-4})\}$$

Next we summarize the extreme value theory briefly,(see, e.g., Resnik(1987), p. 9). Let $X_1,X_2,\ldots,X_n$ be a i.i.d. sequence of random variables with common distribution F(x). Set $M_n=\max_{1\le i\le n} X_i$

Definition 2.4. Suppose there exist $a_n>0$ and $b_n\in R^n_c$ such that $P\{(M_n - b_n)/a_n < x\} \longrightarrow G(x)$ weakly as $n\longrightarrow\infty$ where G(x) is non-degenerate. Then F is said to belong to the domain of attraction of G and G is said a extreme value distribution.

Proposition 2.8 (Gnedenko(1945)). Extreme value distribution is of the type of one of the following three cases:

(1) $\Phi_\alpha(x)= 0$ for $x<0$ and $\exp(-x^{\alpha})$ for $x\ge 0$ with $\alpha>0$

(2) $\Psi_\alpha(x)= \exp(-(-x)^{\alpha})$ for $x<0$ with $\alpha>0$ and 1 for $x\ge 0$

(3) $\Lambda(x)=\exp(-e^{-x})$ $x \in R^1$.

An extension of an extreme value distribution to the maximum of a stationary Gaussian sequence was given by Berman (1964) and extentions to the maximum of a continuous parameter stationary Gaussian process have appeared in various literatures, (Berman(1974,1980), Pickands (1969), etc). Recently Berman(1984) has treated a stationary non-Gaussian case. Here we refer only to a theorem by Pickands (1969).

Proposition 2.9 (Pickands(1969)). Let $Y(t)$, $-\infty<t<\infty$, be a stationary Gaussian process such that $EY(t)=0$ and $Cov(Y(s),Y(s+t))=r(t)$. Let

$Z(T)=\max_{0\le t\le T} Y(t)$. If $r(t)=1-C|t|^{\alpha}+o(|t|^{\alpha})$ and $\lim_{t\to\infty} r(t)\log t=0$,

then for all x, $-\infty<x<\infty$,

$$\lim_{T\to\infty} P\{ a(T)^{-1}(Z(T)-b(T))\le x\}=\exp(-e^{-x}),$$

where $a(T)=(2\log T)^{-1/2}$ and $b(T)=a(T)^{-1}+$

$$a(T)\{((1/\alpha-1/2)\log\log T+\log((2\pi)^{-1/2} C^{1/\alpha} H_{\alpha} 2^{(2-\alpha)/2\alpha}))\}$$

and $$0 < H_{\alpha}=\lim_{T\to\infty} T^{-1}\int_0^{\infty} e^{s} P\{ \sup_{0\le t\le T} Y(t) > s\}ds < \infty,$$

where $Y(t)$is a stationary Gaussian process with means and covariances;

$$EY(t)=-|t|^{\alpha},\ Cov(Y(s),Y(t))=|s|^{\alpha}+ |t|^{\alpha}+ |t-s|^{\alpha}.$$

Remark: In the expression of $b(T)$, $(1/\alpha-1/2)$ is described as $(1/2-1/\alpha)$ in the original paper of Pickands, but from the book of Leadbetter et el (1983), p.237, Theorem 12.3.5 or from the process of the original proof by Pickands we know $(1/\alpha-1/2)$ is a correct one.

For the O-U process $X(t)$, $r(t)=e^{-t}=1-|t|+o(|t|)$, and $C=\alpha=1$. For $\alpha=1$, $H_{\alpha}=1$ (Pickands (1969), p. 77), and we have the next proposition.

Proposition 2.10. Let $Z(T)=\max_{0\leq t\leq T} X(t)$ and

$a(T)=(2\log T)^{-1/2}$ and $b(T)=a(T)^{-1} + a(T)\{(1/2)\log\log T-\log\sqrt{\pi}\}$

Then

$$\lim_{T\to\infty} P\{a(T)^{-1}(Z(T)-b(T))\leq x\}=\exp(-e^{-x}),$$

and moreover

$$\lim_{T\to\infty} P\{ a(T)^{-1}(Z(T)-b^{*}(T))\leq x\}=\exp(-(1/\sqrt{\pi})e^{-x}),$$

where $b^{*}(T)=a(T)^{-1}+(1/2)a(T)\log\log T$.

Proof of Proposition 2.10. The former part is trival from Proposition 2.9. We must show the last part of the proposition. Since $b(T)=b^{*}(t)-a(T)\log\sqrt{\pi}$,

$$\lim_{T\to\infty} P\{a(T)^{-1}(Z(T)-b^{*}(T)) \leq y-\log\sqrt{\pi}\}=\exp(-e^{-y}),$$

and setting $x=y-\log\sqrt{\pi}$, $e^{-y}=(1/\sqrt{\pi})e^{-x}$, which show the last part of Proposition 2.10.

3. A shift in normal means

For a shift in normal means the likelihood ratio test rejects H if T is too large, where $T=\max_{1\leq k\leq n-1} |T_k|$ and

$$T_k =\{S_k/\sqrt{n} - (k/n)S_n/\sqrt{n})\}/\sqrt{(k/n)(1-(k/n))}$$

and for a one-sided shift, T must be replaced by T^{*} where

$T^{*}=\max\{0, \max_{1\leq k\leq n-1} T_k\}$, and $S_k=\sum_{i=1}^{k} X_i$, $k=1,2,\ldots,n$, (see, e.g., Yao and Davis, (1986), p.343).

Hawkins(1977) tried to determine the normalizing constants and the limiting distribution from heuristic arguments, and suggested an approximation by an stationary Gaussian sequence satisfying Berman's condition(1964): $r(n)\log r(n) \to 0$. Yao and Davis(1986) determined exactly the normalizing constants and the limiting distribution.

One of the motivations of this section was to understand the reason why the coefficients $2/\sqrt{\pi}$ or $1/\sqrt{\pi}$ appear in the limiting distribution in Proposition 3.1., which do not appear in the extreme value disrtribution of i.i.d. sequence, $\Lambda(x)=\exp(-e^{-x})$, in Proposition 2.8. Another was to know the origin of the normalizing constants a_n and b_n.

<u>Proposition 3.1</u> (Yao and Davis(1986)).

$$\lim_{n\to\infty} P\{T \leq a_n x + b_n\}=\exp\{-(2/\sqrt{\pi})e^{-x}\},$$

$$\lim_{n\to\infty} P\{T^* \leq a_n x + b_n\}=\exp\{-(1/\sqrt{\pi})e^{-x}\},$$

where $a_n=(2\log\log n)^{-1/2}$ and $b_n=a_n^{-1} + (a_n/2)\log\log\log n$.

For T^* their arguments are briefly reviewed here. First note T_k is represented as

$$T_k = W^0(t_k)/\sqrt{t_k(1-t_k)},\quad t_k=k/n,\ k=1,2,\ldots,n-1,$$

and the distribution of T^* is essentially approximable by that of the maximum of a stationary Gaussian sequence, (see, Hawkins, 1979, p.184). They proceeded as follows. First they showed from the arguments based on the law of the iterated logarithm of the Brownian motion $W(t)$ and the symmetry of the Brownian bridge $W^0(t)$ that

$$P\{\max_{1\leq k\leq n-1} T_k \leq a_n x + b_n\} \sim P\{\max_{1\leq k\leq n/\log n} T_k' \leq a_n x + b_n\}^2$$

where $T_k'=W(t_k)/\sqrt{t_k}$, $k=1,2,\ldots,n/\log n$. Secondly, noting that the sequence $T_k'=W(t_k)/\sqrt{t_k}$, $k=1,2,\ldots,n/\log n$, is equal in distribution to the sequence $T_k''=W(k)/\sqrt{k}$, $k=1,2,\ldots,n/\log n$, they showed that

$$P\{\max_{1\le k\le n/\log n} T_k' \le a_n x + b_n\}^2 \sim P\{\max_{1\le k\le n/\log n} T_k'' \le a_n x + b_n\}^2$$

$$\sim P\{\max_{1\le k\le n/\log n} T_k'' \le a_{n/\log n}\, x + b_{n/\log n}\}^2.$$

Finally the last term was evaluated through the theorem of Darling and Erdos(1956).

The theorem by Darling and Erdos(1956) is now briefly reviewed below. In short, the therom evaluates the maximum of $T_k''=W(k)/\sqrt{k}$, $k=1,2,\ldots,n$, by making the transformation given in Proposition 2.4 , that is, by embedding this into the O-U process $X(t)$. Further the limiting distribution of the maximum of the sequence $X(\tau_k)$, $\tau_k=(1/2)\log k$, $k=1,2,\ldots,n$, is shown to be the same with that of the maximum over the interval $(0,\tau_n)$, and the limiting distribution of the last quantity is evaluated.

<u>Proposition 3.2</u> (Darling and Erdős(1956)).

$$\lim_{n\to\infty} P\{\max_{1\le k\le n-1} S_k/\sqrt{k} \le a_n x+b_n\}=\exp\{(-1/2\sqrt{\pi})e^{-x}\},$$

where $a_n=\sqrt{2\log\log n}$ and $b_n=a_n^{-1} + (a_n/2)\log\log\log n$.

The proof of Proposition 3.2 is sketched as follows. After observing that

$$P\{\max_{1\le k\le n} S_k/\sqrt{k} \le a_n x+b_n\}=P\{\max_{1\le k\le n} X(\tau_k) \le a_n x+b_n\},$$

where X is the O-U process and $\tau_k=(1/2)\log k$, they showed the next lemma.

<u>Lemma</u> (Darling and Erdős(1956)).

(1) $\lim_{c\to\infty} P\{ T(c) > \mu(c)y\} = e^{-y}$.

(2) $(1/2)\log N(c) - T(c) \to 0$ in probability as $c\to\infty$.

where $T(c)=\inf\{t>0;\ X(t)\geq c\}$, $N(c)=\min\{k\geq 1;\ X(\tau_k)\geq c\}$, and

$$\mu(c)=(\sqrt{2\pi}/c)e^{c^2/2}.$$

Note that (1) implies that

$$\lim_{c\to\infty} P\{ \max_{0\leq t\leq \mu(c)y} X(t) \leq c\} = \lim_{c\to\infty} P\{T(c) > \mu(c)y\} = e^{-y},$$

and this is rewritten that

$$\lim_{T\to\infty} P\{ \max_{0\leq t\leq T} X(t) \leq d(T) \} = e^{-y},$$

where $d(T)= a(T)^{-1} + (1/2)a(T)\log\log T - a(T)\log \sqrt{\pi}y$,

and $a(T)=(2\log T)^{-1/2}$. Setting $b(T)=a(T)^{-1} + (1/2)a(T)\log\log T$ and $x=\log \sqrt{\pi}y$, we see that (1) implies

$$\lim_{T\to\infty} P\{ \max_{0\leq t\leq T} X(t) \leq a(T)x + b(T)\} = \exp\{-(1/\sqrt{\pi})e^{-x}\}.$$

This result is coincident with Proposition 2.9.

On the other hand (2) implies

$$\lim_{c\to\infty} P\{ \max_{0\leq t\leq \mu(c)y} X(t) \leq c\} = \lim_{c\to\infty} P\{T(c) > \mu(c)y\} = \lim_{c\to\infty} P\{\tau_{N(c)} \geq \mu(c)y\}$$

$$= \lim_{c\to\infty} P\{ \max_{0\leq \tau_k\leq \mu(c)y} X(\tau_k) \leq c\} = \lim_{c\to\infty} P\{ \max_{0\leq \tau_k\leq T} X(t) \leq a(T)x + b(T)\}$$

$$= \exp\{-(1/\sqrt{\pi})e^{-x}\}.$$

That is, (2) implies the limiting distributions are identical for the maximum of $X(t)$ over the discrete points τ_k, $k=1,2,..n$ and over the interval $[0,\tau_n]$, and (1) is applicable. By solving the equation $\mu(c)y=(1/2)\log n$, the normalizing constants a_n and b_n in Proposition 3.1 was obtained, and this constants are also the normalizing constants for the likelihood ratio test T or T^*, as shown by Yao and Davis.

Finally we close this section after stating a question:

Question : Can we get Proposition 3.1. ?, by observing

$$P\{\max_{1\le k\le n} W^0(t_k)/\sqrt{t_k(1-t_k)}\le c\}=P\{\max_{1\le k\le n} X(\tau_k)\le c\},$$

where $t_k=k/n$ and $\tau_k=(1/2)\log(t_k/(1-t_k))$, and then following the line of the proof of Lemma of Darling and Erdős.

4. A shift in variances

In this section we treat the problem of detecting a change in a sequence of variances of univariate normal populations and study the asymptotic behavior of the modified likelihood ratio test as the number of populations, n, tends to infinity.

Let X_k, $k=1,\ldots,n$ be distributed as $\chi^2(\nu_k)$ multiplied by a constant σ_k^2, where $\chi^2(\nu)$ denotes a chi-square random variable with ν degrees of freedom. Our problem is to test the null hypothesis

$$H:\ \sigma_1^2 = \ldots = \sigma^2_n$$

against the alternative

$$K = \bigcup_{k=1}^{n-1} K_k,\ \text{where } K_k:\ \sigma^2_1 =\ldots= \sigma^2_k \ne \sigma^2_{k+1} = \ldots = \sigma^2_n$$

Let

$$S_k = \sum_{i=1}^{k} X_i,\ \text{and}\ S_k^* = \sum_{i=k+1}^{n} X_i,\ (k=1,\ldots,n-1)\ \text{and}\ S = \sum_{i=1}^{n} X_i$$

and let

$$N_k = \sum_{i=1}^{k}\nu_i,\quad N_k^* = \sum_{i=k+1}^{n}\nu_i,\ \text{and}\ N = \sum_{i=1}^{n}\nu_i.$$

The likelihood ratio test rejects H if $\max_{1\le k\le n-1} T_k$ is too large, where each T_k is the likelihood ratio test for testing H against K_k and given by the following quantity,

$$T_k = N\log(S/N) - N_k\log(S_k/N_k) - N_k\log(S_k^*/N_k^*).$$

In the problem of detecting a change of normal means, Siegmund(1986) proposed the modified likelihood ratio test given by

$$\max_{1<m_0\le k\le m_1<n} \Lambda_k,$$

where Λ_k is the likelihood ratio statistic for testing the null hypotesis against the alternative that a change has occured between the k-th population and (k+1)-th population. He treated the asymptotic approximation of the probabability of the first kind of error for the modified likelihood ratio test statistic when the number of the populations n, and m_0 and m_1 tend to infinity in such a way that

Condition (A): for some $0<t_0<t_1<1$, $m_0/m \longrightarrow t_0$, $m_1/m \longrightarrow t_1$

In this study we take the same modification for the likelihood ratio test. That is,

$$\text{let } T(m_0,m_1) = \max_{1<m_0\le k\le m_1<n} T_k$$

and we call this as the modified likelihood ratio test statistic. In the following we say simply that n tends to infinity if the condition (A) holds, and we consider all the limit theorems under the null hypothesis. Finally it should be also noted that the terminology of "asymptotic equivalence" about two statistic F_1 and F_2 is used for both cases $F_1=F_2+o_p(1)$ and $F_1=(1+o_p(1))F_2$.

First we have the next limit theorem.

Theorem 1. Let $T1(m_0,m_1) = \max_{1<m_0\le k\le m_1<n} T1_k$, where

$$T1_{,k} = (N_k/2)\{1-(S_k/N_k)/(S/N)\}^2 + (N_k^*/2)\{1-(S_k^*/N_k^*)/(S/N)\}^2.$$

Then $T(m_0,m_1)$ and $T1(m_0,m_1)$ is asymptotically equivalent when n tends to infinity.

Proof. Without loss of generality we assume $\sigma_i^2=1$, $i=1,\ldots,n$. Then from Central Limit Theorem, all of S/N, S_k/N_k, and S_k^*/N_k^* converge to 1 with probability 1. Since T_k is rewrited as

$$T_k=-N_k\log[1-\{1-(S_k/N_k)/(S/N)\}] - N_k^*\log[1-\{1-(S_k^*/N_k^*)/(S/N)\}],$$

taking the Tailor expansion of the log function about 1, we obtain

$$T_k = T1_k + o_p(1), \quad k = m_0,\ldots,m_1.$$

This completes the proof of Theorem 1.

Corollary 1. Let $T2(m_0,m_1) = \max_{1<m_0\le k\le m_1<n} T2_k$,

where

$$T2_k = (N_k/2)\{(S/N)-(S_k/N_k)\}^2 + (N_k^*/2)\{(S/N)-(S_k^*/N_k^*)\}^2.$$

Then $T2(m_0,m_1)$ is asymptotically equivalent to $T(m_0,m_1)$ when n tends to infinity.

Proof. This is easy to see if we note that

$$T1(m_0,m_1) = (S/N)^{-2}T2(m_0,m_1)$$

and $(S/N)^{-2}$ converges to 1 with probability 1.

Next simulation results show to what extent the approximations of $T(m_0,m_1)$ by $T1(m_0,m_1)$ and $T2(m_0,m_2)$ are good. Note that the results are based on the 10,000 generations of the statistics.

Table I

The quantiles of $T(m_0,m_1)$, $T1(m_0,m_1)$, and $T2(m_0,m_1)$

$n = 50$
$m_0 = 5$
$m_1 = 45$
$\nu_1 = \ldots$
$\nu_n = 20$

%	$T(m_0,m_1)$	$T1(m_0,m_1)$	$T2(m_0,m_1)$
10	1.3258	1.3156	1.31383
30	2.7112	2.17388	2.1685
50	3.02059	3.0044	3.00169
60	3.56599	3.559	3.5462
70	4.23738	4.1963	4.2154
80	5.14256	5.1173	5.0125
90	6.6528	6.624	6.6366
95	8.2040	8.2562	8.35647

Table II

The quantiles of $T(m_0,m_1)$, $T1(m_0,m_1)$, and $T2(m_0,m_1)$

$n = 100$
$m_0 = 10$
$m_1 = 90$
$\nu_1 = \ldots$
$\nu_k = 20$

%	$T(m_0,m_1)$	$T1(m_0,m_1)$	$T2(m_0,m_1)$
10	1.459	1.4626	1.4506
30	2.3075	2.3042	2.298
50	3.2262	3.2176	3.207
60	3.7859	3.7395	3.7515
70	4.423	4.4177	4.4198
80	5.3976	5.3462	5.3417
90	6.938	6.9399	6.9465
95	8.4808	8.4511	8.4777

5. Convergence of $T2(m_0,m_1)$

The purpose of this section is to give the convergece theorem of $T2(m_0,m_1)$ to a functional of the standared

Brownian bridge. First we state a well-known lemma.

Lemma 5.1(Donskar, see Billingsley(1968), p. 137). Let $\xi_1,...,\xi_n$ be identically and independently distributed random variables with mean 0 and finite variance σ^2. Let $Y_n(t)=S_{[nt]}/\sqrt{n}\sigma$. Then when n tends to infinity $Y_n(t)$ converges to W(t), $0\le t\le 1$, weakly, where W(t) denotes the standard Brownian motion.

Theorem 2. Assume $\nu_1=...=\nu_n=\nu$. When n tends to infinity $T2(m_0,m_1)$ converges to Z in law,

where

$$Z = \max_{t_1<t<t_2} \{ W_0(t)^2/t(1-t)\}$$

Proof. Let $\tilde{S}=S-N$, $\tilde{S}_k=S_k-N_k$ and $\tilde{S}_k{}^*=S_k{}^*-N_k{}^*$. Then $\tilde{S}$, $\tilde{S}_k$, and $\tilde{S}_k{}^*$ are the sums of identically and independently distributed random variables with mean 0 and finite variance $\sqrt{2\nu}$, respectively. It is easy to see $T2_k$ is rewritten as

$$T2_k=(1/2\nu)[(n/k)\{(k/n)(\tilde{S}/\sqrt{n})-\tilde{S}_k/\sqrt{n}\}^2 + (n/k^*)\{ (k^*/n)(\tilde{S}/\sqrt{n})-\tilde{S}_k{}^*/\sqrt{n})^2]$$

$$=(1/2\nu)(n/k + n/k^*)\{(k/n)(\tilde{S}/\sqrt{n})-\tilde{S}_k/\sqrt{n}\}^2$$

Set $Y_n(t)=\tilde{S}_{[nt]}/\sqrt{n}\sqrt{2\nu}$, $0\le t\le 1$. Then $T2_k$ is also rewritten as

$$T2_k= (n/k + n/k^*) \{(k/n)Y_n(1)-Y_n(k/n)\}^2.$$

When n tends to infinity, since $Y_n(t)$ converges to W(t) weakly, $T2(m_0,m_1)$ converges to Z in law, which completes the proof of Theorem 2.

6. Evaluation of the limiting distribution of $T(m_0,m_1)$

We have the following evaluation about the limiting distribution of $T(m_0,m_1)$.

Theorem 3.

$$\lim_{n\to\infty} P\{T(m_0,m_1)\le c^2\} = \sum_{i=1}^{\infty} \alpha_i(c,q)T^{-\beta_i(c,q)}$$

where $q=1/2$ and $T = t_2(1-t_1)/t_1(1-t_2)$

Proof of Theorem 3. This is immediatly deduced from Corollary of Theorem 1 and Propositions 2.5 and 2.6.

Theorem 4. For large c,

$$\lim_{n\to\infty} P\{T(m_0,m_1)\ge c^2\} \sim (ce^{-c^2/2}/\sqrt{2\pi})\{(\log T)(1-c^{-2})+2/c^2+O(c^{-4})\}$$

Proof of Theorem 4. This is an immediate consequence of Corollary of Theorem 1 and Propositions 2.5 and 2.7.

Next table III represents the approximations of the upper tail probability by the formula in Theorm 4.

Table III

n	m_0	m_1	c^2	exact P	approximated P
50	5	45	8.2040	0.05	0.07753
100	10	90	8.4808	0.05	0.0688

Note that the used datas are parts of Tables I and II. The approximation errors seem to be moderate.

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