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On Weak and Classical Solutions of the Two-Dimensional Magnetohydrodynamic Equations

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Let Ω be a bounded domain in R^2 with smooth boundary $\partial\Omega$. In $Q_T := \Omega\times(0, T)$, we consider the following magnetohydrodynamic equations for an ideal incompressible fluid coupled with magnetic field:

$$(*)\quad \begin{cases} \partial_t u + (u, \nabla)u - (B, \nabla)B + \nabla((1/2)|B|^2) + \nabla\pi = f & \text{in } Q_T, \\ \partial_t B - \Delta B + (u, \nabla)B - (B, \nabla)u = 0 & \text{in } Q_T, \\ \operatorname{div} u = 0, \quad \operatorname{div} B = 0 & \text{in } Q_T, \\ u\cdot\nu = 0, \quad B\cdot\nu = 0, \quad \operatorname{rot} B = 0, & \text{on } \partial\Omega\times(0, T), \\ u\big|_{t=0} = u_0, \quad B\big|_{t=0} = B_0. & \end{cases}$$

Here $u = u(x, t) = (u^1(x, t), u^2(x, t))$, $B = B(x, t) = (B^1(x, t), B^2(x, t))$ and $\pi = \pi(x, t)$ denote the unknown velocity field of the fluid, magnetic field and pressure of the fluid, respectively; $f = f(x, t) = (f^1(x, t), f^2(x, t))$ denotes the given external force, $u_0 = u_0(x) = (u_0^1(x), u_0^2(x))$ and B_0

$= B_0(x) = (B_0^1(x),\ B_0^2(x))$ denote the given initial data and ν denotes the unit outward normal on $\partial\Omega$.

The first purpose of this paper is to state the existence and uniqueness of a global weak solution of (*) without restriction on the data. In case B is identically equal to zero, i.e., in the case of the Euler equations, such a problem for global weak and classical solutions was solved by Bardos [1] and Kato [4], respectively. (Kikuchi [5] extended the result of Kato [4] in an exterior domain.) Using the energy method developed by Bardos [1], we can obtain a global weak solution in our case.

Our second purpose is to state the existence and uniqueness of a local classical solution of (*). Although the method of characteristic curves for the vorticity equation plays an important role in a global classical solution of the two-dimensional Euler equations, such a method seems to give us only a local classical solution of (*) because of the additional terms $(B, \nabla)B$ and $(u, \nabla)B - (B, \nabla)u$. Our result on classical solution is considered, however, a generalization of that of Kato [4] in some sense.

1. Notation.

Let us introduce some function spaces. $C_{0,\sigma}^{\infty}(\Omega)$ denotes the set of all C^{∞}-real vector-valued functions $\Phi = (\Phi^1, \Phi^2)$ with compact support in Ω such that $\operatorname{div} \Phi = 0$. H is the completion of $C^{\infty}_{0,\sigma}(\Omega)$ with respect to the L^2-norm $\| \ \|$; $(\ , \ )$ denotes the L^2-inner product. V denotes the set of all vector-valued functions u in $H^1(\Omega)$ with $\operatorname{div} u = 0$ in Ω and $u \cdot \nu = 0$ on $\partial\Omega$. Equipped with the norm $|\!|\!| \ |\!|\!|$:

$$|\!|\!| u |\!|\!|^2 = \|\mathrm{rot}\ u\|^2 + \|u\|^2,$$

V is a Hilbert space. By Duvaut-Lions [2, Chapter 7 Theorem 6.1], we have

$$\|u\|_{H^1(\Omega)} \leqq C(\Omega) |\!|\!| u |\!|\!| \quad \text{for all} \quad u \quad V.$$

Hence the norm $|\!|\!| \ |\!|\!|$ is equivalent to the one usually induced from $H^1(\Omega)$ and V is compactly imbedded into H.

If X is a Hilbert space, then $L^p(0, T; X)$ $(1 \leqq p < \infty)$ denotes the set of all measurable functions $u(t)$ with values in X such that $\int_0^T \|u(t)\|_X^p \, dt < \infty$ ($\| \ \|_X$ is the norm in X). $L^{\infty}(0, T; X)$ denotes the set of all essentially bounded (in the norm of X) measurable functions of t with values in X.

Let $C^m([0, T]; X)$ denote the set of all X-valued m-times continuously differentiable functions of t $(0 \leqq t \leqq T)$. $C^m_0([0, T); X)$ is the set of all X-valued m-times continuously differentiable functions on $[0, T)$ with compact support in $[0, T)$.

$C^{k+\alpha}(\bar{\Omega})$ with integer $k \geqq 0$ and $0 \leqq \alpha < 1$ denotes the usual Hölder space of continuous functions on $\bar{\Omega}$. $|\ |_{k+\alpha}$ denotes the norm in $C^{k+\alpha}(\bar{\Omega})$. $C^{k,j}(\bar{Q}_T)$ with integers $k, j \geqq 0$ is the set of all functions ϕ for which all the $\partial_x^q \partial_t^r \phi$ exist and are continuous on $\bar{Q}_T$ for $0 \leqq |q| \leqq k$, $0 \leqq r \leqq j$. $C^{k+\alpha, j+\beta}(\bar{Q}_T)$ with integers $k, j \geqq 0$ and $0 \leqq \alpha, \beta < 1$ is the subset of $C^{k,j}(\bar{Q}_T)$ containing all functions ϕ for which all the $\partial_x^q \partial_t^r \phi$, $0 \leqq |q| \leqq k$, $0 \leqq r \leqq j$, are Hölder continuous with exponents α in x and β in t. If

$$K^{\alpha,\beta}(\phi) = \sup_{\substack{(x,t)\in\bar{Q}_T \\ (x',t)\in\bar{Q}_T \\ |x-x'|<1}} |\phi(x, t) - \phi(x', t)|/|x - x'|^{\alpha} + \sup_{\substack{(x,t)\in\bar{Q}_T \\ (x,t')\in\bar{Q}_T \\ |t-t'|<1}} |\phi(x, t) - \phi(x, t')|/|t - t'|^{\beta},$$

we define the norm $|\ |_{k+\alpha, j+\beta}$ in $C^{k+\alpha, j+\beta}(\bar{Q}_T)$ by

$$|\phi|_{k+\alpha, j+\beta} = \sup_{(x,t)\in\bar{Q}_T} \sum_{\substack{|q|\leqq k \\ r\leqq j}} |D_x^q \partial_t^r \phi(x, t)| + \sum_{|q|=k} K^{\alpha,\beta}(D_x^q \partial_t^j \phi).$$

For the spaces of vector-valued functions, we shall use the bold-faced letters analogously.

2. Definitions and results.

Our definition of a weak solution of (*) is as follows.

Definition. *Let* $u_0 \in H$, $B_0 \in H$ *and* $f \in L^2(0, T; L^2(\Omega))$. *A pair of measurable vector functions* u *and* B *on* Q_T *is called a weak solution of* (*) *if*

(i) $u \in L^{\infty}(0, T; H) \cap L^2(0, T; V)$, $B \in L^{\infty}(0, T; H) \cap L^2(0, T; V)$;

(ii) $$\int_0^T \{-(u, \partial_t \Phi) + ((u, \nabla)u - (B, \nabla)B, \Phi)\}\, dt$$

$$= (u_0, \Phi(0)) + \int_0^T (f, \Phi)\, dt$$

$$\int_0^T \{-(B, \partial_t \Phi) + (\mathrm{rot}\, B, \mathrm{rot}\, \Phi) + ((u, \nabla)B - (B, \nabla)u, \Phi)\}\, dt$$

$$= (B_0, \Phi(0))$$

for all $\Phi \in C_0^1([0, T); V)$.

Concerning the uniqueness of weak solutions of (*), we have

Proposition. *There exists at most one weak solution of* (*). *If* $\{u, B\}$ *is the weak solution of* (*), *after a suitable redefinition of* $u(t)$ *and* $B(t)$ *on a set of measure zero of the* *time interval* [0, T], *we have that* $u \in C([0, T]; H)$ *and* $B \in C([0, T]; H)$.

For the proof of this proposition, see Temam [6, Chapter 3 Theorem 3.2], we omit it.

Our result on the existence of a weak solution now reads:

Theorem 1. *Let* $u_0 \in V$, $B_0 \in V$ *and* $f \in L^2(0, T; L^2(\Omega))$ *with* $\mathrm{rot}\, f \in L^2(0, T; L^2(\Omega))$. *Then there exists a weak solution* $\{u, B\}$ *of* (*) *such that* $u \in L^\infty(0, T; V) \cap C([0, T]; H)$ *and* $B \in L^2(0, T; H^2(\Omega)) \cap C([0, T]; V)$.

We next proceed to our result on classical solutions. To this end, we make the following assumptions on the domain Ω and the given data u_0, B_0 and f.

Assumption 1. The boundary $\partial\Omega$ of Ω consists of $m + 1$ sufficiently smooth, simple closed curves $S_0, S_1, \dots, S_m$, where $S_j (j = 1, \dots, m)$ are inside of S_0 and outside of one another.

Günter [3, p.22] refers to the above assumption as ''Case J''.

Assumption 2. For some $0 < \theta < 1$, $u_0 \in C^{1+\theta}(\bar{\Omega})$, $B_0 \in C^{2+\theta}(\bar{\Omega})$ and $f \in C^{1+\theta, 0}(\bar{Q}_T)$. Moreover, u_0 and B_0 satisfy the conditions $\operatorname{div} u_0 = 0$, $\operatorname{div} B_0 = 0$ in Ω, $u_0 \cdot \nu = 0$, $B_0 \cdot \nu = 0$ on $\partial\Omega$.

Our result on the existence and uniqueness of classical solutions reads:

Theorem 2. Let the assumptions 1 and 2 hold. Then there is a positive number $C_* = C_*(\Omega, T, |u_0|_{1+\theta}, |f|_{1+\theta,0})$ such that if $|B_0|_{2+\theta} \leq C_*$, there exists a solution $\{u, B, \pi\} \in C^{1,1}(\bar{Q}_T) \times C^{2,1}(\bar{Q}_T) \times C^{1,0}(\bar{Q}_T)$ of (*). Such a solution is unique up to adding an arbitrary function of t to π.

Remark. Taking $B_0 = 0$ in Ω, we may cover the result of Kato [4].

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