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<th>Title</th>
<th>Nonlinear Nonautonomous Differential Equations</th>
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<tbody>
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Nonlinear Nonautonomous Differential Equations

By Naoki TANAKA (田中直和)
(Waseda University, Japan)

Introduction.

Let $X$ be a real Banach space with norm $\| \cdot \|$ and let $C = C([-r,0];X)$, $0 \leq r < \infty$, be the Banach space of all continuous functions from $[-r,0]$ into $X$. We denote the norm of $\phi \in C$ by $\| \phi \|_C$, i.e., $\| \phi \|_C = \sup_{\theta \in [-r,0]} \| \phi(\theta) \|_X$.

This paper is concerned with the abstract nonlinear functional differential equation

$$(FDE; \phi)_s \quad u'_s(t) + A(t)u(t) \equiv F(t,u_t), \quad t \in [s,T], \quad (s \geq 0)$$

where $u:[-r,T] \to X$ is the unknown function; $\{A(t); \; t \in [0,T]\}$ is a given family of operators in $X$; $F:[0,T] \times C \to X$ is a given function; $\phi$ is given in $C$. The symbol $u_t$ denotes the function $u_t(\theta) = u(t+\theta)$, $\theta \in [-r,T]$.

We assume that the following conditions (A.1) -- (A.4) hold:

(A.1) There exists a constant $\alpha_0$ such that for each $t \in [0,T]$, $A(t) + \alpha_0$ is accretive and $R(I + \lambda A(t)) = X$ for $0 < \lambda < 1/\max(0,\alpha_0)$.

(A.2) There are a continuous function $h:[0,T] \to X$ which is of bounded variation on $[0,T]$, and a monotone increasing continuous function $L_1:[0,\infty) \to [0,\infty)$ such that

$$\| A_{\lambda}(t)x - A_{\lambda}(t)x \| \leq \| h(t) - h(t) \| L_1(\| x \|)(1 + \| A_{\lambda}(t)x \|)$$

for $0 < \lambda < 1/\max(0,\alpha_0)$, $t, \tau \in [0,T]$ and $x \in X$, where $J_{\lambda}(t) = (I + \lambda A(t))^{-1}$ and $A_{\lambda}(t) = \lambda^{-1}(I - J_{\lambda}(t))$.

(A.3) There exists a constant $\beta_0 > 0$ such that for $\phi, \psi \in C$ and $t \in [0,T]$, $\| F(t,\phi) - F(t,\psi) \| \leq \beta_0 \| \phi - \psi \|_C$. 

- 1 -
There are a continuous function \( k : [0, T] \to X \) which is of bounded variation on \([0, T]\), and a monotone increasing function \( L_2 : [0, \infty) \to [0, \infty) \) such that for \( t, \tau \in [0, T] \) and \( \phi \in C \),
\[
\| F(t, \phi) - F(\tau, \phi) \| \leq \| k(t) - k(\tau) \| L_2(\| \phi \|_C).
\]

The purpose of this paper is to show the existence of a generalized solution of \((FDE; \phi)_s\). In particular, in case \( X \) is reflexive, we show that the generalized solution is the strong solution of \((FDE; \phi)_s\).

Recently, Kartsatos [6] has proved the existence of the evolution operator associated with \((FDE; \phi)_s\) under the following conditions (B.2) and (B.3) instead of (A.2), (A.3) and (A.4).

(B.2) There exists an increasing continuous function \( L : [0, \infty) \to [0, \infty) \) such that for all \( \lambda > 0 \), \( x \in X \), \( t, \tau \in [0, T] \),
\[
\| A_\lambda(t)x - A_\lambda(\tau)x \| \leq |t - \tau| L(\| x \|_C)(1 + \| A_\lambda(\tau)x \|_C).
\]

(B.3) There exists a positive constant \( b \) such that
\[
\| F(\tau, f_1) - F(t, f_2) \| \leq b(|t - \tau| + \| f_1 - f_2 \|_C)
\]
for every \( t, \tau \in [0, T] \), \( f_1, f_2 \in C \).

In order to apply the method of successive approximations to \((FDE; \phi)_s\), he essentially used conditions (B.2) and (B.3) which imply that \( A_\lambda(t)x \) and \( F(t, f) \) are Lipschitz continuous in \( t \). However this method does not seem to be directly applicable under (A.1) – (A.4). Also, it has not been proved that the generalized solutions in the sense of Kartsatos are weak solutions, except on a small interval in which they are Lipschitz continuous. (For a refined definition of weak solutions, see Definition 2.)

Now, in order to improve these points, we use the nonlinear evolution operator theory of Crandall and Pazy [2] as the main
tool for solving \((FDE;\phi)_s\). Various author have so far considered \((FDE;\phi)_s\) under different setting in nonlinear operator theory. (For example, see [3,4,10].)

This paper consists of three sections. In section 1, we recall the nonlinear evolution operator theory. In section 2, we show that the existence of generalized solutions of \((FDE;\phi)_s\) and it is represented as the uniform limit of a sequence of strong solutions of the approximating equations for \((FDE;\phi)_s\) involving the Yosida approximations. Finally, in section 3, we investigate some properties of generalized solutions and consider weak solutions and give the existence for strong solutions of \((FDE;\phi)_s\) when \(X\) is reflexive.

1. Basic concept of nonlinear evolution operator theory

We discuss briefly some concepts in the nonlinear evolution operator theory. Let \(Y\) be a Banach space with \(\|\cdot\|_Y\). A family \(\{V(t,s); 0\leq s\leq t\leq T\}\) of operators \(V(t,s): Y \to Y\) is said to be a family of operators, if

\[ V(t,t)y = y \quad \text{for all } y \in Y \text{ and } t \in [0,T], \]

\[ V(t,r)V(r,s) = V(t,s) \quad \text{for } 0 \leq s \leq r \leq t \leq T. \]

Let \(\{V(t,s); 0\leq s\leq t\leq T\}\) be an evolution operator and define the operator \(B(t)\) by

\[ D(B(t)) = \{y \in Y; \lim_{h \to 0^+}(1/h)(V(t+h,t)y - y) \text{ exists}\} \]

\[ -B(t)y = \lim_{h \to 0^+}(1/h)(V(t+h,t)y - y) \quad \text{for } y \in D(B(t)). \]

If \(D(B(t))\) is non-empty for each \(t \geq 0\), then the family \(-B(t)\) is said to be the infinitesimal generator of \(V(t,s)\).
Consider the problem \((\text{FDE};\phi)_s\). Suppose that for every \(\phi \in C\) and \(s \geq 0\), \((\text{FDE};\phi)_s\) has the unique solution \(u(s,\phi)(\cdot)\) and that \(A(t)\) and \(F\) are continuous. Then one can find that the infinitesimal generator of the evolution operator \(V(t,s)\), defined by \(V(t,s)\phi = u_t(s,\phi)\) is given by
\[
D(\hat{A}(t)) = \{\phi \in C; \ \phi' \in C, \ \phi(0) \in D(A(t)), \ 
\phi'(0) + A(t)\phi(0) \in F(t,\phi)\}
\]
(1.1) \quad \hat{A}(t)\phi = -\phi'.

Conversely, given the family \(A(t)\), we shall prove that under suitable conditions on \(A(t)\) and \(F\), \(A(t)\) generates an evolution operator \(V(t,s)\) such that \(V(t,s)\phi\) gives the segments of a solution of \((\text{FDE};\phi)_s\). This will rely on the following result due to Crandall – Pazy [2].

A subset \(B\) of \(Y \times Y\) is in class \(A(\omega)\) if for each \(\lambda > 0\) such that \(\lambda \omega > 1\) and each pair \([y_i,z_i]\) \(\in B\), \(i=1,2\), we have
\[
(1.2) \quad \| (y_1 + \lambda z_1) - (y_2 + \lambda z_2) \|_Y \geq (1 - \lambda \omega) \| y_1 - y_2 \|_Y.
\]

\(B\) is called accretive if \(B \in A(0)\). Also, (1.2) implies that \((I + \lambda B)^{-1}\) exists on \(R(I + \lambda B)\) and is a Lipschitzian with constant \((1 - \lambda \omega)^{-1}\). Let \(B \in A(\omega)\) and \(R(I + \lambda B) = Y\) for all \(0 < \lambda \leq \lambda_0\).
Define \(|B|\) by \(|B| = \lim_{\lambda \to 0^+} \| B_\lambda \|_Y\), where \(J_\lambda = (I + \lambda B)^{-1}\) and \(B_\lambda = \lambda^{-1}(I - J_\lambda)\). (Note that this limit exists, although it may be infinite.) For such \(B\) we define \(\hat{D}(B) = \{y \in Y; \ |B| < \infty\}\) which is called a generalized domain of \(B\).

Theorem 1. (Crandall-Pazy). Let \(T > 0\) and \(\omega\) be real number and assume that \(B(t)\) satisfies the following conditions:

(C.1) \(B(t) \in A(\omega)\) for \(0 \leq t \leq T\),
(C.2) \( R(I + \lambda B(t)) = Y \) for \( 0 \leq t \leq T \) and \( 0 < \lambda < \lambda_0 \), where \( \lambda_0 > 0 \) and \( \lambda_0 \omega < 1 \).

(C.3) There are a continuous function \( f:[0,T] \to Y \) which is of bounded variation on \([0,T]\), and a monotone increasing function \( L: [0,\infty) \to [0,\infty) \) such that

\[
\| B_\lambda(t) y - B_\lambda(\tau) y \|_Y \leq \| f(t) - f(\tau) \|_Y L(\| y \|_Y)(1 + \| B_\lambda(\tau) y \|_Y)
\]

for \( 0 < \lambda < \lambda_0 \), \( 0 \leq t, \tau \leq T \) and \( y \in Y \).

Then

\[
V(t,s)y = \lim_{n \to \infty} \prod_{i=1}^{n} (I + (t - s)B_n(s) + i(t - s))^{-1} y
\]

exists for \( y \in \overline{D(B(t))} \) and \( 0 \leq s < t \leq T \). The \( V(t,s) \) defined by (1.3) for \( 0 \leq s < t \leq T \) and by \( V(t,t) = I \) for \( 0 \leq t \leq T \) is an evolution operator on \( \overline{D(B(t))} \).

2. On the existence of generalized solutions of (FDE;\( \phi \)_s)

We define for each \( t \in [0,T] \) an operator \( \hat{A}(t): D(\hat{A}(t)) \subset C \to C \) by (1.1).

Proposition 1. Suppose that conditions (A.1)-(A.4) hold. If \( \{\hat{A}(t); \ t \in [0,T]\} \) is the family of operators defined in C by (1.1), then there exists a family of nonlinear evolution operators \( \overline{V(t,s)}: D(\hat{A}(t)) \subset C \to C \) such that for all \( \phi \in D(\hat{A}(t)) \)

\[
(2.1) \quad V(t,s)\phi = \begin{cases} 
\lim_{n \to \infty} \prod_{i=1}^{n} (I + (t - s)\hat{A}(s) + i(t - s))^{-1} \phi \\
0 \leq s < t \leq T, \\
\phi \\
0 \leq s = t \leq T.
\end{cases}
\]

Proof. We are going to apply Theorem 1 for \( B(t) = \hat{A}(t) \) and \( Y = C \). Under assumptions (A.1) and (A.3) we can apply [11, Proposition 1] to show that \( \hat{A}(t) \in \mathfrak{A}(\omega_0) \) for \( t \in [0,T] \) and \( R(I + \lambda \hat{A}(t)) = C \) for \( 0 < \lambda < 1/\omega_0 \), where \( \omega_0 = \max(0, \alpha_0 + \beta_0) \). Thus
conditions (C.1) and (C.2) hold for $\hat{A}(t)$. Next, by using the same argument as in [4, Theorems 12 and 13] and the inequality

$$||h(t) - h(\tau)|| + ||k(t) - k(\tau)|| \leq |g(t) - g(\tau)|,$$

where $g(t) = \text{Var}([0,t]; h) + \text{Var}([0,t]; k)$ and $\text{Var}([0,t]; h)$ denotes the total variation of $h$ on $[0,t]$, we will show that $\hat{A}(t)$ satisfies (C.3) with $B(t) = \hat{A}(t)$ and $f(t) = g(t)I$, where $I$ denotes the identity in $X$. To this end, set $\phi(t, \cdot) = (I + \lambda A(t))^{-1} \psi, \psi \in C$. Then we have

$$\phi(t, \theta) = e^{\theta/\lambda} \phi(t, 0) + \int_0^\theta \frac{1}{\lambda} e^{-(s-\theta)/\lambda} \psi(s) \, ds,$$

and by $\phi(t, \cdot) \in D(\hat{A}(t))$, we have $\phi(t, 0) = \psi(0) + \lambda \phi'(t, 0) = \psi(0) - \lambda A(t) \phi(t, 0) + \lambda F(t, \phi(t, \cdot))$, i.e., $\phi(t, 0) = (I + \lambda A(t))^{-1} (\psi(0) + \lambda F(t, \phi(t, \cdot)))$.

Now, for $0 < \lambda < 1$ with $\lambda a_0 < 1/2$,

$$|| \phi(t, \cdot) - \phi(\tau, \cdot) ||_C = || \phi(t, 0) - \phi(\tau, 0) ||$$

$$= || (I + \lambda A(t))^{-1} (\psi(0) + \lambda F(t, \phi(t, \cdot)))$$

$$- (I + \lambda A(\tau))^{-1} (\psi(0) + \lambda F(\tau, \phi(\tau, \cdot))) ||$$

$$\leq \lambda (1 - \lambda a_0)^{-1} ||F(t, \phi(t, \cdot)) - F(\tau, \phi(\tau, \cdot))||$$

$$+ \lambda ||h(t) - h(\tau)||_1 (||\psi(0) + \lambda F(\tau, \phi(\tau, \cdot))||)$$

$$\times (1 + ||A_\lambda(\tau)(\psi(0) + \lambda F(\tau, \phi(\tau, \cdot)))||.)$$

But,

$$||A_\lambda(\tau)(\psi(0) + \lambda F(\tau, \phi(\tau, \cdot)))||$$

$$= \lambda^{-1} ||\psi(0) + \lambda F(\tau, \phi(\tau, \cdot)) - J_\lambda(\tau)(\psi(0) + \lambda F(\tau, \phi(\tau, \cdot)))||$$

$$\leq ||\hat{A}_\lambda(\tau)\psi||_C + ||F(\tau, \phi(\tau, \cdot))||,$$

which implies that

$$||\phi(t, \cdot) - \phi(\tau, \cdot)||_C$$

$$\leq \lambda (1 - \lambda a_0)^{-1} [\beta_0 ||\phi(t, \cdot) - \phi(\tau, \cdot)||_C$$

$$+ ||k(t) - k(\tau)||_1 (||\phi(\tau, \cdot)||_C)] +$$
Thus there exists a constant $K_1$ such that
\[(2.2) \quad \|\phi(t, \cdot) - \phi(\tau, \cdot)\|_C \leq K_1 \lambda |g(t) - g(\tau)| \left[ 1 + \|\hat{A}_\lambda(\tau)\psi\|_C \right] \left[ L_2(\|\phi(\tau, \cdot)\|_C) \right. \\
+ \left. (1 + \|F(\tau, \phi(\tau, \cdot))\|) L_1(\|\psi(0)\| + \lambda \|F(\tau, \phi(\tau, \cdot))\|) \right].\]

Suppose that $\chi \in C$ and $\phi_0 \in D(\hat{A}(0))$. Then $\|F(\tau, \chi)\| \leq \\
\theta_0 [\|x\|_C + \|\phi_0\|_C] + \|k(t) - k(0)\| L_2(\|\phi_0\|_C) + \|F(0, \phi_0)\|$
and hence $\|F(\tau, \chi)\|$ is bounded by an increasing function of
$\|x\|_C$. It remains to prove that $\|\phi(\tau, \cdot)\|_C \leq L_3(\|\psi\|_C)$ for
some monotone increasing function $L_3$. From (2.2), $\|\phi(\tau, \cdot)\|_C \leq \\
\leq \|\phi(0, \cdot)\|_C + K_1 \lambda |g(\tau) - g(0)| \left[ 1 + \|\hat{A}_\lambda(0)\psi\|_C \right] \\
\times \left[ L_2(\|\phi(0, \cdot)\|_C) \right. \\
+ \left. (1 + \|F(0, \phi(0, \cdot))\|) L_1(\|\psi(0)\| + \lambda \|F(0, \phi(0, \cdot))\|) \right].$

However $\lambda \|\hat{A}_\lambda(0)\psi\|_C = \|\psi - \hat{J}_\lambda(0)\psi\|_C \leq \|\psi\|_C + \|\phi(0, \cdot)\|_C$
and if $\phi_0 \in D(\hat{A}(0))$ then
$\|\phi(0, \cdot)\|_C = \|(1 + \lambda \hat{A}(0))^{-1}\psi\|_C$
$\leq (1 - \lambda \omega_0)^{-1} [\|\psi - \phi_0\|_C + \lambda \|\hat{A}(0)\phi_0\|_C] + \|\phi_0\|_C$
$\leq K_2 \left[ \|\psi\|_C + \|\phi_0\|_C + \|\hat{A}(0)\phi_0\|_C \right]$ for some $K_2$,
which implies that
$\|\phi(0, \cdot)\|_C$ is bounded by a monotone increasing function of
$\|\psi\|_C$. Thus $\hat{A}(t)$ satisfies (C.3) with $B(t) = \hat{A}(t)$ and $f(t) = g(t)I$. Therefore, the conclusion of the proposition follows from
Theorem 1.

Q.E.D.
Note that, as was proved in [5], \( \hat{D}(\hat{A}(t)) \) is independent of \( t \) because \( \hat{A}(t) \) satisfies (C.3) and also \( \hat{D}(A(t)) \) is independent of \( t \) because of (A.2). In what follows, \( \hat{D}_0 \) and \( \hat{D} \) stand for a generalized domain of \( \hat{A}(0) \) and \( A(0) \), respectively.

As in [3, Proposition 1], we have the following

Proposition 2. Suppose that conditions (A.1)-(A.4) hold. If \( u(s,\phi)(\cdot) \) for each \( \phi \in \hat{D}_0 \) and \( s \geq 0 \) is defined by

\[
(2.3) \quad u(s,\phi)(t) = \begin{cases} 
\phi(t-s) & s-r \leq t \leq s, \\
(V(t,s)\phi)(0) & s \leq t \leq T,
\end{cases}
\]

where \( V(t,s) \) is as constructed by Proposition 1, then \( u(s,\phi)(\cdot) \in C([s-r,T];X) \) and \( V(t,s)\phi = u_t(s,\phi) \) for \( t \in [s,T] \).

Remark. We introduce the following stronger conditions than (A.1) and (A.2):

(A.1)' There exists a constant \( \alpha_1 > 0 \) such that for \( x, y \in X, \)

\[ \| A(t)x - A(t)y \| \leq \alpha_1 \| x - y \|. \]

(A.2)' There are a continuous function \( h: [0,T] \rightarrow X \) which is of bounded variation on \( [0,T] \) and a monotone increasing continuous function \( L_4: [0,\infty) \rightarrow [0,\infty) \) such that

\[ \| A(t)x - A(\tau)x \| \leq \| h(t) - h(\tau) \| \cdot L_4( \| x \| ) (1 + \| A(\tau)x \| ) \]

for all \( t, \tau \in [0,T] \) and \( x \in X. \)

Since (A.1)' and (A.2)' imply (A.1) and (A.2), Propositions 1 and 2 hold, although (A.1) and (A.2) are replaced by (A.1)' and (A.2)'.

Next, we recall the following expression for \( \hat{D}_0 \).

Lemma 1 ([4, Theorem 10]). Let \( A(t) \) and \( F(t,\phi) \) satisfy conditions (A.1) and (A.3). Then \( \hat{D}_0 = \{ \phi \in C; \phi \) is Lipschitz continuous function and \( \phi(0) \in \hat{D} \} \).
Remark. If $\phi$ is Lipschitz continuous function and $\phi(0) \in \hat{D}$, then the function defined by (2.3) is a Lipschitzian. In fact, for such $\phi$, by [2, Proposition 2.3] and Lemma 1, there exists a constant $K$ such that for $0 \leq s \leq t, \tau \leq T$, $||V(t,s)\phi - V(\tau,s)\phi||_C \leq K|t - \tau|$. So that our assertion holds.

Definition 1. A function $u(s,\phi)(\cdot) \in C([-r,T];X)$ is said to be a strong solution of $(FDE;\phi)_S$ if it is an absolutely continuous function is which differentiable a.e. on $[s,T]$ and satisfies $(FDE;\phi)_S$ a.e. on $[s,T]$.

We shall first prove the following uniqueness result for strong solutions of $(FDE;\phi)_S$.

Proposition 3. Assume that $\{A(t); t \in [0,T]\}$ and $F: [0,T] \times C \to X$ satisfy conditions (A.1) and (A.3). Then there exists at most one strong solution of $(FDE;\phi)_S$.

Proof. Let $u(s,\phi)(t)$ and $v(s,\phi)(t)$ be two strong solutions of $(FDE;\phi)_S$. Then $||u(s,\phi)(t) - v(s,\phi)(t)||$ is differentiable a.e. $t$ and $(d/dt) ||u(s,\phi)(t) - v(s,\phi)(t)||$

\[\begin{align*}
&= [u(s,\phi)(t) - v(s,\phi)(t), u'(s,\phi)(t) - v'(s,\phi)(t)]^+ \\
&\leq [u(s,\phi)(t) - v(s,\phi)(t), F(t,u_t(s,\phi)) - F(t,v_t(s,\phi))]^+ \\
&\quad - [u(s,\phi)(t) - v(s,\phi)(t), F(t,u_t(s,\phi)) - u'(s,\phi)(t) \\
&\quad \quad - F(t,v_t(s,\phi)) + v'(s,\phi)(t)]^+.
\end{align*}\]

By $A(t) \in A(\alpha_0)$ and (A.3), we obtain that

\[\begin{align*}
&(d/dt) ||u(s,\phi)(t) - v(s,\phi)(t)|| \\
&\leq (\alpha_0 + \beta_0)||u_t(s,\phi) - v_t(s,\phi)||_C \quad \text{a.e. } t \in [s,T],
\end{align*}\]
which yields that for \( t \in [s,T] \),
\[
\sup_{\theta \in [s-r,t]} \| u(s,\phi)(\theta) - v(s,\phi)(\theta) \| \\
\left\{ \begin{array}{ll}
(\alpha_0 + \beta_0) \int_0^t \sup_{\theta \in [s-r,\tau]} \| u(s,\phi)(\theta) - v(s,\phi)(\theta) \| \, d\tau \\
0 & \text{if } \alpha_0 + \beta_0 > 0, \\
& \text{otherwise.}
\end{array} \right.
\]
By Gronwall's inequality, we have that
\[
\sup_{\theta \in [s-r,T]} \| u(s,\phi)(\theta) - v(s,\phi)(\theta) \| = 0, \text{ i.e., } u(s,\phi) = v(s,\phi).
\]
Q.E.D.

We next prove the existence of strong solutions to \((\text{FDE} ; \phi)\)'s under stronger conditions than those in Propositions 1 and 2.

Proposition 4. Suppose that conditions (A.1)',(A.2)',(A.3) and (A.4) hold. If \( u(s,\phi)(\cdot) \) is the function defined by (2.3), then \( u(s,\phi)(\cdot) \in C^1([s-r,T] ; X) \) and satisfies
\[
(2.4) \quad u'(s,\phi)(t) + A(t)(u(s,\phi)(t)) = F(t,u_t(t,\phi))
\]
for \( t \in [s,T] \) and for all \( \phi \in \text{Lip} \equiv \{ \phi \in C; \phi \text{ is Lipschitz continuous} \} \).

Proof. By Remark after Proposition 2, \( \{ V(t,s); 0 \leq s \leq t \leq T \} \) defined by (2.1) is an evolution operator. We approximate \( V(t,s) \)
by the evolution operator \( V_\lambda(t,s) \) generated by \( \hat{A}_\lambda(t) = \hat{A}(t) \hat{J}_\lambda(t) = \lambda^{-1}(I - \hat{J}_\lambda(t)) \). From [2, Lemma 4.2], we see that for \( \phi \in \bar{D}_0^\ast \),
\[
\lim_{\lambda \to 0^+} V_\lambda(t,s)\phi = V(t,s)\phi \text{ uniformly in } t \in [s,T].
\]

Also, the approximate problem
\[
u'(t) + \hat{A}_\lambda(t)u_\lambda(t) = 0, \quad t \in [s,T], \quad u_\lambda(s) = \phi,
\]
has a unique continuously differentiable solution \( u_\lambda(t) = V_\lambda(t,s)\phi \).
Hence, we have that
\[
V_\lambda(t,s)\phi = \phi - \int_s^t \hat{A}_\lambda(\tau)V_\lambda(\tau,s)\phi \, d\tau = \phi - \int_s^t \hat{A}(\tau)\hat{J}_\lambda(\tau)V_\lambda(\tau,s)\phi \, d\tau.
\]
Taking account of the definition of \( D(\hat{A}(\tau)) \), we obtain that

\[
(2.5) \quad (V_\lambda(t,s)\phi)(0) = \phi(0)
\]

\[
- \int_s^t \left[ A(\tau) (\hat{J}_\lambda(\tau)V_\lambda(\tau,s)\phi)(0) - F(\tau,\hat{J}_\lambda(\tau)V_\lambda(\tau,s)\phi) \right] d\tau.
\]

Now, by (A.1)' and (A.3), we see that

\[
I_1 = \int_s^t \| A(\tau) (\hat{J}_\lambda(\tau)V_\lambda(\tau,s)\phi)(0) - A(\tau)(V(\tau,s)\phi)(0) \| \, d\tau
\]

\[
\leq \alpha_1 \int_s^t \| \hat{J}_\lambda(\tau)V_\lambda(\tau,s)\phi - V(\tau,s)\phi \|_C \, d\tau
\]

and

\[
I_2 = \int_s^t \| F(\tau,\hat{J}_\lambda(\tau)V_\lambda(\tau,s)\phi) - F(\tau,V(\tau,s)\phi) \| \, d\tau
\]

\[
\leq \beta_0 \int_s^t \| \hat{J}_\lambda(\tau)V_\lambda(\tau,s)\phi - V(\tau,s)\phi \|_C \, d\tau.
\]

Let \( \phi \in \hat{D}_0 \); note here that \( \phi \in \text{L}ip \) by \( D(A(t)) = X \) and Lemma 1. For each \( \tau \in [s,T] \), we have for \( \lambda \) with \( \lambda \omega_1 < 1 \),

\[
I_3 = \| \hat{J}_\lambda(\tau)V_\lambda(\tau,s)\phi - V(\tau,s)\phi \|_C
\]

\[
\leq (1 - \lambda \omega_1)^{-1} \| V_\lambda(\tau,s)\phi - V(\tau,s)\phi \|_C + \| \hat{J}_\lambda(\tau)V(\tau,s)\phi - V(\tau,s)\phi \|_C,
\]

where \( \omega_1 = \alpha_1 + \beta_0 \).

By [2, Proposition 2.4], \( V(\tau,s)\phi \in \hat{D}_0 \) for \( \phi \in \hat{D}_0 \). This implies that the second term of the above inequality tends to zero as \( \lambda \to 0^+ \).

Hence \( I_3 \to 0 \) as \( \lambda \to 0^+ \).

Next, we note that

\[
(2.6) \quad \| \hat{J}_\lambda(\tau)V_\lambda(\tau,s)\phi \|_C
\]

\[
\leq (1 - \lambda \omega_1)^{-1} \| V_\lambda(\tau,s)\phi - \phi \|_C + \| \hat{J}_\lambda(\tau)\phi \|_C
\]

Since (C.3) is satisfied with \( B(t) = \hat{A}(t) \) and \( f(t) = g(t)I \), it follows that
\( (2.7) \quad \| \hat{J}_\lambda (\tau) \phi \|_C \leq \| \hat{J}_\lambda (s) \phi \|_C + \lambda \| g(\tau) - g(s) \| L(\| \phi \|_C) (1 + \| \hat{A}_\lambda (s) \phi \|_C) \)

\( \leq \lambda \| \hat{A}_\lambda (s) \phi \|_C + \| \phi \|_C \)

\( + \lambda \| (\tau - g(s) \| L(\| \phi \|_C) (1 + \| \hat{A}_\lambda (s) \phi \|_C) \)

\( \leq \lambda (1 - \lambda \omega_1)^{-1} \| \hat{A}(s) \phi \| + \| \phi \|_C \)

\( + \lambda \| (\tau - g(s) \| L(\| \phi \|_C) (1 + (1 - \lambda \omega_1)^{-1} \| \hat{A}(s) \phi \|) \).

Besides, since \( \lim_{\lambda \to 0^+} \{ \sup_{\tau \in [s, T]} \| V_\lambda (\tau, s) \phi - V(\tau, s) \phi \|_C \} = 0 \), we see that there exists \( \lambda_1 \) such that if \( 0 < \lambda \leq \lambda_1 \),

\( \sup_{\tau \in [s, T]} \| V_\lambda (\tau, s) \phi - V(\tau, s) \phi \|_C < 1 \). Thus it follows from (2.6) and (2.7) that \( \sup_{0 < \lambda < \lambda_1} \left( \sup_{\tau \in [s, T]} \| \hat{J}_\lambda (\tau) V_\lambda (\tau, s) \phi \|_C \right) \) is bounded. By the Lebesgue's dominated convergence theorem, we obtain that \( I_1 \to 0 \) and \( I_2 \to 0 \) as \( \lambda \to 0^+ \). Therefore, letting \( \lambda \to 0^+ \) in (2.5) yields (2.4). Q.E.D.

Remark. In general setting \( u(s, \phi)(\cdot) \) defined by (2.3) need not have a strong derivative. We may have regard the function \( u(s, \phi)(t) \) as a generalized solution of (FDE;\( \phi \))_s and investigate the meaning of generalized solutions. For convenience, the function \( u(s, \phi)(t) \) defined by (2.3) is called a generalized solution.

Now, we consider the approximate problem

\[
(FDE; \phi)_s^\beta \quad u_\beta^\prime (t) + A_\beta^\prime (t) u_\beta (t) = F(t, u_\beta (t)) \quad t \in [s, T],
\]

where \( A_\beta^\prime (t) \) is the Yosida approximation of \( A(t) \).

We define \( \hat{A}_\beta^\prime (t) : D(\hat{A}_\beta^\prime (t)) \subset C \to C \) by

\[
\hat{A}_\beta^\prime (t) \phi = -\phi',
\]

\[
D(\hat{A}_\beta^\prime (t)) = \{ \phi \in C; \phi' \in C, \phi'(0) + A_\beta^\prime (t) \phi(0) = F(t, \phi) \}.
\]
Clearly $A_\beta(t)$ satisfies the conditions of Proposition 4 with $\alpha_1 = \beta^{-1}(1 + (1 - \beta \alpha_0)^{-1})$; see [2, Lemma 1.2]. Therefore, there exists a family of nonlinear evolution operators $\{V_\beta(t,s); 0 \leq s \leq t \leq T\}$ generated by $\hat{A}^\beta(t)$. If $u_\beta(s,\phi)(\cdot)$ is defined by

$$u_\beta(s,\phi)(t) = \begin{cases} \phi(t - s) & s - r \leq t \leq s, \\ (V_\beta(t,s)\phi)(0) & s \leq t \leq T, \end{cases}$$

then $u_\beta(s,\phi)(t)$ is the strong solution of $(FDE;\phi)^\beta_s$ and by Proposition 2, $V_\beta(t,s)\phi = u_\beta(t,s,\phi)$ for $s \leq t \leq T$ and $\phi \in \text{Lip}$. By the proof of [2, Lemma 4.2], $\lim_{\beta \to 0^+}(1 + \lambda A^\beta(t))^{-1} \chi = (1 + \lambda A(t))^{-1} \chi$ for $\chi \in X$ and sufficiently small $\lambda$. Thus, by [10, Lemma 3.2], we obtain that $\lim_{\beta \to 0^+}(1 + \lambda \hat{A}^\beta(t))^{-1} \phi = (1 + \lambda \hat{A}(t))^{-1} \phi$ for $\phi \in C$ and small $\lambda$. Also, it follows from [2, Lemma 4.1] that $A_\beta(t)$ satisfies (A.1) and (A.2) uniformly in $\beta$, sufficiently small and hence $\hat{A}^\beta(t)$ satisfies (C.1)-(C.3) uniformly in $\beta$, sufficiently small. (To speak more carefully, by the same way as Proposition 1, we have that

$$\|\phi_\beta(t,\cdot) - \phi_\beta(\tau,\cdot)\|_C \leq K_{3\lambda} |g(t) - g(\tau)| \left[1 + \|\hat{A}^\beta(\tau)\psi\|_C \right] \left[L_2(\|\phi_\beta(\tau,\cdot)\|_C) + \right]
+ (1 + \|F(\tau,\phi_\beta(\tau,\cdot))\|_C) L_{1}(\|\psi(0)\| + \lambda \|F(\tau,\phi_\beta(\tau,\cdot))\|_C),$$

where $\phi_\beta(\tau,\cdot) = (1 + \lambda \hat{A}^\beta(t))^{-1} \psi$, $\psi \in C$,

and if $\chi \in C$ and $\phi_0 \in D(\hat{A}(0))$ then

$$\|F(\tau,\chi)\|$$

is bounded by an increasing function of $\|\chi\|_C$.

Now, in this case, we must prove that

$$(2.8) \quad \|\phi_\beta(\tau,\cdot)\|_C \leq L_{5}(\|\psi\|_C)$$

for some monotone increasing function $L_5$. However, since

$$\lim_{\beta \to 0^+}(1 + \lambda \hat{A}^\beta(t))^{-1} \phi = (1 + \lambda \hat{A}(t))^{-1} \phi$$

for all $\phi \in C$,

$$\|\phi_\beta(\tau,\cdot)\|_C \leq \|\phi(\tau,\cdot)\|_C + 1$$

for all small $\beta$. Therefore,
using $\| \phi(t, \cdot) \|_C \leq L_3(\| \psi \|_C)$ (see, Proposition 1), (2.8) is proved and hence $\hat{A}^\beta(t)$ satisfies (C.3) uniformly in $\beta$. We can apply the Crandall-Pazy approximation theorem [2, Theorem 4.1] to give $\lim_{\beta \to 0^+} V_\beta(t, s) \phi = V(t, s) \phi$ for all $\phi \in \hat{D}_0$. Therefore, by Proposition 2 and Lemma 1, we have that

**Theorem 2.** Let $\phi \in \text{Lip}$ with $\phi(0) \in \hat{D}$. Suppose that $\{A(t); t \in [0, T] \}$ and $F: [0, T] \times C \rightarrow X$ satisfy conditions (A.1)-(A.4). If $u(s, \phi)(\cdot)$ is a generalized solution of $(\text{FDE}; \phi)_s$ then $u(s, \phi)(t) = \lim_{\beta \to 0^+} u_\beta(s, \phi)(t)$ uniformly in $t \in [s, T]$, where $u_\beta(s, \phi)(\cdot)$ is the strong solution of $(\text{FDE}; \phi)_s^\beta$.


Our first result in this section is on the comparision of two generalized solutions.

**Theorem 3.** Let $\phi_i \in \text{Lip}$ with $\phi_i(0) \in \hat{D}$ for $i = 1, 2$. If $u(s, \phi_i)(\cdot)$ is a generalized solution of $(\text{FDE}; \phi_i)_s$, then we have

$$(3.1) \quad e^{-\alpha_0 t} \| u(s, \phi_1)(t) - u(s, \phi_2)(t) \| - e^{-\alpha_0 \tau} \| u(s, \phi_1)(\tau) - u(s, \phi_2)(\tau) \|$$

$$\leq \int_{\tau}^{t} e^{-\alpha_0 \xi} [u(s, \phi_1)(\xi) - u(s, \phi_2)(\xi), F(\xi, u_\xi(s, \phi_1)) - F(\xi, u_\xi(s, \phi_2))]_+ \, d$$

for $s \leq \tau \leq t \leq T$, where the symbol $[x, y]_+$ is defined by $[x, y]_+ = \lim_{\lambda \to 0^+} \lambda^{-1} \left( \| x + \lambda y \| - \| x \| \right)$ for $x, y \in X$.

**Proof.** Let $u_\beta(s, \phi_i)(t)$ be the strong solution of $(\text{FDE}; \phi_i)_s^\beta$ such that $\lim_{\beta \to 0^+} u_\beta(s, \phi_i)(t) = u(s, \phi_i)(t)$ uniformly for $t \in [s, T]$. 
Then \( \| u_\beta(s,\phi_1)(t) - u_\beta(s,\phi_2)(t) \| \) is differentiable a.e. \( t \in [s,T] \) and 
\[
\frac{d}{dt}\| u_\beta(s,\phi_1)(t) - u_\beta(s,\phi_2)(t) \| 
= \left[ u_\beta(s,\phi_1)(t) - u_\beta(s,\phi_2)(t), -A_\beta(t)(u_\beta(s,\phi_1)(t)) + F(t,u_{\beta t}(s,\phi_1)) \right.
+ A_\beta(t)(u_\beta(s,\phi_2)(t)) - F(t,u_{\beta t}(s,\phi_2)) \right]_+,
\]
where \([x,y]_- = -[x,-y]_+\). Since \([x - y, A_\beta(t)x - A_\beta(t)y]_+ \leq \alpha_0(1 - \beta \alpha_0)^{-1}\| x - y \|\), it follows that 
\[
\frac{d}{dt}\| u_\beta(s,\phi_1)(t) - u_\beta(s,\phi_2)(t) \| 
\leq \alpha_0(1 - \beta \alpha_0)^{-1}\| u_\beta(s,\phi_1)(t) - u_\beta(s,\phi_2)(t) \| 
+ \left[ u_\beta(s,\phi_1)(t) - u_\beta(s,\phi_2)(t), F(t,u_{\beta t}(s,\phi_1)) - F(t,u_{\beta t}(s,\phi_2)) \right]_+.
\]
Integrating the above inequality, we have for \( s \leq t \leq T \), 
\[
\| u_\beta(s,\phi_1)(t) - u_\beta(s,\phi_2)(t) \| 
\leq \alpha_0(1 - \beta \alpha_0)^{-1} \int_t^T \| u_\beta(s,\phi_1)(\xi) - u_\beta(s,\phi_2)(\xi) \| \, d\xi 
+ \int_t^T \left[ u_\beta(s,\phi_1)(\xi) - u_\beta(s,\phi_2)(\xi), F(\xi,u_{\beta \xi}(s,\phi_1)) - F(\xi,u_{\beta \xi}(s,\phi_2)) \right]_+ \, d\xi.
\]
Letting \( \beta \to 0^+ \) in this inequality, we see that for \( s \leq \tau \leq t \leq T \),
\[
(3.2) \quad \| u(s,\phi_1)(t) - u(s,\phi_2)(t) \| 
\leq \alpha_0 \int_t^\tau \| u(s,\phi_1)(\xi) - u(s,\phi_2)(\xi) \| \, d\xi 
+ \int_t^\tau \left[ u(s,\phi_1)(\xi) - u(s,\phi_2)(\xi), F(\xi,u_{\xi}(s,\phi_1)) - F(\xi,u_{\xi}(s,\phi_2)) \right]_+ \, d\xi.
\]
By the standard argument one can prove that (3.2) implies (3.1). (For example, see [9].) Q.E.D.

The following theorem gives the existence of integral solutions.
Theorem 4. Let \( u(s,\phi)(\cdot) \) be a generalized solution of \((\text{FDE};\phi)_s\).

Then the following inequality holds:

\[
(3.4) \quad e^{-\alpha_0 t} \| u(s,\phi)(t) - x \| - e^{-\alpha_0 \tau} \| u(s,\phi)(\tau) - x \|
\leq \int_{\tau}^{t} e^{-\alpha_0 \xi} \left\{ \left[ u(s,\phi)(\xi) - x, F(\xi, u_\xi(s,\phi)) - y \right]_+ + \theta(\xi, r) \right\} \, d\xi
\]

for \( s \leq \tau \leq t \), \([x,y] \in A(r), r \in [0,T]\),

where \( \theta(\xi, r) = L_1(||x||) \| h(\xi) - h(r) \| (1 + ||y||) \).

Proof. Let \( u(s,\phi)(\cdot) \) be a generalized solution of \((\text{FDE};\phi)_s\).

By Theorem 2, \( \lim_{\beta \to 0^+} u_\beta(s,\phi)(t) = u(s,\phi)(t) \) uniformly for \( t \in [s,T] \), where \( u_\beta(s,\phi)(t) \) is the strong solution of \((\text{FDE};\phi)_s\). Let

\( [x,y] \in A(r) \) and set \( x_\beta = x + \beta y \). Note that \( x = J_\beta(r)x_\beta \) and \( y = A_\beta(r)x_\beta \), where \( J_\beta(r) \) and \( A_\beta(r) \) are the resolvent and the Yosida approximation of \( A(r) \), respectively. Then

\[
\frac{d}{dt} \| u_\beta(s,\phi)(t) - x_\beta \|
\leq [u_\beta(s,\phi)(t) - x_\beta, -A_\beta(t)(u_\beta(s,\phi)(t)) + F(t,u_{\beta t}(s,\phi))]
\leq [u_\beta(s,\phi)(t) - x_\beta, -A_\beta(t)(u_\beta(s,\phi)(t)) + y]_+
\leq [u_\beta(s,\phi)(t) - x_\beta, F(t,u_{\beta t}(s,\phi)) - y]_+
\leq \beta^{-1}(\| u_\beta(s,\phi)(t) - x_\beta + \beta(-A_\beta(t)(u_\beta(s,\phi)(t)) + y) \|
- \| u_\beta(s,\phi)(t) - x_\beta \| + [u_\beta(s,\phi)(t) - x_\beta, F(t,u_{\beta t}(s,\phi)) - y]_+
\leq \beta^{-1}(\| J_\beta(t)(u_\beta(s,\phi)(t)) - J_\beta(r)x_\beta \| - \| u_\beta(s,\phi)(t) - x_\beta \|
+ [u_\beta(s,\phi)(t) - x_\beta, F(t,u_{\beta t}(s,\phi)) - y]_+
\leq L_1(||x_\beta||) \| h(t) - h(r) \| (1 + ||y||)
+ \alpha_0(1 - \beta \alpha_0)^{-1} \| u_\beta(s,\phi)(t) - x_\beta \|
+ [u_\beta(s,\phi)(t) - x_\beta, F(t,u_{\beta t}(s,\phi)) - y]_+
\leq A(t) \in \mathcal{Y}(\alpha_0) \) and \((A.2)).\)
Integrating these inequality over \([\tau, t] \subseteq [s, T]\),
\[
\|u_\beta(s,\phi)(t) - x_\beta\| - \|u_\beta(s,\phi)(\tau) - x_\beta\| \\
\leq \int_\tau^t (L_1(||x_\beta||) \|h(\xi) - h(r)|| (1 + ||y||) \\
+ \alpha_0(1 - \beta \alpha_0)^{-1} \|u_\beta(s,\phi)(\xi) - x_\beta\| \\
+ [u_\beta(s,\phi)(\xi) - x_\beta, F(\xi, u_{\beta \xi}(s,\phi)) - y]_+) \, d\xi.
\]
Letting \(\beta \to 0^+\), we see that for \(s \leq \tau \leq t \leq T\),
\[
(3.5) \quad \|u(s,\phi)(t) - x\| - \|u(s,\phi)(\tau) - x\| \\
\leq \int_\tau^t \{[u(s,\phi)(\xi) - x, F(\xi, u_{\xi}(s,\phi)) - y]_+ + \theta(\xi, r)\} \, d\xi \\
+ \alpha_0 \int_\tau^t \|u(s,\phi)(\xi) - x\| \, d\xi,
\]
which yields (3.4).

Q.E.D.

Next, we recall the definition of weak solutions in the sense of Kartsatos and Parrott [6,7] and consider the existence of weak solutions of \((FDE;\phi)_0\).

Definition 2. A function \(u(t) \in C([-r,T];X)\) is said to be a weak solution of \((FDE;\phi)_0\) if \(u(t) = \phi(t)\) for \(t \in [-r,0]\) and
\[
(\text{DE}) \quad v'(t) + A(t)v(t) \ni F(t,u_\xi), \quad t \in [0,T] \\
v(0) = \phi(0)
\]
has a solution \(v(t)\) in the sense of Evans [5] such that \(v(t) = u(t)\) for \(t \in [0,T]\).

Remark. By definition and [5, Theorem 3], there exists at most one weak solution of \((FDE;\phi)_0\). Indeed, if \(u_1(t)\) and \(u_2(t)\) are two weak solutions, they satisfy the integral inequality

- 17 -
\[ \| u_1(t) - u_2(t) \| \leq \int_0^t \| F(\tau, u_1) - F(\tau, u_2) \| \, d\tau. \]  
(See [5, (8.3)].)

Thus, by (A.3) and the Gronwall inequality, \( u_1(t) = u_2(t) \) for \( t \in [0, T] \).

**Theorem 5.** Suppose that \( \{ A(t); t \in [0,T] \} \) satisfy (A.1) with \( \alpha_0 = 0 \) and (A.2) and \( F: [0,T] \times C \to X \) satisfy (A.3) and (A.4). If \( \phi \in \text{Lip} \) and \( \phi(0) = \hat{0} \), then \( (\text{FDE}; \phi)_0 \) has a unique weak solution.

**Proof.** It suffices to show a generalized solution \( u(0,\phi)(t) \) of \( (\text{FDE}; \phi)_0 \) is a weak solution. Note that \( t \mapsto F(t, u_t(0,\phi)) \) is of bounded variation by (A.3) and (A.4) because \( u(0,\phi)(t) \) is Lipschitz continuous. Then (DE) has a solution \( v(t) \) in the sense of Evans, i.e., there exist sequence \( \{ t^n_k \} \) and \( \{ u^n_k \} \) such that

1) \[ \frac{u^n_k - u^{n-1}_k}{h^n_k} + A(t^n_k, u^n_k) \equiv F(t^n_k, u^n_{t^n_k}(0,\phi)), \text{ where } h^n_k = t^n_k - t^{n-1}_k, \]

2) the step functions \( v^n(t) \) (\( \equiv u^n_k \) on \( (t^{n-1}_k, t^n_k) \)) converge uniformly on \( [0, T] \) to \( v(t) \).

Note here that

\[ M \equiv \max \left\{ \sup \| u^n_k \|, \sup \| \frac{u^n_k - u^{n-1}_k}{h^n_k} - F(t^n_k, u^n_{t^n_k}(0,\phi)) \| \right\} < \infty. \]

(See [5, Proof of Theorem 2].)

Let \( v^n_k \in A(t^n_k)u^n_k \). By (3.5) we see that

\[ \| u(0,\phi)(t) - u^n_k \| - \| u(0,\phi)(\tau) - u^n_k \| \]

\[ \leq \int_\tau^t \left\{ [u(0,\phi)(\xi) - u^n_k, F(\xi, u_\xi(0,\phi)) - v^n_k]_+ + \theta_1(\xi, t^n_k) \right\} d\xi 
\]

\[ + \alpha_0 \int_\tau^t \| u(0,\phi)(\xi) - u^n_k \| d\xi, \]

- 18 -
where \( \theta_1(\xi, r) = M_1\| h(\xi) - h(r) \| \) and \( M_1 = L_1(M)(1 + M) \).

Since

\[
\begin{align*}
& h^n_k[u(0, \phi)(\xi)] - u^n_k, F(\xi, u(0, \phi)) - v^n_k, \\
& \leq \| u(0, \phi)(\xi) - u^n_{k-1} \| - \| u(0, \phi)(\xi) - u^n_k \| \\
& \quad + h^n_k \| F(\xi, u(0, \phi)) - F(t^n_k, u^n_k(0, \phi)) \|
\end{align*}
\]

it follows by the standard argument that

\[
\int_{t^n_j}^{t^n_i} \left( \| u(0, \phi)(t) - v^n(\eta) \| - \| u(0, \phi)(t) - v^n(\eta) \| \right) \, d\eta
\]

\[
\leq \int_{\tau}^{t} \left( \| u(0, \phi)(\xi) - v^n(t^n_j) \| - \| u(0, \phi)(\xi) - v^n(t^n_i) \| \right) \, d\xi
\]

\[
+ \int_{t^n_j}^{t^n_i} \int_{t^n_j}^{t} \{ \alpha_0 \| u(0, \phi)(\xi) - v^n(\eta) \| + \theta_1^n(\xi, \eta) \\
& \quad + \| F(\xi, u(0, \phi)) - F^n(\eta) \| \} \, d\xi \, d\eta,
\]

where \( \theta_1^n \) and \( F^n \) are functions defined by

\[
\theta_1^n(\xi, \eta) = \theta_1(\xi, t^n_k) \quad \text{for} \quad \eta \in (t^n_{k-1}, t^n_k)
\]

and

\[
F^n(\eta) = F(t^n_k, u^n_k(0, \phi)) \quad \text{for} \quad \eta \in (t^n_{k-1}, t^n_k), \text{ respectively}.
\]

Letting \( t^n_j \to t^j, t^n_i \to t^i \) as \( n \to \infty \) and applying [8, Proposition 2.5] we obtain that \( u(0, \phi)(t) = v(t) \) for \( t \in [0, T] \). Q.E.D.

Finally, we consider the existence of strong solutions of \( (\text{FDE}; \phi)_s \).

Corollary 1. Let \( \phi \in \text{Lip} \) with \( \phi(0) \in \hat{D} \). Assume that \( \{A(t) ; t \in [0, T]\} \) and \( F: [0, T] \times C \to X \) satisfy conditions (A.1)-(A.4). If \( X \) is reflexive, or, more generally, \( X \) satisfies the Radon-Nikodym property, then \( (\text{FDE}; \phi)_s \) has a unique strong solution.
Proof. By virtue of Theorem 2, there exists a generalized solution \( u(s,\phi)(t) \) and by the Remark after Lemma 1, \( u(s,\phi)(t) \) is Lipschitz continuous and hence \( u(s,\phi)(t) \) is differentiable a.e. \( t \in [s,T] \). Now, let \( h > 0 \) and \( t_0 \) be any point at which \( u(s,\phi)(\cdot) \) is differentiable. Putting \( \tau = \tau = t_0 \) and \( t = t_0 + h \) in (3.5), we see that

\[
\|u(s,\phi)(t_0 + h) - x\| - \|u(s,\phi)(t_0) - x\| \\
\leq \int_{t_0}^{t_0+h} \left\{ [u(s,\phi)(\xi) - x, F(\xi, u_\xi(s,\phi)) - y]_+ + \theta(\xi, t_0) \right\} \, d\xi \\
+ \alpha_0 \int_{t_0}^{t_0+h} \|u(s,\phi)(\xi) - x\| \, d\xi \quad \text{for} \quad [x,y] \in A(t_0).
\]

Dividing the above inequality by \( h \) and letting \( h \to 0 \), it follows

\[
[u(s,\phi)(t_0) - x, u'(s,\phi)(t_0)]_+ \\
\leq [u(s,\phi)(t_0) - x, F(t_0, u_{t_0}(s,\phi)) - y]_+ + \alpha_0 \|u(s,\phi)(t_0) - x\|,
\]

i.e., for \([x,y] \in A(t_0)\)

(3.6) \( [u(s,\phi)(t_0) - x, -u'(s,\phi)(t_0) + F(t_0, u_{t_0}(s,\phi)) \\
+ \alpha_0 u(s,\phi)(t_0) - (\alpha_0 x + y)]_+ \geq 0. \)

By condition (A.1), it is easy to see that \( A(t_0) + \alpha_0 \) is \( m \)-accretive. Therefore, by (3.6), we see that

\[
u'(s,\phi)(t_0) + A(t_0)(u(s,\phi)(t_0)) \ni F(t_0, u_{t_0}(s,\phi)).
\]

Q.E.D.

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