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Nonlinear Nonautonomous Differential Equations

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Introduction.

Let X be a real Banach space with norm $\|\cdot\|$ and let $C = C([-r,0];X)$, $0 \le r < \infty$, be the Banach space of all continuous functions from $[-r,0]$ into X. We denote the norm of $\phi \in C$ by $\|\phi\|_C$, i.e., $\|\phi\|_C = \sup_{\theta\in[-r,0]}\|\phi(\theta)\|$.

This paper is concerned with the abstract nonlinear functional differential equation

$$(FDE;\phi)_s \quad \begin{cases} u'(t) + A(t)u(t) \ni F(t,u_t), \ t \in [s,T] \ (s \ge 0) \\ u_s = \phi, \end{cases}$$

where $u:[-r,T] \to X$ is the unknown function; $\{A(t);\ t \in [0,T]\}$ is a given family of operators in X; $F:[0,T]\times C \to X$ is a given function; ϕ is given in C. The symbol u_t denotes the function $u_t(\theta) = u(t+\theta)$, $\theta \in [-r,T]$.

We assume that the following conditions (A.1) – (A.4) hold:

(A.1) There exists a constant α_0 such that for each $t \in [0,T]$, $A(t) + \alpha_0$ is accretive and $R(I + \lambda A(t)) = X$ for $0 < \lambda < 1/\max(0,\alpha_0)$.

(A.2) There are a continuous function $h:[0,T] \to X$ which is of bounded variation on $[0,T]$, and a monotone increasing continuous function $L_1:[0,\infty) \to [0,\infty)$ such that

$$\|A_\lambda(t)x - A_\lambda(\tau)x\| \le \|h(t) - h(\tau)\| L_1(\|x\|)(1 + \|A_\lambda(\tau)x\|)$$

for $0 < \lambda < 1/\max(0,\alpha_0)$, $t,\tau \in [0,T]$ and $x \in X$, where $J_\lambda(t) = (I + \lambda A(t))^{-1}$ and $A_\lambda(t) = \lambda^{-1}(I - J_\lambda(t))$.

(A.3) There exists a constant $\beta_0 > 0$ such that for $\phi,\psi \in C$ and $t \in [0,T]$, $\|F(t,\phi) - F(t,\psi)\| \le \beta_0\|\phi - \psi\|_C$.

(A.4) There are a continuous function $k:[0,T] \to X$ which is of bounded variation on $[0,T]$, and a monotone increasing function $L_2:[0,\infty) \to [0,\infty)$ such that for $t,\tau \in [0,T]$ and $\phi \in C$,
$\|F(t,\phi) - F(\tau,\phi)\| \le \|k(t) - k(\tau)\| L_2(\|\phi\|_C)$.

The purpose of this paper is to show the existence of a generalized solution of $(FDE;\phi)_s$. In particular, in case X is reflexive, we show that the generalized solution is the strong solution of $(FDE;\phi)_s$.

Recently, Kartsatos [6] has proved the existence of the evolution operator associated with $(FDE;\phi)_s$ under the following conditions (B.2) and (B.3) instead of (A.2), (A.3) and (A.4).

(B.2) There exists an increasing continuous function $L:[0,\infty) \to [0,\infty)$ such that for all $\lambda > 0$, $x \in X$, $t,\tau \in [0,T]$,
$\|A_\lambda(t)x - A_\lambda(\tau)x\| \le |t-\tau| \, L(\|x\|)(1 + \|A_\lambda(\tau)x\|)$.

(B.3) There exists a positive constant b such that
$\|F(\tau,f_1) - F(t,f_2)\| \le b(|t-\tau| + \|f_1 - f_2\|_C)$
for every $t,\tau \in [0,T]$, $f_1,f_2 \in C$.

In order to apply the method of successive approximations to $(FDE;\phi)_s$, he essentially used conditions (B.2) and (B.3) which imply that $A_\lambda(t)x$ and $F(t,f)$ are Lipschitz continuous in t. However this method does not seem to be directly applicable under (A.1) – (A.4). Also, it has not been proved that the generalized solutions in the sense of Kartsatos are weak solutions, except on a small interval in which they are Lipschitz continuous. (For a refined definition of weak solutions, see Definition 2.)

Now, in order to improve these points, we use the nonlinear evolution operator theory of Crandall and Pazy [2] as the main tool for solving $(FDE;\phi)_s$. Various author have so far considered $(FDE;\phi)_s$ under different setting in nonlinear operator theory. (For example, see [3,4,10].)

This paper consists of three sections. In section 1, we recall the nonlinear evolution operator theory. In section 2, we show that the existence of generalized solutions of $(FDE;\phi)_s$ and it is represented as the uniform limit of a sequence of strong solutions of the approximating equations for $(FDE;\phi)_s$ involving the Yosida approximations. Finally, in section 3, we investigate some properties of generalized solutions and consider weak solutions and give the existence for strong solutions of $(FDE;\phi)_s$ when X is reflexive.

1. Basic concept of nonlinear evolution operator theory

We discuss briefly some concepts in the nonlinear evolution operator theory. Let Y be a Banach space with $\|\ \|_Y$. A family $\{V(t,s);\ 0\le s\le t\le T\}$ of operators $V(t,s): Y\to Y$ is said to be a family of operators, if

$$V(t,t)y = y \text{ for all } y\in Y \text{ and } t\in[0,T],$$

$$V(t,r)V(r,s) = V(t,s) \text{ for } 0\le s\le r\le t\le T.$$

Let $\{V(t,s);\ 0\le s\le t\le T\}$ be an evolution operator and define the operator $B(t)$ by

$$D(B(t)) = \{y\in Y;\ \lim_{h\to 0+}(1/h)(V(t+h,t)y - y) \text{ exists}\}$$

$$-B(t)y = \lim_{h\to 0+}(1/h)(V(t+h,t)y - y) \text{ for } y\in D(B(t)).$$

If $D(B(t))$ is non-empty for each $t\ge 0$, then the family $-B(t)$ is said to be the infinitesimal generator of $V(t,s)$.

Consider the problem $(FDE;\phi)_s$. Suppose that for every $\phi \in C$ and $s \geq 0$, $(FDE;\phi)_s$ has the unique solution $u(s,\phi)(\cdot)$ and that $A(t)$ and F are continuous. Then one can find that the infinitesimal generator of the evolution operator $V(t,s)$, defined by $V(t,s)\phi = u_t(s,\phi)$ is given by

$$
(1.1)\qquad \begin{aligned} D(\hat{A}(t)) &= \{\phi \in C;\ \phi' \in C,\ \phi(0) \in D(A(t)), \\ &\qquad \phi'(0) + A(t)\phi(0) \ni F(t,\phi)\} \\ \hat{A}(t)\phi &= -\phi'. \end{aligned}
$$

Conversely, given the family $A(t)$, we shall prove that under suitable conditions on $A(t)$ and F, $A(t)$ generates an evolution operator $V(t,s)$ such that $V(t,s)\phi$ gives the segments of a solution of $(FDE;\phi)_s$. This will rely on the following result due to Crandall — Pazy [2].

A subset B of $Y \times Y$ is in class $\mathcal{A}(\omega)$ if for each $\lambda > 0$ such that $\lambda\omega > 1$ and each pair $[y_i, z_i] \in B$, $i=1,2$, we have

$$
(1.2)\qquad \|(y_1 + \lambda z_1) - (y_2 + \lambda z_2)\|_Y \geq (1 - \lambda\omega)\|y_1 - y_2\|_Y.
$$

B is called accretive if $B \in \mathcal{A}(0)$. Also, (1.2) implies that $(I + \lambda B)^{-1}$ exists on $R(I + \lambda B)$ and is a Lipschitzian with constant $(1 - \lambda\omega)^{-1}$. Let $B \in \mathcal{A}(\omega)$ and $R(I + \lambda B) = Y$ for all $0 < \lambda \leq \lambda_0$. Define $|By|$ by $|By| = \lim_{\lambda \to 0+} \|B_\lambda y\|_Y$, where $J_\lambda = (I + \lambda B)^{-1}$ and $B_\lambda = \lambda^{-1}(I - J_\lambda)$. (Note that this limit exists, although it may be infinite.) For such B we define $\hat{D}(B) = \{y \in Y;\ |By| < \infty\}$ which is called a generalized domain of B.

Theorem 1. (Crandall-Pazy). Let $T > 0$ and ω be real number and assume that $B(t)$ satisfies the following conditions:

(C.1) $B(t) \in \mathcal{A}(\omega)$ for $0 \leq t \leq T$, (C.2) $R(I + \lambda B(t)) = Y$ for $0 \le t \le T$ and $0 < \lambda < \lambda_0$, where $\lambda_0 > 0$ and $\lambda_0\omega < 1$,

(C.3) There are a continuous function $f:[0,T] \to Y$ which is of bounded variation on $[0,T]$, and a monotone increasing function $L: [0,\infty) \to [0,\infty)$ such that

$$\|B_\lambda(t)y - B_\lambda(\tau)y\|_Y \le \|f(t) - f(\tau)\|_Y L(\|y\|_Y)(1 + \|B_\lambda(\tau)y\|_Y)$$

for $0 < \lambda < \lambda_0$, $0 \le t,\tau \le T$ and $y \in Y$.

Then

$$(1.3) \quad V(t,s)y = \lim_{n\to\infty} \prod_{i=1}^{n} \left(I + \left(\frac{t-s}{n}\right)B\left(s + i\left(\frac{t-s}{n}\right)\right)\right)^{-1} y$$

exists for $y \in \overline{D(B(t))}$ and $0 \le s < t \le T$. The $V(t,s)$ defined by (1.3) for $0 \le s < t \le T$ and by $V(t,t) = I$ for $0 \le t \le T$ is an evolution operator on $\overline{D(B(t))}$.

2. On the existence of generalized solutions of $(FDE;\phi)_s$

We define for each $t \in [0,T]$ an operator $\hat{A}(t):D(\hat{A}(t)) \subset C \to C$ by (1.1).

Proposition 1. Suppose that conditions (A.1)-(A.4) hold. If $\{\hat{A}(t);\ t \in [0,T]\}$ is the family of operators defined in C by (1.1), then there exists a family of nonlinear evolution operators $V(t,s)$: $\overline{D(\hat{A}(t))} \subset C \to C$ such that for all $\phi \in \overline{D(\hat{A}(t))}$

$$(2.1) \quad V(t,s)\phi = \begin{cases} \lim\limits_{n\to\infty} \prod\limits_{i=1}^{n} \left(I + \left(\frac{t-s}{n}\right)\hat{A}\left(s + i\left(\frac{t-s}{n}\right)\right)\right)^{-1}\phi & 0 \le s < t \le T, \\ \phi & 0 \le s = t \le T. \end{cases}$$

Proof. We are going to apply Theorem 1 for $B(t) = \hat{A}(t)$ and $Y = C$. Under assumptions (A.1) and (A.3) we can apply [11, Proposition 1] to show that $\hat{A}(t) \in \mathcal{A}(\omega_0)$ for $t \in [0,T]$ and $R(I + \lambda\hat{A}(t)) = C$ for $0 < \lambda < 1/\omega_0$, where $\omega_0 = \max(0,\alpha_0+\beta_0)$. Thus conditions (C.1) and (C.2) hold for $\hat{A}(t)$. Next, by using the same argument as in [4, Theorems 12 and 13] and the inequality $\|h(t) - h(\tau)\| + \|k(t) - k(\tau)\| \le |g(t) - g(\tau)|$, where $g(t) = \mathrm{Var}([0,t];h) + \mathrm{Var}([0,t];k)$ and $\mathrm{Var}([0,t];h)$ denotes the total variation of h on $[0,t]$, we will show that $\hat{A}(t)$ satisfies (C.3) with $B(t) = \hat{A}(t)$ and $f(t) = g(t)I$, where I denotes the identity in X. To this end, set $\phi(t,\cdot) = (I + \lambda A(t))^{-1}\psi, \psi \in C$. Then we have $\phi(t,\theta) = e^{\theta/\lambda}\phi(t,0) + \int_\theta^0 \frac{1}{\lambda} e^{-(s-\theta)/\lambda}\psi(s)\,ds$, and by $\phi(t,\cdot) \in D(\hat{A}(t))$, we have $\phi(t,0) = \psi(0) + \lambda\phi'(t,0) = \psi(0) - \lambda A(t)\phi(t,0) + \lambda F(t,\phi(t,\cdot))$, i.e., $\phi(t,0) = (I + \lambda A(t))^{-1}(\psi(0) + \lambda F(t,\phi(t,\cdot)))$.

Now, for $0 < \lambda < 1$ with $\lambda\omega_0 < 1/2$,

$$\begin{aligned}
&\|\phi(t,\cdot) - \phi(\tau,\cdot)\|_C = \|\phi(t,0) - \phi(\tau,0)\| \\
&= \|(I + \lambda A(t))^{-1}(\psi(0) + \lambda F(t,\phi(t,\cdot))) \\
&\qquad\qquad - (I + \lambda A(\tau))^{-1}(\psi(0) + \lambda F(\tau,\phi(\tau,\cdot)))\| \\
&\le \lambda(1 - \lambda\alpha_0)^{-1}\|F(t,\phi(t,\cdot)) - F(\tau,\phi(\tau,\cdot))\| \\
&\qquad + \lambda\|h(t) - h(\tau)\|\, L_1(\|\psi(0) + \lambda F(\tau,\phi(\tau,\cdot))\|) \\
&\qquad\qquad \times (1 + \|A_\lambda(\tau)(\psi(0) + \lambda F(\tau,\phi(\tau,\cdot)))\|).
\end{aligned}$$

But,

$$\begin{aligned}
&\|A_\lambda(\tau)(\psi(0) + \lambda F(\tau,\phi(\tau,\cdot)))\| \\
&= \lambda^{-1}\|\psi(0) + \lambda F(\tau,\phi(\tau,\cdot)) - J_\lambda(\tau)(\psi(0) + \lambda F(\tau,\phi(\tau,\cdot)))\| \\
&\le \|\hat{A}_\lambda(\tau)\psi\|_C + \|F(\tau,\phi(\tau,\cdot))\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|\phi(t,\cdot) - \phi(\tau,\cdot)\|_C \\
&\le \lambda(1 - \lambda\alpha_0)^{-1}[\beta_0\|\phi(t,\cdot) - \phi(\tau,\cdot)\|_C \\
&\qquad\qquad + \|k(t) - k(\tau)\|\, L_2(\|\phi(\tau,\cdot)\|_C)] \quad +
\end{aligned}$$

$$+ \lambda \|h(t) - h(\tau)\| L_1(\|\psi(0)\| + \lambda\|F(\tau,\phi(\tau,\cdot))\|) \times (1 + \|\hat{A}_\lambda(\tau)\psi\|_C + \|F(\tau,\phi(\tau,\cdot))\|).$$

Thus there exists a constant K_1 such that

$$(2.2) \quad \|\phi(t,\cdot) - \phi(\tau,\cdot)\|_C$$

$$\leq K_1\lambda|g(t) - g(\tau)|[1 + \|\hat{A}_\lambda(\tau)\psi\|_C][L_2(\|\phi(\tau,\cdot)\|_C)$$

$$+ (1 + \|F(\tau,\phi(\tau,\cdot))\|)L_1(\|\psi(0)\| + \lambda\|F(\tau,\phi(\tau,\cdot))\|)].$$

Suppose that $\chi \in C$ and $\phi_0 \in D(\hat{A}(0))$. Then $\|F(\tau,\chi)\| \leq$

$\beta_0[\|\chi\|_C + \|\phi_0\|_C] + \|k(\tau) - k(0)\| L_2(\|\phi_0\|_C) + \|F(0,\phi_0)\|$

and hence $\|F(\tau,\chi)\|$ is bounded by an increasing function of $\|\chi\|_C$. It remains to prove that $\|\phi(\tau,\cdot)\|_C \leq L_3(\|\psi\|_C)$ for some monotone increasing function L_3. From (2.2), $\|\phi(\tau,\cdot)\|_C \leq$

$$\leq \|\phi(0,\cdot)\|_C + K_1\lambda|g(\tau) - g(0)|[1 + \|\hat{A}_\lambda(0)\psi\|_C] \times$$

$$\times [L_2(\|\phi(0,\cdot)\|_C) +$$

$$+ (1 + \|F(0,\phi(0,\cdot))\|)L_1(\|\psi(0)\| + \lambda\|F(0,\phi(0,\cdot))\|)].$$

However $\lambda\|\hat{A}_\lambda(0)\psi\|_C = \|\psi - \hat{J}_\lambda(0)\psi\|_C \leq \|\psi\|_C + \|\phi(0,\cdot)\|_C$ and if $\phi_0 \in D(\hat{A}(0))$ then

$$\|\phi(0,\cdot)\|_C = \|(1 + \lambda\hat{A}(0))^{-1}\psi\|_C$$

$$\leq (1 - \lambda\omega_0)^{-1}[\|\psi - \phi_0\|_C + \lambda\|\hat{A}(0)\phi_0\|_C] + \|\phi_0\|_C$$

$$\leq K_2[\|\psi\|_C + \|\phi_0\|_C + \|\hat{A}(0)\phi_0\|_C] \text{ for some } K_2,$$

which implies that

$\|\phi(0,\cdot)\|_C$ is bounded by a monotone increasing function of $\|\psi\|_C$. Thus $\hat{A}(t)$ satisfies (C.3) with $B(t) = \hat{A}(t)$ and $f(t) = g(t)I$. Therefore, the conclusion of the proposition follows from Theorem 1. Q.E.D.

Note that, as was proved in [5], $\hat{D}(\hat{A}(t))$ is independent of t because $\hat{A}(t)$ satisfies (C.3) and also $\hat{D}(A(t))$ is independent of t because of (A.2). In what follows, $\hat{D}_0$ and $\hat{D}$ stand for a generalized domain of $\hat{A}(0)$ and $A(0)$, respectively.

As in [3, Proposition 1], we have the following

Proposition 2. Suppose that conditions (A.1)-(A.4) hold. If $u(s,\phi)(\cdot)$ for each $\phi \in \hat{D}_0$ and $s \geq 0$ is defined by

$$(2.3) \quad u(s,\phi)(t) = \begin{cases} \phi(t-s) & s-r \leq t \leq s, \\ (V(t,s)\phi)(0) & s \leq t \leq T, \end{cases}$$

where $V(t,s)$ is as constructed by Proposition 1, then $u(s,\phi)(\cdot) \in C([s-r,T];X)$ and $V(t,s)\phi = u_t(s,\phi)$ for $t \in [s,T]$.

Remark. We introduce the following stronger conditions than (A.1) and (A.2):

(A.1)' There exists a constant $\alpha_1 > 0$ such that for $x,y \in X$, $\| A(t)x - A(t)y\| \leq \alpha_1 \| x - y\|$.

(A.2)' There are a continuous function $h:[0,T] \to X$ which is of bounded variation on $[0,T]$ and a monotone increasing continuous function $L_4:[0,\infty) \to [0,\infty)$ such that

$\| A(t)x - A(\tau)x\| \leq \|h(t) - h(\tau)\| L_4(\|x\|)(1 + \| A(\tau)x\|)$

for all $t,\tau \in [0,T]$ and $x \in X$.

Since (A.1)' and (A.2)' imply (A.1) and (A.2), Propositions 1 and 2 hold, although (A.1) and (A.2) are replaced by (A.1)' and (A.2)'.

Next, we recall the following expression for $\hat{D}_0$.

Lemma 1 ([4, Theorem 10]). Let $A(t)$ and $F(t,\phi)$ satisfy conditions (A.1) and (A.3). Then

$\hat{D}_0 = \{\phi \in C;\ \phi$ is Lipschitz continuous function and $\phi(0) \in \hat{D}\}$.

Remark. If ϕ is Lipschitz continuous function and $\phi(0) \in \hat{D}$, then the function defined by (2.3) is a Lipschitzian. In fact, for such ϕ, by [2, Proposition 2.3] and Lemma 1, there exists a constant K such that for $0 \le s \le t, \tau \le T$, $\|V(t,s)\phi - V(\tau,s)\phi\|_C \le$ $\le K|t - \tau|$. So that our assertion holds.

Definition 1. A function $u(s,\phi)(\cdot) \in C([-r,T];X)$ is said to be a strong solution of $(FDE;\phi)_s$ if it is an absolutely continuous function is which differentiable a.e. on $[s,T]$ and satisfies $(FDE;\phi)_s$ a.e. on $[s,T]$.

We shall first prove the following uniqueness result for strong solutions of $(FDE;\phi)_s$.

Proposition 3. Assume that $\{A(t);\ t \in [0,T]\}$ and $F:[0,T] \times C \to X$ satisfy conditions (A.1) and (A.3). Then there exists at most one strong solution of $(FDE;\phi)_s$.

Proof. Let $u(s,\phi)(t)$ and $v(s,\phi)(t)$ be two strong solutions of $(FDE;\phi)_s$. Then $\|u(s,\phi)(t) - v(s,\phi)(t)\|$ is differentiable a.e.t and $(d/dt)\ \|u(s,\phi)(t) - v(s,\phi)(t)\|$

$$= [u(s,\phi)(t) - v(s,\phi)(t), u'(s,\phi)(t) - v'(s,\phi)(t)]_-$$

$$\le [u(s,\phi)(t) - v(s,\phi)(t), F(t,u_t(s,\phi)) - F(t,v_t(s,\phi))]_+$$

$$- [u(s,\phi)(t) - v(s,\phi)(t), F(t,u_t(s,\phi)) - u'(s,\phi)(t)$$

$$- F(t,v_t(s,\phi)) + v'(s,\phi)(t)]_+.$$

By $A(t) \in \mathcal{A}(\alpha_0)$ and (A.3), we obtain that

$(d/dt)\ \|u(s,\phi)(t) - v(s,\phi)(t)\|$

$$\le (\alpha_0 + \beta_0)\|u_t(s,\phi) - v_t(s,\phi)\|_C \quad \text{a.e.}t \in [s,T],$$

which yields that for $t \in [s,T]$,

$\sup_{\theta \in [s-r,t]} \| u(s,\phi)(\theta) - v(s,\phi)(\theta) \|$

$$\le \begin{cases} (\alpha_0 + \beta_0) \int_0^t \sup_{\theta \in [s-r,\tau]} \|u(s,\phi)(\theta) - v(s,\phi)(\theta)\| \; d\tau & \text{if } \alpha_0 + \beta_0 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

By Grownwall's inequality, we have that

$\sup_{\theta \in [s-r,T]} \|u(s,\phi)(\theta) - v(s,\phi)(\theta)\| = 0$, i.e., $u(s,\phi) = v(s,\phi)$.

Q.E.D.

We next prove the existence of strong solutions to $(FDE;\phi)_s$ under stronger conditions than those in Propositions 1 and 2.

Proposition 4. Suppose that conditions (A.1)',(A.2)',(A.3) and (A.4) hold. If $u(s,\phi)(\cdot)$ is the function defined by (2.3), then $u(s,\phi)(\cdot) \in C^1([s-r,T];X)$ and satisfies

(2.4) $\quad u'(s,\phi)(t) + A(t)(u(s,\phi)(t)) = F(t,u_t(s,\phi))$

for $t \in [s,T]$ and for all $\phi \in \mathrm{Lip} \equiv \{\phi \in C;\ \phi$ is Lipschitz continuous$\}$.

Proof. By Remark after Proposition 2, $\{V(t,s);\ 0 \le s \le t \le T\}$ defined by (2.1) is an evolution operator. We approximate $V(t,s)$ by the evolution operator $V_\lambda(t,s)$ generated by $\hat{A}_\lambda(t) = \hat{A}(t)\hat{J}_\lambda(t) = \lambda^{-1}(I - \hat{J}_\lambda(t))$. From [2, Lemma 4.2], we see that for $\phi \in \overline{D}_0$,

$\lim_{\lambda \to 0+} V_\lambda(t,s)\phi = V(t,s)\phi$ uniformly in $t \in [s,T]$.

Also, the approximate problem

$$u'(t) + \hat{A}_\lambda(t)u_\lambda(t) = 0,\ t \in [s,T],\quad u_\lambda(s) = \phi,$$

has a unique continuously differentiable solution $u_\lambda(t) = V_\lambda(t,s)\phi$. Hence, we have that

$$V_\lambda(t,s)\phi = \phi - \int_s^t \hat{A}_\lambda(\tau)V_\lambda(\tau,s)\phi \, d\tau = \phi - \int_s^t \hat{A}(\tau)\hat{J}_\lambda(\tau)V_\lambda(\tau,s)\phi \, d\tau.$$

Taking account of the definition of $D(\hat{A}(\tau))$, we obtain that

$$(2.5)\quad (V_\lambda(t,s)\phi)(0) = \phi(0) - \int_s^t [A(\tau)(\hat{J}_\lambda(\tau)V_\lambda(\tau,s)\phi)(0) - F(\tau,\hat{J}_\lambda(\tau)V_\lambda(\tau,s)\phi)]\, d\tau.$$

Now, by (A.1)' and (A.3), we see that

$$I_1 = \int_s^t \| A(\tau)(\hat{J}_\lambda(\tau)V_\lambda(\tau,s)\phi)(0) - A(\tau)(V(\tau,s)\phi)(0)\| \, d\tau \le \alpha_1 \int_s^t \|\hat{J}_\lambda(\tau)V_\lambda(\tau,s)\phi - V(\tau,s)\phi\|_C \, d\tau$$

and

$$I_2 = \int_s^t \|F(\tau,\hat{J}_\lambda(\tau)V_\lambda(\tau,s)\phi - F(\tau,V(\tau,s)\phi)\| \, d\tau \le \beta_0 \int_s^t \| \hat{J}_\lambda(\tau)V_\lambda(\tau,s)\phi - V(\tau,s)\phi\|_C \, d\tau.$$

Let $\phi \in \hat{D}_0$; note here that $\phi \in \mathrm{Lip}$ by $D(A(t)) = X$ and Lemma 1. For each $\tau \in [s,T]$, we have for λ with $\lambda\omega_1 < 1$,

$$I_3 = \| \hat{J}_\lambda(\tau)V_\lambda(\tau,s)\phi - V(\tau,s)\phi\|_C \le (1 - \lambda\omega_1)^{-1}\| V_\lambda(\tau,s)\phi - V(\tau,s)\phi\|_C + \|\hat{J}_\lambda(\tau)V(\tau,s)\phi - V(\tau,s)\phi\|_C, \text{ where } \omega_1 = \alpha_1 + \beta_0.$$

By [2, Proposition 2.4], $V(\tau,s)\phi \in \hat{D}_0$ for $\phi \in \hat{D}_0$. This implies that the second term of the above inequality tends to zero as $\lambda \to 0+$. Hence $I_3 \to 0$ as $\lambda \to 0+$.

Next, we note that

$$(2.6)\quad \|\hat{J}_\lambda(\tau)V_\lambda(\tau,s)\phi\|_C \le (1 - \lambda\omega_1)^{-1}\| V_\lambda(\tau,s)\phi - \phi\|_C + \| \hat{J}_\lambda(\tau)\phi\|_C.$$

Since (C.3) is satisfied with $B(t) = \hat{A}(t)$ and $f(t) = g(t)I$, it follows that

$$(2.7)\quad \|\hat{J}_\lambda(\tau)\phi\|_C$$
$$\le \|\hat{J}_\lambda(s)\phi\|_C + \lambda|g(\tau) - g(s)|L(\|\phi\|_C)(1 + \|\hat{A}_\lambda(s)\phi\|_C)$$
$$\le \lambda\|\hat{A}_\lambda(s)\phi\|_C + \|\phi\|_C$$
$$+ \lambda|g(\tau) - g(s)|L(\|\phi\|_C)(1 + \|\hat{A}_\lambda(s)\phi\|_C)$$
$$\le \lambda(1 - \lambda\omega_1)^{-1}|\hat{A}(s)\phi| + \|\phi\|_C$$
$$+ \lambda|g(\tau) - g(s)|L(\|\phi\|_C)(1 + (1 - \lambda\omega_1)^{-1}|\hat{A}(s)\phi|).$$

Besides, since $\mathrm{Lim}_{\lambda\to 0+}\{\sup_{\tau\in[s,T]}\|V_\lambda(\tau,s)\phi - V(\tau,s)\phi\|_C\} = 0$, we see that there exists λ_1 such that if $0 < \lambda \le \lambda_1$, $\sup_{\tau\in[s,T]}\|V_\lambda(\tau,s)\phi - V(\tau,s)\phi\|_C < 1$. Thus it follows from (2.6) and (2.7) that $\sup_{0<\lambda<\lambda_1}(\sup_{\tau\in[s,T]}\|\hat{J}_\lambda(\tau)V_\lambda(\tau,s)\phi\|_C)$ is bounded. By the Lebesgue's dominated convergence theorem, we obtain that $I_1 \to 0$ and $I_2 \to 0$ as $\lambda \to 0+$. Therefore, letting $\lambda \to 0+$ in (2.5) yields (2.4). Q.E.D.

Remark. In general setting $u(s,\phi)(\cdot)$ defined by (2.3) need not have a strong derivative. We may have regard the function $u(s,\phi)(t)$ as a generalized solution of $(\mathrm{FDE};\phi)_s$ and investigate the meaning of generalized solutions. For convenience, the function $u(s,\phi)(t)$ defined by (2.3) is called a generalized solution.

Now, we consider the approximate problem

$$(\mathrm{FDE};\phi)_s^\beta \quad \begin{array}{ll} u_\beta'(t) + A_\beta(t)u_\beta(t) = F(t,u_{\beta_t}) & t\in[s,T] \\ u_{\beta_s} = \phi, & \end{array}$$

where $A_\beta(t)$ is the Yosida approximation of $A(t)$.
We define $\hat{A}^\beta(t)\colon D(\hat{A}^\beta(t)) \subset C \to C$ by

$$\hat{A}^\beta(t)\phi = -\phi'$$
$$D(\hat{A}^\beta(t)) = \{\phi\in C;\ \phi'\in C,\ \phi'(0) + A_\beta(t)\phi(0) = F(t,\phi)\}.$$

Clearly $A_\beta(t)$ satisfies the conditions of Proposition 4 with $\alpha_1 = \beta^{-1}(1 + (1 - \beta\alpha_0)^{-1})$; see [2, Lemma 1.2]. Therefore, there exists a family of nonlinear evolution operators $\{V_\beta(t,s);\ 0 \le s \le t \le T\}$ generated by $\hat{A}^\beta(t)$. If $u_\beta(s,\phi)(\cdot)$ is defined by

$$u_\beta(s,\phi)(t) = \begin{cases} \phi(t-s) & s-r \le t \le s, \\ (V_\beta(t,s)\phi)(0) & s \le t \le T, \end{cases}$$

then $u_\beta(s,\phi)(t)$ is the strong solution of $(\mathrm{FDE};\phi)_s^\beta$ and by Proposition 2, $V_\beta(t,s)\phi = u_{\beta_t}(s,\phi)$ for $s \le t \le T$ and $\phi \in \mathrm{Lip}$. By the proof of [2, Lemma 4.2], $\lim_{\beta \to 0+}(1 + \lambda A_\beta(t))^{-1}x = (1 + \lambda A(t))^{-1}x$ for $x \in X$ and sufficiently small λ. Thus, by [10, Lemma 3.2], we obtain that $\lim_{\beta \to 0+}(1 + \lambda\hat{A}^\beta(t))^{-1}\phi = (1 + \lambda\hat{A}(t))^{-1}\phi$ for $\phi \in C$ and small λ. Also, it follows from [2, Lemma 4.1] that $A_\beta(t)$ satisfies (A.1) and (A.2) uniformly in β, sufficiently small and hence $\hat{A}^\beta(t)$ satisfies (C.1)-(C.3) uniformly in β, sufficiently small. (To speak more carefully, by the same way as Proposition 1, we have that

$$\begin{aligned} &\|\phi_\beta(t,\cdot) - \phi_\beta(\tau,\cdot)\|_C \\ &\le K_3\lambda|g(t) - g(\tau)|[1 + \|\hat{A}_\lambda^\beta(\tau)\psi\|_C][L_2(\|\phi_\beta(\tau,\cdot)\|_C) + \\ &\qquad + (1 + \|F(\tau,\phi_\beta(\tau,\cdot))\|)L_1(\|\psi(0)\| + \lambda\|F(\tau,\phi_\beta(\tau,\cdot))\|)], \end{aligned}$$

where $\phi_\beta(t,\cdot) = (1 + \lambda\hat{A}^\beta(t))^{-1}\psi$, $\psi \in C$,
and if $\chi \in C$ and $\phi_0 \in D(\hat{A}(0))$ then
$\|F(\tau,\chi)\|$ is bounded by an increasing function of $\|\chi\|_C$.

Now, in this case, we must prove that

$$(2.8) \qquad \|\phi_\beta(\tau,\cdot)\|_C \le L_5(\|\psi\|_C)$$

for some monotone increasing function L_5. However, since $\lim_{\beta \to 0+}(1 + \lambda\hat{A}^\beta(t))^{-1}\phi = (1 + \lambda\hat{A}(t))^{-1}\phi$ for all $\phi \in C$, $\|\phi_\beta(\tau,\cdot)\|_C \le \|\phi(\tau,\cdot)\|_C + 1$ for all small β. Therefore, using $\|\phi(\tau,\cdot)\|_C \leq L_3(\|\psi\|_C)$ (see, Proposition 1), (2.8) is proved and hence $\hat{A}^\beta(t)$ satisfies (C.3) uniformly in β.) We can apply the Crandall-Pazy approximation theorem [2, Theorem 4.1] to give $\lim_{\beta\to 0+} V_\beta(t,s)\phi = V(t,s)\phi$ for all $\phi \in \hat{D}_0$. Therefore, by Proposition 2 and Lemma 1, we have that

Theorem 2. Let $\phi \in \mathrm{Lip}$ with $\phi(0) \in \hat{D}$. Suppose that $\{A(t);\ t \in [0,T]\}$ and $F:[0,T]\times C \to X$ satisfy conditions (A.1)-(A.4). If $u(s,\phi)(\cdot)$ is a generalized solution of $(FDE;\phi)_s$ then $u(s,\phi)(t) = \lim_{\beta\to 0+} u_\beta(s,\phi)(t)$ uniformly in $t \in [s,T]$, where $u_\beta(s,\phi)(\cdot)$ is the strong solution of $(FDE;\phi)_s^\beta$.

3. Properties for generalized solutions and existence of weak solutions and strong solutions.

Our first result in this section is on the comparision of two generalized solutions.

Theorem 3. Let $\phi_i \in \mathrm{Lip}$ with $\phi_i(0) \in \hat{D}$ for i= 1,2. If $u(s,\phi_i)(\cdot)$ is a generalized solution of $(FDE;\phi_i)_s$, then we have

$$(3.1)\quad e^{-\alpha_0 t}\|u(s,\phi_1)(t) - u(s,\phi_2)(t)\| - e^{-\alpha_0\tau}\|u(s,\phi_1)(\tau) - u(s,\phi_2)(\tau)\|$$

$$\leq \int_\tau^t e^{-\alpha_0\xi}[u(s,\phi_1)(\xi) - u(s,\phi_2)(\xi), F(\xi,u_\xi(s,\phi_1)) - F(\xi,u_\xi(s,\phi_2))]_+\, d$$

for $s \leq \tau \leq t \leq T$, where the symbol $[x,y]_+$ is defined by $[x,y]_+ = \lim_{\lambda\to 0+}\lambda^{-1}(\|x+\lambda y\| - \|x\|)$ for $x,y \in X$.

Proof. Let $u_\beta(s,\phi_i)(t)$ be the strong solution of $(FED;\phi_i)_s^\beta$ such that $\lim_{\beta\to 0+} u_\beta(s,\phi_i)(t) = u(s,\phi_i)(t)$ uniformly for $t \in [s,T]$.

Then $\| u_\beta(s,\phi_1)(t) - u_\beta(s,\phi_2)(t)\|$ is differentiable a.e. $t \in [s,T]$ and $(d/dt)\| u_\beta(s,\phi_1)(t) - u_\beta(s,\phi_2)(t)\|$

$$= [u_\beta(s,\phi_1)(t) - u_\beta(s,\phi_2)(t), -A_\beta(t)(u_\beta(s,\phi_1)(t)) + F(t,u_{\beta_t}(s,\phi_1)) + A_\beta(t)(u_\beta(s,\phi_2)(t)) - F(t,u_{\beta_t}(s,\phi_2))]_-,$$

where $[x,y]_- = -[x,-y]_+$.

Since $[x - y, A_\beta(t)x - A_\beta(t)y]_+ \le -\alpha_0(1 - \beta\alpha_0)^{-1}\| x - y\|$, it follows that

$$(d/dt)\| u_\beta(s,\phi_1)(t) - u_\beta(s,\phi_2)(t)\|$$
$$\le \alpha_0(1 - \beta\alpha_0)^{-1}\| u_\beta(s,\phi_1)(t) - u_\beta(s,\phi_2)(t)\|$$
$$+ [u_\beta(s,\phi_1)(t) - u_\beta(s,\phi_2)(t), F(t,u_{\beta_t}(s,\phi_1)) - F(t,u_{\beta_t}(s,\phi_2))]_+.$$

Integrating the above inequality, we have for $s \le \tau \le t \le T$,

$$\| u_\beta(s,\phi_1)(t) - u_\beta(s,\phi_2)(t)\| - \| u_\beta(s,\phi_1)(\tau) - u_\beta(s,\phi_2)(\tau)\|$$
$$\le \alpha_0(1 - \beta\alpha_0)^{-1} \int_\tau^t \| u_\beta(s,\phi_1)(\xi) - u_\beta(s,\phi_2)(\xi)\| \, d\xi$$
$$+ \int_\tau^t [u_\beta(s,\phi_1)(\xi) - u_\beta(s,\phi_2)(\xi), F(\xi,u_{\beta_\xi}(s,\phi_1)) - F(\xi,u_{\beta_\xi}(s,\phi_2))]_+ \, d\xi.$$

Letting $\beta \to 0+$ in this inequality, we see that for $s \le \tau \le t \le T$,

$$(3.2) \quad \| u(s,\phi_1)(t) - u(s,\phi_2)(t)\| - \| u(s,\phi_1)(\tau) - u(s,\phi_2)(\tau)\|$$
$$\le \alpha_0 \int_\tau^t \| u(s,\phi_1)(\xi) - u(s,\phi_2)(\xi)\| \, d\xi$$
$$+ \int_\tau^t [u(s,\phi_1)(\xi) - u(s,\phi_2)(\xi), F(\xi,u_\xi(s,\phi_1)) - F(\xi,u_\xi(s,\phi_2))]_+ \, d\xi.$$

By the standard argument one can prove that (3.2) implies (3.1). (For example, see [9].) Q.E.D.

The following theorem gives the existence of integral solutions.

Theorem 4. Let $u(s,\phi)(\cdot)$ be a generalized solution of $(FDE;\phi)_s$. Then the following inequality holds:

$$(3.4)\quad e^{-\alpha_0 t}\|u(s,\phi)(t) - x\| - e^{-\alpha_0 \tau}\|u(s,\phi)(\tau) - x\|$$

$$\le \int_\tau^t e^{-\alpha_0 \xi}\{[u(s,\phi)(\xi) - x, F(\xi,u_\xi(s,\phi)) - y]_+ + \theta(\xi,r)\}\,d\xi$$

for $s \le \tau \le t$, $[x,y] \in A(r)$, $r \in [0,T]$,
where $\theta(\xi,r) = L_1(\|x\|)\|h(\xi) - h(r)\|(1 + \|y\|)$.

Proof. Let $u(s,\phi)(\cdot)$ be a generalized solution of $(FDE;\phi)_s$. By Theorem 2, $\lim_{\beta\to 0+} u_\beta(s,\phi)(t) = u(s,\phi)(t)$ uniformly for $t \in [s,T]$, where $u_\beta(s,\phi)(t)$ is the strong solution of $(FDE;\phi)_s^\beta$. Let $[x,y] \in A(r)$ and set $x_\beta = x + \beta y$. Note that $x = J_\beta(r)x_\beta$ and $y = A_\beta(r)x_\beta$, where $J_\beta(r)$ and $A_\beta(r)$ are the resolvent and the Yosida approximation of $A(r)$, respectively. Then

$$(d/dt)\|u_\beta(s,\phi)(t) - x_\beta\|$$

$$= [u_\beta(s,\phi)(t) - x_\beta, -A_\beta(t)(u_\beta(s,\phi)(t)) + F(t,u_{\beta t}(s,\phi))]_-$$

$$\le [u_\beta(s,\phi)(t) - x_\beta, -A_\beta(t)(u_\beta(s,\phi)(t)) + y]_-$$

$$+ [u_\beta(s,\phi)(t) - x_\beta, F(t,u_{\beta t}(s,\phi)) - y]_+$$

$$\le \beta^{-1}(\|u_\beta(s,\phi)(t) - x_\beta + \beta(-A_\beta(t)(u_\beta(s,\phi)(t)) + y)\|$$

$$- \|u_\beta(s,\phi)(t) - x_\beta\|) + [u_\beta(s,\phi)(t) - x_\beta, F(t,u_{\beta t}(s,\phi)) - y]_+$$

$$= \beta^{-1}(\|J_\beta(t)(u_\beta(s,\phi)(t)) - J_\beta(r)x_\beta\| - \|u_\beta(s,\phi)(t) - x_\beta\|)$$

$$+ [u_\beta(s,\phi)(t) - x_\beta, F(t,u_{\beta t}(s,\phi)) - y]_+$$

$$\le L_1(\|x_\beta\|)\|h(t) - h(r)\|(1 + \|y\|)$$

$$+ \alpha_0(1 - \beta\alpha_0)^{-1}\|u_\beta(s,\phi)(t) - x_\beta\|$$

$$+ [u_\beta(s,\phi)(t) - x_\beta, F(t,u_{\beta t}(s,\phi)) - y]_+ \quad \text{by } A(t) \in \mathcal{A}(\alpha_0) \text{ and (A.2).}$$

Integrating these inequality over $[\tau,t] \subset [s,T]$,

$$\|u_\beta(s,\phi)(t) - x_\beta\| - \|u_\beta(s,\phi)(\tau) - x_\beta\|$$

$$\leq \int_\tau^t \{L_1(\|x_\beta\|)\|h(\xi) - h(r)\|(1 + \|y\|)$$

$$+ \alpha_0(1 - \beta\alpha_0)^{-1}\|u_\beta(s,\phi)(\xi) - x_\beta\|$$

$$+ [u_\beta(s,\phi)(\xi) - x_\beta, F(\xi,u_{\beta_\xi}(s,\phi)) - y]_+\}\, d\xi.$$

Letting $\beta \to 0+$, we see that for $s \leq \tau \leq t \leq T$,

$$(3.5) \quad \|u(s,\phi)(t) - x\| - \|u(s,\phi)(\tau) - x\|$$

$$\leq \int_\tau^t \{[u(s,\phi)(\xi) - x, F(\xi,u_\xi(s,\phi)) - y]_+ + \theta(\xi,r)\}\, d\xi$$

$$+ \alpha_0 \int_\tau^t \|u(s,\phi)(\xi) - x\|\, d\xi,$$

which yields (3.4).

Q.E.D.

Next, we recall the definition of weak solutions in the sense of Kartsatos and Parrott [6,7] and consider the existence of weak solutions of $(FDE;\phi)_0$.

Definition 2. A function $u(t) \in C([-r,T];X)$ is said to be a weak solution of $(FDE;\phi)_0$ if $u(t) = \phi(t)$ for $t \in [-r,0]$ and

$$(DE) \quad \begin{cases} v'(t) + A(t)v(t) \ni F(t,u_t), & t \in [0,T] \\ v(0) = \phi(0) \end{cases}$$

has a solution $v(t)$ in the sense of Evans [5] such that $v(t) = u(t)$ for $t \in [0,T]$.

Remark. By definition and [5, Theorem 3], there exists at most one weak solution of $(FDE;\phi)_0$. Indeed, if $u_1(t)$ and $u_2(t)$ are two weak solutions, they satisfy the integral inequality

$$\|u_1(t) - u_2(t)\| \le \int_0^t \|F(\tau,u_{1_\tau}) - F(\tau,u_{2_\tau})\| \, d\tau. \text{ (See [5, (8.3)].)}$$

Thus, by (A.3) and the Grownwall inequality, $u_1(t) = u_2(t)$ for $t \in [0,T]$.

Theorem 5. Suppose that $\{A(t);\ t \in [0,T]\}$ satisfy (A.1) with $\alpha_0 = 0$ and (A.2) and $F:[0,T] \times C \to X$ satisfy (A.3) and (A.4). If $\phi \in \mathrm{Lip}$ and $\phi(0) \in \hat{D}$, then $(FDE;\phi)_0$ has a unique weak solution.

Proof. It suffices to show a generalized solution $u(0,\phi)(t)$ of $(FDE;\phi)_0$ is a weak solution. Note that $t \to F(t,u_t(0,\phi))$ is of bounded variation by (A.3) and (A.4) because $u(0,\phi)(t)$ is Lipschitz continuous. Then (DE) has a solution $v(t)$ in the sense of Evans, i.e., there exist sequence $\{t_k^n\}$ and $\{u_k^n\}$ such that

i) $\dfrac{u_k^n - u_{k-1}^n}{h_k^n} + A(t_k^n)u_k^n \ni F(t_k^n, u_{t_k^n}(0,\phi))$, where $h_k^n = t_k^n - t_{k-1}^n$,

ii) the step functions $v^n(t)$ $(\equiv u_k^n$ on $(t_{k-1}^n, t_k^n])$ converge uniformly on $[0,T]$ to $v(t)$.

Note here that

$$M \equiv \max\{\sup \|u_k^n\| \ ,\ \sup \left\| \frac{u_k^n - u_{k-1}^n}{h_k^n} - F(t_k^n, u_{t_k^n}(0,\phi))\right\| \} < \infty.$$

(See [5, Proof of Theorem 2].)

Let $v_k^n \in A(t_k^n)u_k^n$. By (3.5) we see that

$$\|u(0,\phi)(t) - u_k^n\| - \|u(0,\phi)(\tau) - u_k^n\|$$

$$\le \int_\tau^t \{[u(0,\phi)(\xi) - u_k^n,\ F(\xi,u_\xi(0,\phi)) - v_k^n]_+ + \theta_1(\xi,t_k^n)\}\, d\xi$$

$$+ \alpha_0 \int_\tau^t \|u(0,\phi)(\xi) - u_k^n\| \, d\xi,$$

where $\theta_1(\xi,r) = M_1\|h(\xi) - h(r)\|$ and $M_1 = L_1(M)(1 + M)$.

Since $h_k^n[u(0,\phi)(\xi) - u_k^n, F(\xi,u_\xi(0,\phi)) - v_k^n]_+$

$$\leq \|u(0,\phi)(\xi) - u_{k-1}^n\| - \|u(0,\phi)(\xi) - u_k^n\|$$

$$+ h_k^n\|F(\xi,u_\xi(0,\phi)) - F(t_k^n,u_{t_k^n}(0,\phi))\|,$$

it follows by the standard argument that

$$\int_{t_j^n}^{t_i^n} (\|u(0,\phi)(t) - v^n(\eta)\| - \|u(0,\phi)(\tau) - v^n(\eta)\|)\, d\eta$$

$$\leq \int_\tau^t (\|u(0,\phi)(\xi) - v^n(t_j^n)\| - \|u(0,\phi)(\xi) - v^n(t_i^n)\|)\, d\xi$$

$$+ \int_{t_j^n}^{t_i^n}\int_\tau^t \{\alpha_0\|u(0,\phi)(\xi) - v^n(\eta)\| + \theta_1^n(\xi,\eta)$$

$$+ \|F(\xi,u_\xi(0,\phi)) - F^n(\eta)\|\}\, d\xi\, d\eta,$$

where θ_1^n and F^n are functions defined by

$$\theta_1^n(\xi,\eta) = \theta_1(\xi,t_k^n) \quad \text{for } \eta \in (t_{k-1}^n,t_k^n]$$

and

$$F^n(\eta) = F(t_k^n,u_{t_k^n}(0,\phi)) \quad \text{for } \eta \in (t_{k-1}^n,t_k^n], \text{ respectively.}$$

Letting $t_i^n \to t'$, $t_j^n \to \tau'$ as $n \to \infty$ and applying [8, Proposition 2.5] we obtain that $u(0,\phi)(t) = v(t)$ for $t \in [0,T]$. Q.E.D.

Finally, we consider the existence of strong solutions of $(FDE;\phi)_s$.

Corollary 1. Let $\phi \in Lip$ with $\phi(0) \in \hat{D}$. Assume that $\{A(t); t \in [0,T]\}$ and $F:[0,T] \times C \to X$ satisfy conditions (A.1)-(A.4). If X is reflexive, or, more generally, X satisfies the Radon-Nikodym property, then $(FDE;\phi)_s$ has a unique strong solution.

Proof. By virtue of Theorem 2, there exists a generalized solution $u(s,\phi)(t)$ and by the Remark after Lemma 1, $u(s,\phi)(t)$ is Lipschitz continuous and hence $u(s,\phi)(t)$ is differentiable a.e. $t \in [s,T]$. Now, let $h > 0$ and t_0 be any point at which $u(s,\phi)(\cdot)$ is differentiable. Putting $\tau = r = t_0$ and $t = t_0 + h$ in (3.5), we see that

$$\|u(s,\phi)(t_0 + h) - x\| - \|u(s,\phi)(t_0) - x\|$$

$$\leq \int_{t_0}^{t_0+h} \{[u(s,\phi)(\xi) - x, F(\xi,u_\xi(s,\phi)) - y]_+ + \theta(\xi,t_0)\} \, d\xi$$

$$+ \alpha_0 \int_{t_0}^{t_0+h} \|u(s,\phi)(\xi) - x\| \, d\xi \quad \text{for } [x,y] \in A(t_0).$$

Dividing the above inequality by h and letting $h \downarrow 0$, it follows

$$[u(s,\phi)(t_0) - x, u'(s,\phi)(t_0)]_+$$

$$\leq [u(s,\phi)(t_0) - x, F(t_0,u_{t_0}(s,\phi)) - y]_+ + \alpha_0 \|u(s,\phi)(t_0) - x\|,$$

i.e., for $[x,y] \in A(t_0)$

$$[u(s,\phi)(t_0) - x, -u'(s,\phi)(t_0) + F(t_0,u_{t_0}(s,\phi)) + \alpha_0 u(s,\phi)(t_0) - (\alpha_0 x + y)]_+ \geq 0. \tag{3.6}$$

By condition (A.1), it is easy to see that $A(t_0) + \alpha_0$ is m-accretive. Therefore, by (3.6), we see that

$$u'(s,\phi)(t_0) + A(t_0)(u(s,\phi)(t_0)) \ni F(t_0, u_{t_0}(s,\phi)).$$

Q.E.D.

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References

[1] Crandall M., A generalized domain for semigroup generators, Proc. Am. Math. Soc. 37, (1973) 434 – 440.

[2] Crandall M. & Pazy A., Nonlinear evolution equations in Banach spaces, Israel J. Math. 11, (1972) 57 – 94.

[3] Dyson J. & Villela Bressan R., Functional differential equations and nonlinear evolution operators, Proc. Roy. Soc. Edinburgh 75 A, (1975/76) 223 – 234.

[4] ____, Semigroups of translations associated with functional and functional differential equations, Pro. Roy. Soc. Edinburgh 82 A, (1979) 171 – 188.

[5] Evans L., Nonlinear evolution equations in an arbitary Banach space, Israel J. Math. 26, (1977) 1 – 42.

[6] Kartsatos A. G., A direct method for the existence of evolution operators in general Banach spaces, to appear in Funkcialaj Ekvacioj.

[7] Kartsatos A. G. & Parrott M. E., A simplified approach to the existence and stability problem of a functional evolution equation in a general Banach space., Lecture Notes in Math., 1076, Springer-Verlag, Berlin, 1984.

[8] Kobayasi K., Kobayashi Y. & Oharu S., Nonlinear evolution operators in Banach spaces, Osaka J. Math. 21, (1984) 281 – 310.

[9] Kobayashi Y., Difference approximation of Cauchy problems for quasi-dissipative operators and generation of nonlinear semigroups, J. Math. Soc. Japan 27, (1975), 640 – 665.

[10] Parrott M. E., Representation and approximation of generalized solutions of a nonlinear functional differential equation, Nonlinear analysis 6, (1982) 307 – 318.

[11] Webb G., Asymtotic stability for abstract nonlinear functional differential equations, Proc. Am. Math. Soc. 54, (1976) 225 – 230.