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On the effectiveness of the method of regularization
- In the Case of Numerical Harmonic Continuation -

Takashi Kitagawa

北川高嗣・愛媛大学理学部

1. Introduction. We consider the problem of harmonic continuation of a function which shall be stated that given a harmonic function $h(r,\theta)$ in the unit circle with known $r < 1$, find $h(r,\theta)$ for $r=1$. If we set

$$g(\theta) = h(r,\theta)$$

and

$$f(\phi) = h(1,\phi),$$

then this problem can be formulated by the Fredholm integral equations of the first kind of the form

$$(1.1) \qquad \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1-2r\cos(s-t)+r^2} f(t)\, dt = g(s).$$

from which we directly recognize the problem is ill-posed. This can be viewed as direct inversion of Poisson integral to find $g(s)$ in the unit disk from the boundary value of $f(t)$ for t on the unit circle which we denote $C(1)$.

The difficulty of this problem can be understood from the relation of the Fourier coefficients $\{a_k\}$ and $\{b_k\}$ of f and g that if

$$f(\theta) = a_0 + \sum_{k=1}^{\infty} (a_k\cos(k\theta) + b_k\sin(k\theta)),$$

then

$$g(\theta) = a_0 + \sum_{k=1}^{\infty} r^k(a_k\cos(k\theta) + b_k\sin(k\theta)).$$

This explains that if $r < 1$ is very small and the g is contaminated by some perturbation, say Δg, the perturbation introduced to f can be magnified by the factor of $1/r^k$.

In the present paper we propose a method for this problem based on the fundamental solution method and the method of regularization which is applicable for harmonic continuation not only to the unit disk but also to any simply connected region. In Section 2 we give a brief introduction of the fundamental solution method and basic result on convergence. In Section 3, we present numerical scheme for harmonic continuation and we discuss the convergence of the method in Section 4. In Sections 5 and 6, which is the main part of this paper, we examine the numerical stability of the method based on our previous papers [10] and [11]. Some numerical examples are included in the last two sections.

2. Fundamental Solution Method. To illustrate the idea of fundamental solution method, we first consider the Dirichlet problem of Laplace equation of the form

(2.1) $$\Delta u = 0 \quad \text{in} \quad \Omega$$

(2.2) $$u = g \quad \text{on} \quad \partial\Omega ,$$

where $\Omega = \{ \omega \in R^2 \mid \|\omega\|_2 < \rho \}$.

The fundamental solution method approximates the solution $u(x)$ by

(2.3) $$u_n(x) = \sum_{k=1}^{n} c_k \; G(x,y_k) \quad , \qquad x \in \Omega$$

where $G(x,y)$ is the Green's function for (Δ,Ω),

$$G(x,y) = -\frac{1}{2}\log\|x-y\|_2 \quad , \ x,y \in R^2,$$

points y_k's, called charge points, are chosen appropriately and c_k's are constants to be determined. The vector $c = (c_1, c_2, \ldots, c_n)^t \in R^n$ is called charge and determined in such a way that $u_n(x)$ satisfies the boundary condition

$$(2.4) \qquad u_n(\hat{x}_j) = g(\hat{x}_j) \qquad j = 1,2,\ldots,n,$$

where $\hat{x}_j$'s are properly chosen n collocation points on the boundary. Let the charge points $y_1, y_2, \ldots y_n$ be on the auxiliary boundary which is the outer circle with radius R (with "outer" we imply $R > \rho$).

With the collocation points $\hat{x}_k = \rho\, e^{\frac{2\pi}{n}(k-1)i}$ and the charge points $y_k = R\, e^{\frac{2\pi}{n}(k-1)i}$, k=1,2,...,n, the following results are known.

Theorem 2.1.(Katsurada[8]) a) The approximate solution u_n converges to the solution u exponentially with respect to n. More precisely

$$(2.5) \qquad \|u - u_n\|_\infty \leq$$

$$\sup_{\|x\|_2 = r_0} |u(x)| \frac{2}{1-\rho/r_0} \{(1+C(R,\rho))\,(\rho/r_0)^{n/3} + 4(\rho/R)^{n/3}\},$$

where we suppose that the harmonic extension of u exists in $\Omega_{r_0} = \{\omega \mid \|\omega\|_2 < r_0\}$ with $\rho < r_0$. $C(R,\rho)$ is a constant depends on R and ρ.

b) The condition number of the coefficients matrix of the equation (2.4) which determines the charge c grows exponentially with respect to n. Approximately the condition number Cond(n,R) can be estimated by

$$(2.6) \qquad \mathrm{Cond}(n,R) \sim \frac{\log R}{2} \; n \; \left(\frac{R}{\rho}\right)^{n/2} .$$

The estimate (2.6) follows from the fact that the coefficient matrix for the particular location of $\hat{x}_k$ and y_k is circulant. For the properties of circulant matrices, see e.g. Davis[1]. Numerical stability of this method is studied in Kitagawa[10].

3. Numerical Scheme for Harmonic Continuation

The numerical scheme proposed here makes use of the fundamental solution method and the method of regularization. Suppose that the harmonic function h(x), $x \in R^2$ to be continuated is given on a circle $C(\rho)$ with radius ρ and we shall seek its harmonic continuation on the circle $C(\gamma)$ with radius $\gamma > \rho$. We also assume that the function $h(x) = h(r,\theta)$ is harmonic for $r < r_0$ and $r_0 > \gamma > \rho$. The process of the fundamental solution method for this problem of harmonic continuation shall be as follows.

STEP 1. Let R be the radius of the auxiliary circle C(R) on which we scatter the charge point $y = (y_1, y_2, \ldots, y_n)$ with $y_k = R\, e^{\frac{2\pi}{n}(k-1)i}$, $k=1,2,\ldots,n$, where i denotes the imaginary unit. We determine the charge $c = (c_1, c_2, \ldots, c_n)$ in such a way that

(3.1) $$h_n(w_j) \equiv \sum_{k=1}^{n} c_k \; G(w_j, y_k) = g(w_j) \;, j = 1,2,\ldots,n,$$

where w_j's are properly chosen n collocation points on the circle $C(\rho)$.

STEP 2. We approximates the values of the harmonic continuation $\bar{h}(x)$ of $h(x)$ on the circle $C(\gamma)$ by the formula

(3.2) $$\bar{h}_n(x) \equiv \sum_{k=1}^{n} c_k \; G(x, y_k) \;, \quad x \in C(\gamma)$$

One may consider this approach is quite straight forward and the method should work out well as in the case of the Dirichlet problem. The situation, however, is completely different from the case and the direct application of above numerical method produces unacceptable results. This is especially true when the ratio R/ρ is relatively large. In the rest of the paper, we deal with the convergence and the numerical stability of numerical procedure proposed above in detail.

4. Convergence of the Method. Since we have the convergence result in the case of the Dirichlet problem, the convergence of the numerical method for harmonic continuation proposed in Section 3 is rather straight forward. See [12] for details.

5. Numerical Stability of the Method.

5.1. Formulation and Basic Results. The method of Section 3 reduces to the numerical process of the following two steps:

1) We solve an ill-conditioned linear system of (3.1) in the form of

(5.1) $$\Gamma c = g$$

for given data g which may be contaminated by some perturbation Δg, where $g \in Y = R^m$, $c \in X = R^n$ and $\Gamma: X \to Y$ with

$$(\Gamma c)_j \equiv \sum_{k=1}^{n} c_k G(w_j, y_k) = g(w_j), \quad w_j \in C(\rho), \quad j = 1,2,\ldots,n.$$

2) We use the intermediate solution c to obtain the final result f by

(5.2) $$f = \Lambda c ,$$

where $f \in X$ and $\Lambda : X \to Y$ with

$$f_j \equiv \bar{h}_n(x_j) = \sum_{k=1}^{n} c_k G(x_j, y_k) \equiv (\Lambda c)_j, \quad x_j \in C(\gamma), \quad j = 1,2,\ldots,n.$$

Due to the ill-conditioning of (5.1), some 'large' perturbation Δc may be introduced to the intermediate solution c. One may assume intuitively that the error $\|\Delta f\|$ in the final result f is on a level with $\|\Delta c\|$ or as large as $\|\Lambda\|\ \|\Delta c\|$. If this is the case, the method of regularization Groetsch[6] and Tikhonov et al.[14], applied to (5.1) may be very effective. But this is not always true. Even if the error $\|\Delta c\|$ is very large, $\|\Delta f\|$ can be very small. In this case, we do not necessarily have to use the method. In some cases, we may have worse result by using the method. To examine whether the method of regularization is effective or not for this class of numerical procedures, we have the following results.

We assume that given data $\bar{g} = g + \Delta g$ and the intermediate solution $\bar{c} = c + \Delta c$. We have $\Gamma \bar{c} = \bar{g}$ as well as (5.1). Let $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$ be singular values of Γ and $\{u_i\}$ $i=1,2,\ldots,m$, $\{v_j\}$ $j=1,2,\ldots n$ be singular vectors of Γ. Reflecting the ill-conditioning of Γ, we assume that $\sigma_n \to 0$ as $n \to \infty$. We can see that $\|\Delta c\| \to \infty$ as $\sigma_n \to 0$ from

(5.3) $$\Delta c = \sum_{i=1}^{n} \frac{1}{\sigma_i} (\Delta g, u_i)\, v_i .$$

Next, we suppose that the final result $\bar{f} = f + \Delta f$. We have $\bar{f} = \Lambda \bar{c}$ as well as (5.2). Let $\hat{\sigma}_1 \geq \hat{\sigma}_2 \geq \dots \geq \hat{\sigma}_n \geq 0$ be singular values of Λ and $\{\hat{u}_i\}$ $i=1,2,\dots,m$, $\{\hat{v}_j\}$ $j=1,2,\dots n$ be singular vectors of Λ. As for $\|\Delta f\|$, we have the following result from Theorem 4.1 in Kitagawa[10].

Theorem 5.1

(5.4) $$\|\Delta f\| \leq \|\Xi * \Theta\|_F \, \|\Delta g\|$$

where

(5.5a) $$\Xi = (\xi_{ij}) , \quad \xi_{ij} = \hat{\sigma}_i / \sigma_j$$

(5.5b) $$\Theta = (\theta_{ij}) , \quad \theta_{ij} = (\hat{v}_i, v_j), \ i,j=1,2,\dots,n.$$

$\Xi * \Theta$ represents the Hadamard product of the matrices Ξ and Θ and $\|\cdot\|_F$ denotes the Frobenius norm.

We here construct the matrices Θ and Ξ to examine the numerical stability of the method of Section 3. We compare the matrices with those of the case of the Dirichlet problem and conclude that, unlike the case of the Dirichlet problem,

(i) the whole numerical process of 1) and 2) for harmonic continuation shall be very unstable as the ratio R/ρ becomes large

(ii) we need to employ the method of regularization to stabilize the ill-conditioned linear system of (4.1).

5.2. Matrices Ξ and Θ for Harmonic Continuation. The elements ξ_{ij} of Ξ, which we called explosive factor matrix in [10-11], represent the upper bound of magnification of the u_i-component $(\Delta g, u_i)$ of perturbation Δg to $\hat{u}_j$- component $(\Delta f, \hat{u}_j)$ of Δf. For instance, the largest element ξ_{1n} gives the upper bound of $\hat{\sigma}_1/\sigma_n$ which coincides with the straight forward upper bound with the spectral norm $\|\cdot\|_s$ of matrix given by $\|\Delta f\| \leq \|\Gamma^{-1}\|_s \|\Lambda\|_s \|\Delta g\|$, since $\|\Gamma^{-1}\|_s \|\Lambda\|_s = \hat{\sigma}_1/\sigma_n$. On the other hand, the elements θ_{ij} of Θ, which we call distortion coefficients matrix, represents the actual ratio of propagation of $(\Delta g, u_i)$ to $(\Delta f, \hat{u}_j)$. The actual magnification of propagation of $(\Delta g, u_i)$ to $(\Delta f, \hat{u}_j)$ is given by $\xi_{ij} \times \theta_{ij}$ and the upper bound of the total propagation of Δg to Δf is given by the square root of the sum of squares of $\xi_{ij} \times \theta_{ij}$, or $\|\Xi * \Theta\|_F$.

Fig. 5.1. and 5.2. represents typical patterns of Ξ and Θ respectively in the case of Dirichlet problem. The elements of matrices Ξ and Θ are given by the rounding off the fractions of logarithm with basis 10. For instance, an element $a_{ij} = 5$ stands for $10^{4.5} \leq a_{ij} < 10^{5.5}$. We set the parameters $R = 4$, $\rho = 0.5$ and $n = 16$. As is shown in [10], the elements ξ_{ij} grow large for small i's and large j's (upper right corner of Ξ) and the diagonal elements ξ_{ii}'s are almost unity, while the elements θ_{ij} almost vanishes except for near diagonal elements and the diagonal

Fig.5.1. EXPLOSIVE FACTOR MATRIX FOR DIRICHLET PROBLEM

0	1	1	3	3	4	4	5	5	6	6	7	7	8	8	9
-1	0	0	1	1	2	2	3	3	4	4	5	5	6	6	7
-1	0	0	1	1	2	2	3	3	4	4	5	5	6	6	7
-3	-1	-1	0	0	1	1	2	2	3	3	4	4	5	5	6
-3	-1	-1	0	0	1	1	2	2	3	3	4	4	5	5	6
-4	-2	-2	-1	-1	0	0	1	1	2	2	3	3	4	4	5
-4	-2	-2	-1	-1	0	0	1	1	2	2	3	3	4	4	5
-5	-3	-3	-2	-2	-1	-1	0	0	1	1	2	2	3	3	4
-5	-3	-3	-2	-2	-1	-1	0	0	1	1	2	2	3	3	4
-6	-4	-4	-3	-3	-2	-2	-1	-1	0	0	1	1	2	2	3
-6	-4	-4	-3	-3	-2	-2	-1	-1	0	0	1	1	2	2	3
-7	-5	-5	-4	-4	-3	-3	-2	-2	-1	-1	0	0	1	1	2
-7	-5	-5	-4	-4	-3	-3	-2	-2	-1	-1	0	0	1	1	2
-8	-6	-6	-5	-5	-4	-4	-3	-3	-2	-2	-1	-1	0	0	1
-8	-6	-6	-5	-5	-4	-4	-3	-3	-2	-2	-1	-1	0	0	1
-9	-7	-7	-6	-6	-5	-5	-4	-4	-3	-3	-2	-2	-1	-1	0

Fig.5.2. DISCO MATRIX FOR DIRICHLET PROBLEM

0	-7	-8	-8	-8	-8	-7	-7	-7	-7	-8	-7	-8	-7	-8	-7
-7	0	-8	-7	-9	-7	-7	-8	-7	-7	-7	-7	-7	-7	-7	-8
-8	-8	0	-8	-8	-7	-7	-8	-9	-7	-7	-7	-8	-8	-7	-7
-8	-7	-8	0	-7	-7	-8	-7	-7	-8	-7	-7	-7	-7	-8	-7
-8	-9	-8	-7	0	-8	-7	-8	-7	-7	-7	-8	-7	-8	-7	-7
-8	-7	-7	-7	-8	0	-7	-7	-8	-7	-8	-7	-7	-9	-7	-7
-7	-7	-7	-8	-7	-7	0	-7	-7	-7	-8	-7	-8	-7	-7	-7
-7	-8	-8	-7	-8	-7	-7	0	-7	-7	-7	-7	-7	-7	-8	-7
-7	-7	-9	-7	-7	-8	-7	-7	0	-7	-7	-7	-7	-7	-7	-8
-7	-7	-7	-8	-7	-7	-7	-7	-7	0	-7	-7	-7	-7	-6	-7
-8	-7	-7	-7	-7	-8	-8	-7	-7	-7	0	-7	-8	-7	-7	-8
-7	-7	-7	-7	-8	-7	-7	-7	-7	-7	-7	0	-7	-8	-6	-8
-8	-7	-8	-7	-7	-7	-8	-7	-7	-7	-8	-7	0	-7	-7	-7
-7	-7	-8	-7	-8	-9	-7	-7	-7	-7	-7	-8	-7	0	-7	-7
-8	-7	-7	-8	-7	-7	-7	-8	-7	-6	-7	-6	-7	-7	0	-7
-7	-8	-7	-7	-7	-7	-7	-7	-8	-7	-8	-8	-7	-7	-7	0

Fig. 5.3. MAGNIFIER MATRIX FOR DIRICHLET PROBLEM

0	-6	-7	-5	-5	-4	-4	-3	-3	-2	-2	0	-1	1	-1	2
-9	0	-8	-6	-8	-4	-5	-5	-4	-3	-3	-2	-2	-1	-1	0
-9	-8	0	-7	-7	-5	-4	-4	-6	-3	-3	-2	-3	-1	-1	0
-10	-9	-9	0	-7	-6	-7	-5	-5	-5	-4	-3	-3	-2	-3	-1
-10	-10	-9	-7	0	-6	-6	-5	-5	-4	-4	-4	-3	-3	-2	-1
-11	-9	-9	-8	-9	0	-7	-6	-7	-5	-6	-4	-4	-5	-3	-2
-11	-9	-9	-9	-8	-7	0	-6	-6	-5	-6	-4	-5	-3	-3	-2
-12	-11	-11	-9	-10	-8	-8	0	-7	-6	-6	-5	-5	-4	-5	-3
-12	-10	-13	-10	-9	-9	-8	-7	0	-6	-6	-5	-5	-4	-4	-4
-13	-11	-12	-11	-10	-10	-9	-8	-8	0	-7	-6	-6	-5	-5	-4
-14	-11	-11	-10	-10	-10	-10	-8	-8	-7	0	-6	-7	-5	-5	-5
-14	-13	-12	-11	-12	-10	-10	-9	-9	-8	-8	0	-7	-7	-6	-6
-14	-12	-13	-11	-11	-10	-11	-9	-9	-8	-9	-7	0	-6	-6	-5
-15	-13	-14	-12	-13	-13	-12	-10	-10	-9	-9	-9	-8	0	-7	-6
-16	-13	-13	-13	-12	-11	-11	-11	-10	-8	-9	-7	-8	-7	0	-6
-16	-15	-14	-13	-13	-12	-12	-11	-12	-10	-11	-10	-9	-8	-8	0

elements are again almost unity. Consequently, the large elements of ξ_{ij} cancel out by multiplying the corresponding elements of θ_{ij} (Fig. 5.3).

Fig.5.4 and 5.5 represents the typical patterns of the matrices Ξ and Θ respectively in the case of harmonic continuation. The parameters R, ρ and n are identical to those of the case of Dirichlet problem and the radius γ of the circle where the harmonic continuation of h(x) shall be sought is set to 2. The matrix Ξ shows quite different feature from that of the Dirichlet problem. The elements ξ_{ij} for large j (right half) do not decrease to unity even for near diagonal elements. This is because the decay of the singular values σ_i of Γ is much faster than the singular values $\hat{\sigma}_i$ of Λ. Table 5.1 and 5.2 shows the decay of both singular values of $\{\sigma_i\}$ and $\{\hat{\sigma}_j\}$ respectively. The corresponding elements θ_{ij} of near diagonal elements are close to unity as is in the case of Dirichlet problem. Thus the large elements of ξ_{ij} remains at lower right corner of $\Xi*\Theta$ as is shown in Fig. 5.6, and accordingly the perturbation $(\Delta g,u_i)$ for $i \approx n$ shall be significantly magnified and propagates to $(\Delta f,\hat{u}_j)$ for $j \approx n$. This explains the numerical instability of the fundamental solution method for harmonic continuation and implies the necessity of employing the method of regularizaiton.

6. Effectiveness of the Method of Regularization

The method of regularization applied to the equation (5.1) with perturbation Δg can be written as

FIG. 5.4. EXPLOSIVE FACTOR MATRIX FOR HARMONIC CONTINUATION

0	1	1	3	3	4	4	5	5	6	6	7	7	8	8	9
-1	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8
-1	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8
-1	0	0	1	1	2	2	3	3	4	4	5	5	6	6	7
-1	0	0	1	1	2	2	3	3	4	4	5	5	6	6	7
-2	0	0	1	1	2	2	3	3	4	4	5	5	6	6	7
-2	0	0	1	1	2	2	3	3	4	4	5	5	6	6	7
-2	-1	-1	0	0	1	1	2	2	3	3	4	4	5	5	6
-2	-1	-1	0	0	1	1	2	2	3	3	4	4	5	5	6
-3	-1	-1	0	0	1	1	2	2	3	3	4	4	5	5	6
-3	-1	-1	0	0	1	1	2	2	3	3	4	4	5	5	6
-3	-2	-2	0	0	1	1	2	2	3	3	4	4	5	5	6
-3	-2	-2	0	0	1	1	2	2	3	3	4	4	5	5	6
-3	-2	-2	-1	-1	0	0	1	1	2	2	3	3	4	4	5
-3	-2	-2	-1	-1	0	0	1	1	2	2	3	3	4	4	5
-3	-2	-2	-1	-1	0	0	1	1	2	2	3	3	4	4	5

Fig. 5.5. DISCO MATRIX FOR HARMONIC CONTINUATION

0	-9	-7	-7	-8	-8	-7	-7	-7	-8	-8	-7	-7	-7	-7	-7
-7	0	0	-8	-7	-7	-7	-8	-8	-7	-7	-8	-7	-7	-7	-7
-7	0	0	-8	-7	-7	-7	-7	-7	-7	-6	-8	-7	-8	-7	-7
-7	-7	-7	0	-1	-7	-6	-6	-7	-6	-8	-6	-6	-6	-6	-6
-8	-8	-7	-1	0	-7	-7	-6	-6	-7	-6	-7	-6	-7	-7	-6
-7	-6	-6	-6	-7	0	0	-5	-6	-5	-5	-5	-6	-5	-7	-6
-7	-6	-6	-6	-6	0	0	-6	-6	-5	-6	-5	-5	-6	-6	-5
-7	-7	-7	-6	-6	-5	-5	0	0	-4	-4	-4	-4	-6	-4	-4
-7	-7	-8	-6	-7	-5	-6	0	0	-4	-4	-4	-4	-4	-4	-4
-7	-7	-8	-6	-7	-5	-5	-4	-4	0	-1	-3	-3	-3	-4	-4
-7	-8	-7	-7	-6	-5	-5	-4	-5	-1	0	-4	-4	-4	-3	-4
-7	-8	-7	-6	-6	-5	-6	-4	-4	-3	-4	0	0	-2	-2	-2
-7	-8	-8	-7	-7	-5	-6	-4	-6	-3	-4	0	0	-2	-2	-2
-8	-7	-7	-6	-7	-5	-5	-4	-4	-3	-3	-2	-2	0	-1	-2
-7	-8	-8	-6	-7	-6	-6	-4	-4	-5	-3	-2	-3	-1	0	-1
-7	-7	-7	-6	-6	-5	-5	-4	-5	-4	-4	-3	-2	-1	-1	0

Fig. 5.6. MAGNIFIER MATRIX WITHOUT REGULARIZATION

0	-7	-6	-5	-5	-4	-4	-2	-3	-2	-3	-1	-1	0	0	2
-8	0	1	-6	-6	-4	-4	-4	-4	-2	-2	-2	-1	0	0	1
-8	1	0	-6	-5	-4	-4	-3	-3	-2	-2	-2	-1	-1	0	1
-9	-7	-7	1	0	-5	-4	-3	-4	-2	-3	-1	-1	0	0	1
-9	-8	-7	0	1	-4	-4	-3	-3	-2	-2	-2	-1	-1	-1	1
-9	-7	-7	-6	-6	2	2	-2	-3	-1	-1	0	-1	1	-1	1
-9	-7	-7	-6	-6	2	2	-3	-3	-2	-2	0	-1	0	0	2
-10	-8	-8	-6	-6	-4	-4	2	2	-1	-1	0	0	0	2	2
-9	-8	-9	-6	-6	-4	-5	2	2	-1	-1	0	0	1	1	2
-10	-8	-9	-7	-7	-4	-4	-2	-2	3	2	1	1	2	1	2
-10	-9	-8	-7	-6	-4	-4	-2	-3	2	3	0	0	1	2	2
-10	-10	-8	-7	-7	-4	-5	-3	-2	-1	-1	3	3	3	2	3
-10	-10	-9	-7	-7	-5	-5	-3	-4	0	-1	3	3	2	3	3
-11	-9	-9	-7	-7	-5	-5	-3	-3	-1	-1	1	1	4	4	3
-11	-10	-10	-7	-8	-6	-6	-2	-3	-3	-1	2	1	4	4	4
-11	-9	-10	-7	-7	-5	-5	-3	-3	-1	-2	0	1	3	3	5

(6.1) $$(\Gamma^t\Gamma + \mu I)\,\bar{c} = \Gamma^t(g + \Delta g)\ .$$

We write the solution of (6.1) $c(\mu,\Delta g)$. To examine the effectiveness of the method of regularization, we have the next result from Theorem 3.1 in Kitagawa[11]. We use the notations of $f(\mu,\Delta g) = \Lambda\, c(\mu,\Delta g)$ and $\Delta f(\mu,\Delta g) \equiv f(\mu,\Delta g) - f(0,0)$ in the theorem.

Theorem 6.1

(6.2) $$\|\Delta f(\mu,\Delta g)\| \leq \|\Xi_\zeta * \Theta\|_F\, \|g\| + \|\Xi_\rho * \Theta\|_F\, \|\Delta g\|$$

where

(6.3a) $$\Xi_\zeta = (\xi^\zeta_{ij})\ ,\quad \xi^\zeta_{ij} = \hat{\sigma}_i\, \mu / (\sigma_j(\sigma_j^2 + \mu))$$

(6.3b) $$\Xi_\rho = (\xi^\rho_{ij})\ ,\quad \xi^\rho_{ij} = \hat{\sigma}_i\, \sigma_j / (\sigma_j^2 + \mu)$$

and the rest of the symbols are the same as Theorem 5.1.

Based on the Theorems 5.1 and 6.1, we can examine the effectiveness of the method of regularization very clearly. Letting

(6.4) $$\zeta(\mu) = f(\mu,0) - f(0,0)$$

and

(6.5) $$\rho(\mu,\Delta g) = f(\mu,\Delta g) - f(\mu,0),$$

we have

(6.6) $$\|\Delta f(\mu,\Delta g)\| \leq \|\rho(\mu,\Delta g)\| + \|\zeta(\mu)\|.$$

$\rho(\mu,\Delta g)$ defined by (6.5) represents the error due to Δg to the solution f with regularization. If we compare the error due to Δg with that of f without regularization, we can recognize when the regularization is effective.

More specifically, we have

(6.7) $$\|\rho(\mu,\Delta g)\| \leq \|\Xi_\rho * \Theta\|_F \|\Delta g\|$$

and ,from Theorem 5.1,

(6.8) $$\|\Delta f\| \leq \|\Xi * \Theta\|_F \|\Delta g\|.$$

Checking corresponding elements of Ξ_ρ ,Ξ and Θ, we can examine the effectiveness of the regularization. We also have

(6.9) $$\|\varsigma(\mu)\| \leq \|\Xi_\varsigma * \Theta\|_F \|g\|.$$

This explains that we should avoid using the method of regularization when it is not effective and we should choose the regularization parameter μ carefully.

We actually construct the matrices Ξ_ς ,Ξ_ρ and Θ and we examine how the method of regularization stabilizes the numerical process and how we choose the regularization parameter.

6.2. Matrices Ξ_ρ and Ξ_ς for Harmonic Continuation. We first note that since the elements θ_{ij} of the matrix Θ is independent of the regularization parameter μ, the distortion coefficients matrix Θ is common with that without regularization given by Fig. 5.5. We set the parameters $R = 4$, $\rho = 0.5$, $\gamma = 2$ and $n = 16$ which are identical with those of Section 5.2. The regularization parameter is set as $\mu = 10^{-9}$ for the first set of figures from Fig. 6.1 to 6.3 and Table 6.1. We also note that the matrices Ξ_ς and Ξ_ρ as well as Ξ and Θ do not depend on g or Δg at all and, accordingly, we do not have to construct these matrices for different functions of g.

First we examine the elements ξ^{ρ}_{ij} of matrix Ξ_{ρ} due to perturbation Δg to study how the method of regularization stabilizes the numerical process 1) and 2) of Section 5.1. The elements are again given by the rounding off the fractions of logarithm with basis 10. Fig. 6.1 represents the explosive factor matrix Ξ_{ρ} with regularization. The elements ξ^{ρ}_{ij} of critical part of lower right corner ($i \approx n$ and $j \approx n$) are less or equal to 0 and significantly smaller than those of Ξ without regularization of Fig. 5.4. This can be understood very easily if we compare the elements ξ^{ρ}_{ij} and ξ_{ij} of Ξ and Ξ_{ρ}. As we have seen in Section 5.2 the elements ξ_{ij} grow large for large i and j mainly because the denominator σ_j approaches to zero as $j \to n$.

On the contrast, the denominator $(\sigma^2_j + \mu)$ of the elements ξ^{ρ}_{ij} do not approach to zero even if $j \to n$ and σ_j approaches to zero as far as the regularization parameter $\mu > 0$. Since the numerator of the elements ξ^{ρ}_{ij} are irrespective to μ, the elements ξ^{ρ}_{ij} for large j's do not grow large as in the case of ξ_{ij} of Ξ without regularization. Accordingly, as is shown in Fig. 6.2, the corresponding elements $\xi^{\rho}_{ij} * \theta_{ij}$ in lower right corner of matrix $\Xi_{\rho} * \Theta$ are much smaller than those of the matrix $\Xi * \Theta$ (of Fig. 5.6). The harmful elements larger than 2 have disappeared in Fig.6.2.

Moreover the Frobenius norm of $\Xi_{\rho} * \Theta$ reduces to 2.38×10^2 from 1.54×10^5 of $\|\Xi_{\rho} * \Theta\|_F$. This explains that the method of regularization

Table 5.1. SINGULAR VALUES OF GAMMA	Table 5.2. SINGULAR VALUES OF LAMBDA
0.177E+01	0.177E+01
0.796E-01	0.318E+00
0.796E-01	0.318E+00
0.497E-02	0.796E-01
0.497E-02	0.796E-01
0.414E-03	0.265E-01
0.414E-03	0.265E-01
0.389E-04	0.996E-02
0.389E-04	0.996E-02
0.389E-05	0.401E-02
0.388E-05	0.401E-02
0.408E-06	0.172E-02
0.405E-06	0.172E-02
0.389E-07	0.849E-03
0.475E-07	0.849E-03
0.413E-08	0.622E-03

Fig. 6.1. EXPLOSIVE RHO MATRIX DUE TO DELTA G

0	1	1	3	3	4	4	4	4	4	4	3	3	2	2	1
-1	1	1	2	2	3	3	4	4	3	3	2	2	1	1	0
-1	1	1	2	2	3	3	4	4	3	3	2	2	1	1	0
-1	0	0	1	1	2	2	3	3	2	2	2	2	0	1	0
-1	0	0	1	1	2	2	3	3	2	2	2	2	0	1	0
-2	0	0	1	1	2	2	3	3	2	2	1	1	0	0	-1
-2	0	0	1	1	2	2	3	3	2	2	1	1	0	0	-1
-2	-1	-1	0	0	1	1	2	2	2	2	1	1	0	0	-1
-2	-1	-1	0	0	1	1	2	2	2	2	1	1	0	0	-1
-3	-1	-1	0	0	1	1	2	2	1	1	0	0	-1	-1	-2
-3	-1	-1	0	0	1	1	2	2	1	1	0	0	-1	-1	-2
-3	-2	-2	0	0	1	1	1	1	1	1	0	0	-1	-1	-2
-3	-2	-2	0	0	1	1	1	1	1	1	0	0	-1	-1	-2
-3	-2	-2	-1	-1	0	0	1	1	1	1	0	0	-1	-1	-2
-3	-2	-2	-1	-1	0	0	1	1	1	1	0	0	-1	-1	-2
-3	-2	-2	-1	-1	0	0	1	1	0	0	-1	-1	-2	-2	-3

Fig. 6.2. RHO-MAGNIFIER MATRIX WITH REGULARIZATION

0	-7	-6	-5	-5	-4	-4	-3	-3	-4	-4	-4	-5	-6	-5	-6
-8	0	1	-6	-6	-4	-4	-4	-4	-3	-4	-6	-5	-6	-6	-6
-8	1	0	-6	-5	-4	-4	-3	-3	-3	-3	-6	-5	-6	-5	-7
-9	-7	-7	1	0	-5	-4	-3	-4	-4	-5	-5	-5	-5	-5	-6
-9	-8	-7	0	1	-4	-4	-3	-3	-4	-3	-5	-5	-6	-6	-7
-9	-7	-7	-6	-6	2	2	-2	-3	-3	-3	-4	-5	-5	-7	-7
-9	-7	-7	-6	-6	2	2	-3	-3	-3	-4	-4	-4	-6	-6	-6
-10	-8	-8	-6	-6	-4	-4	2	2	-3	-3	-3	-4	-6	-4	-5
-9	-8	-9	-6	-6	-4	-5	2	2	-2	-3	-4	-4	-5	-4	-5
-10	-8	-9	-7	-7	-4	-4	-2	-2	1	0	-3	-3	-4	-4	-5
-10	-9	-8	-7	-6	-4	-4	-2	-3	0	1	-3	-4	-4	-4	-6
-10	-10	-8	-7	-7	-4	-5	-3	-2	-2	-3	0	0	-3	-3	-5
-10	-10	-9	-7	-7	-5	-5	-3	-4	-2	-3	0	0	-3	-3	-5
-11	-9	-9	-7	-7	-5	-5	-3	-3	-3	-3	-2	-3	-1	-2	-5
-11	-10	-10	-7	-8	-6	-6	-2	-3	-4	-3	-2	-3	-2	-1	-3
-11	-9	-10	-7	-7	-5	-5	-3	-4	-3	-4	-3	-3	-3	-3	-3

RHO-NORM WITH REGULA = 238.0664

significantly reduces the magnification of the propagation of the perturbation Δg to the final approximation f in the case of harmonic continuation. This situation is totally different from the case of Dirichlet problem in which the method of regularization is not effective.

Another factor of error $\varsigma(\mu)$ which is defined by (6.4), however, shall be inevitably introduced when we employ the method of regularization. Though the upper bound of the error $\varsigma(\mu)$ is given in (6.9), its interpretation is somewhat more delicate than the case of Ξ_ρ . The element $\xi^{\varsigma}_{ij} \times \theta_{ij}$ of the matrix $\Xi_\varsigma * \Theta$ involved in (6.9) represents the magnification of the propagation of (g,u_i) to $(\Delta f,\hat{u}_j)$ due to introduction of the regularization parameter μ. The size of $(\Delta g,u_i)$ may not differ much among different i's, but the fourier coefficients (g,u_i) of g may be quite different in size. This is because the function g is harmonic and very smooth, which may results in very rapid convergence of the coefficients (g,u_i) to zero.

For our example for harmonic function $h(x) \equiv h(s,t) = s^2 - t^2 + 2s - 2t + 1$, the Fourier coefficients (g,u_i) are given in Fig.6.4 which shows that the coefficients from (g,u_1) to (g,u_5) are significant and the rest of them are numerical zero. This means that from the first to the fifth columns (left part indicated by rectangle in Fig. 6.3) of the matrix $\Xi_\varsigma *$ Θ are of great significance. Fig. 6.3 represents the matrix $\Xi_\varsigma * \Theta$, which we temporarily call zeta-magnifier matrix, for the regularization parameter $\mu = 10^{-9}$.

Fig. 6.3. ZETA-MAGNIFIER MATRIX WITH REGULARIZAITON

-9	-14	-13	-9	-9	-7	-6	-3	-3	-2	-3	-1	-1	0	0	2
-17	-7	-6	-10	-10	-6	-6	-4	-5	-2	-2	-2	-1	0	0	1
-17	-6	-7	-10	-10	-6	-6	-4	-4	-2	-2	-2	-1	-1	0	1
-18	-14	-14	-3	-4	-7	-6	-3	-4	-2	-3	-1	-1	0	0	1
-19	-15	-14	-4	-3	-6	-7	-3	-3	-2	-2	-2	-1	-1	-1	1
-18	-14	-14	-10	-10	-1	-1	-3	-4	-1	-1	0	-1	1	-1	1
-18	-13	-14	-10	-10	-1	-1	-3	-3	-2	-2	0	-1	0	0	2
-19	-15	-15	-10	-10	-6	-6	2	2	-1	-1	0	0	0	2	2
-19	-15	-16	-10	-11	-6	-7	2	2	-1	-1	0	0	1	1	2
-19	-15	-16	-11	-11	-7	-6	-2	-3	3	2	1	1	2	1	2
-19	-16	-15	-11	-10	-6	-7	-2	-3	2	3	0	0	1	2	2
-20	-16	-15	-11	-11	-7	-7	-3	-3	-1	-1	3	3	3	2	3
-19	-17	-16	-12	-12	-7	-8	-3	-4	0	-1	3	3	2	3	3
-20	-16	-16	-11	-12	-7	-7	-3	-3	-1	-1	1	1	4	4	3
-20	-17	-17	-11	-13	-8	-8	-3	-3	-3	-1	2	1	4	4	4
-20	-16	-16	-11	-11	-7	-7	-3	-4	-1	-2	0	1	3	3	5

Fig. 6.4. FOURIER COEFFICIENTS OF G

-0.400E+01
-0.302E+01
0.262E+01
-0.683E+00
0.185E+00
0.793E-06
0.131E-06
-0.277E-06
-0.119E-05
-0.103E-05
-0.105E-05
0.231E-06
0.838E-08
0.190E-06
0.319E-06
0.611E-06

Table 6.1. REGULARIZED SOLUTION vs. NONREGULARIZED SOLUTION

REGULARIZATION PARAMETER = 1.E-9

TRUE FUNC.	W/O REGULA.	WITH REGULA.	ERR. W/O REG.	ERR. WITH REG.
0.90000E+01	0.90256E+01	0.90000E+01	-0.25608E-01	-0.29564E-04
0.59932E+01	0.59987E+01	0.59933E+01	-0.54388E-02	-0.68188E-04
0.10000E+01	0.96555E+00	0.99997E+00	0.34452E-01	0.32425E-04
-0.39932E+01	-0.39365E+01	-0.39933E+01	-0.56679E-01	0.73671E-04
-0.70000E+01	-0.70700E+01	-0.70000E+01	0.69967E-01	-0.19073E-04
-0.70547E+01	-0.69828E+01	-0.70546E+01	-0.71892E-01	-0.61989E-04
-0.46569E+01	-0.47202E+01	-0.46569E+01	0.63382E-01	0.31471E-04
-0.13978E+01	-0.13526E+01	-0.13979E+01	-0.45197E-01	0.10872E-03
0.10000E+01	0.98016E+00	0.99995E+00	0.19843E-01	0.54777E-04
0.16636E+01	0.16551E+01	0.16637E+01	0.85666E-02	-0.21219E-04
0.10000E+01	0.10364E+01	0.10000E+01	-0.36360E-01	0.29206E-05
0.33636E+00	0.27723E+00	0.33633E+00	0.59132E-01	0.24438E-04
0.10000E+01	0.10729E+01	0.10001E+01	-0.72886E-01	-0.58770E-04
0.33978E+01	0.33214E+01	0.33979E+01	0.76459E-01	-0.98944E-04
0.66569E+01	0.67259E+01	0.66569E+01	-0.69032E-01	0.47684E-05
0.90547E+01	0.90032E+01	0.90546E+01	0.51465E-01	0.61989E-04

If there exist some large elements, say on a level with 0 or larger, within the columns 1-5 of the zeta-magnifier matrix $\Xi_{\varsigma} * \Theta$, it is fatal for our example and the approximation of harmonic continuation shall be instantly demolished, since (g, u_i) for $i=1, \ldots, 5$ are much larger than $(\Delta g, u_i)$ and error as large as $\|g\|$ shall be introduced. This is the case when we choose the regularization parameter too large. For the zeta magnifier matrix of Fig. 6.3 , since the maximal element in the first 5 columns is -3, we can expect the method of regularization works out well.

The Table 6.1 presents actual numerical results on accuracy attained without regularization (denoted by " w/o reg." in the table) and with regularization (denoted by " with reg." in the table) for $\mu = 10^{-9}$. "True func." in Table 6.1 gives the exact value of the harmonic continuation on the circle $C(\gamma)$ and "err." stands for error. Numerical result without regularization attains accuracy of only 1 or 2 decimal digits, while that of with regularization attains 4 or 5, which shows the examinations of matrices above are reliable and the method of regulazization is effective.

The last set of Table 6.2 and Fig. 6.5 shows the case when we choose the regularization parameter too large and the parameter $\mu = 10^{-2}$. In this case, Frobenius norm of the matrix $\Xi_{\rho} * \Theta$ due to perturbation Δg is as small as 2.4×10^{0} and there shall be virtually no magnification of propagation of Δg to f. Table 6.2, however, apparently shows that the regularized solution is completely destroyed. Fig.6.5 of zeta-magnifier matrix $\Xi_{\varsigma} * \Theta$ explains the reason clearly. The first 5 columns of $\Xi_{\varsigma} * \Theta$ of Fig. 6.5 include the elements of 0 and 1, which is fatal as is mentioned above and the regularized solution immediately breaks down.

Table 6.2. REGULARIZED SOLUTION vs. NONREGULARIZED SOLUTION

REG.PARAMETER = 1.E-2

TRUE FUNC.	W/O REGULA.	WITH REGULA.	ERR. W/O REG.	ERR. WITH REG.
0.90000E+01	0.90256E+01	0.25576E+01	-0.25608E-01	0.64424E+01
0.59932E+01	0.59987E+01	0.18431E+01	-0.54388E-02	0.41501E+01
0.10000E+01	0.96555E+00	0.99680E+00	0.34452E-01	0.31997E-02
-0.39932E+01	-0.39365E+01	0.15047E+00	-0.56679E-01	-0.41437E+01
-0.70000E+01	-0.70700E+01	-0.56399E+00	0.69967E-01	-0.64360E+01
-0.70547E+01	-0.69828E+01	-0.10365E+01	-0.71892E-01	-0.60181E+01
-0.46569E+01	-0.47202E+01	-0.11965E+01	0.63382E-01	-0.34603E+01
-0.13978E+01	-0.13526E+01	-0.10226E+01	-0.45197E-01	-0.37524E+00
0.10000E+01	0.98016E+00	-0.54424E+00	0.19843E-01	0.15442E+01
0.16636E+01	0.16551E+01	0.16443E+00	0.85666E-02	0.14992E+01
0.10000E+01	0.10364E+01	0.99680E+00	-0.36360E-01	0.31998E-02
0.33636E+00	0.27723E+00	0.18292E+01	0.59132E-01	-0.14928E+01
0.10000E+01	0.10729E+01	0.25378E+01	-0.72886E-01	-0.15378E+01
0.33978E+01	0.33214E+01	0.30162E+01	0.76459E-01	0.38164E+00
0.66569E+01	0.67259E+01	0.31901E+01	-0.69032E-01	0.34667E+01
0.90547E+01	0.90032E+01	0.30301E+01	0.51465E-01	0.60245E+01

Fig. 6.5. ZETA-MAGNIFIER MATRIX WITH REGULARIZAITON

-2	-8	-6	-5	-5	-4	-4	-2	-3	-2	-3	-1	-1	0	0	2
-10	0	0	-6	-6	-4	-4	-4	-4	-2	-2	-2	-1	0	0	1
-10	0	0	-6	-5	-4	-4	-3	-3	-2	-2	-2	-1	-1	0	1
-11	-7	-7	1	0	-5	-4	-3	-4	-2	-3	-1	-1	0	0	1
-12	-8	-7	0	1	-4	-4	-3	-3	-2	-2	-2	-1	-1	-1	1
-11	-7	-7	-6	-6	2	2	-2	-3	-1	-1	0	-1	1	-1	1
-11	-7	-7	-6	-6	2	2	-3	-3	-2	-2	0	-1	0	0	2
-12	-8	-8	-6	-6	-4	-4	2	2	-1	-1	0	0	0	2	2
-12	-8	-9	-6	-6	-4	-5	2	2	-1	-1	0	0	1	1	2
-12	-8	-9	-7	-7	-4	-4	-2	-2	3	2	1	1	2	1	2
-12	-9	-8	-7	-6	-4	-4	-2	-3	2	3	0	0	1	2	2
-13	-10	-8	-7	-7	-4	-5	-3	-2	-1	-1	3	3	3	2	3
-12	-10	-9	-7	-7	-5	-5	-3	-4	0	-1	3	3	2	3	3
-13	-9	-9	-7	-7	-5	-5	-3	-3	-1	-1	1	1	4	4	3
-13	-10	-10	-7	-8	-6	-6	-2	-3	-3	-1	2	1	4	4	4
-13	-9	-10	-7	-7	-5	-5	-3	-3	-1	-2	0	1	3	3	5

This implies that the matrices Ξ_ς ,Ξ_ρ and Θ give us an idea on the choice of the regularization parameter. We should choose μ in such a way that

i) we reduce the size of element ξ^{ρ}_{ij} of Ξ_ρ whose corresponding elements of θ_{ij} of Θ are close to unity

ii) we avoid contaminating the elements ξ^{ς}_{ij} of Ξ_ς whose corresponding elements of θ_{ij} of Θ are close to unity and the corresponding j-th Fourier coefficients (g,u_j) are significant.

7. Concluding Remarks. We considered a method of numerical harmonic continuation using the fundamental solution method and the method of regularization. It is demonstrated in this paper that Theorems 5.1 and 6.1 can be useful not only to examine the numerical stability of a class of numerical procedures given by 1) and 2) of Section 5.1 but also, as a tool for those who design a numerical method, to study whether the method of regularization is effective or not and how to choose the regularization parameter when one applies it.

REFERENCES

[1] P.J. Davis, Circulant Matrices, John Wiley & Sons, 1979.

[2] G.F.Fairweather and R.L.Johnston, The method of fundamental solutions for problems in potential theory, in Treatment of Integral Equations by Numerical Methods.(eds. C.T.H.Baker and G.F.Miller), Academic Press,1982.

[3] J.N.Franklin, Well-posed stochastic extensions of ill-posed linear problems, J. Math. Anal. Appl., 31(1970), 682-716

[4] G.H.Golub, Singular value decomposition and least squares solutions, Numer. Math. 14(1970), 206-216

[5] G.H.Golub and C.F.Van Loan, Matrix Computations. Johns Hopkins University Press, 1983.

[6] C.W.Groetsch, The Theory of Tikhonov Regularization for Fredholm Equations of the First Kind, Pitman 1984.

[7] P. Henrici, Applied and Computational Complex Analysis, Vol.1, John Wiley & Sons, New York, 1974.

[8] M. Katsurada, An Analysis on Charge Simulation Method.(In Japanese) Master thesis, Department of Mathematics, Tokyo University, 1986.

[9] T. Kitagawa, A deterministic approach to the optimal regularization, to appear in Japan J. Appl. Math., vol.4, no.4(1987)

[10] T. Kitagawa, On the numerical stability of the method of fundamental solution applied to Dirichlet problem, to appear in Japan J. Appl. Math.,vol.5,no.1(1988)

[11] T. Kitagawa, On the effectiveness of the method of regularization for a certain class of numerical procedures, to appear in Japan J. Appl. Math.(1988)

[12] T.Kitagawa, Numerical harmonic continuation by Fundamental solution method and the method of regularization, submitted to publication

[13] M.M.Lavrent'ev et al., Ill-Posed Problems of Mathematical Physics and Analysis, American Mathematical Society, 1986.

[14] S.Murashima, Charge Simulation Method and its application.(In Japanese) Morikita Shuppan, Tokyo, 1983.

[15] A.N.Tikhonov and V.W. Arsenin, Solutions of Ill-Posed Problems. Winston-Wiley, New York, 1977.

[16] J.M.Varah, On the numerical solution of ill-conditioned linear systems with applications to ill-posed problems, SIAM J. Numer. Anal.(1973),257-267