On a question of expansion of closure-preserving families (General Topology, Dimension and Set Theory)

Author(s)
Mizokami, Takemi

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Kyoto University
On a question of expansion of closure-preserving families

溝上武男 (Takemi Mizokami)

( Joetsu University of Education )

Through this paper, all spaces are assumed to be $T_1$ topological spaces.

We introduce the $d$-IP-expandability and IP-expandability as follows:

Definition 1. We call a space $X$ $d$-IP-expandable if for a discrete family $\mathcal{F} = \{ F_\lambda : \lambda \in \Lambda \}$ of closed subsets of $X$ and a family $\mathcal{U} = \{ U_\lambda : \lambda \in \Lambda \}$ of open subsets of $X$ such that $F_\lambda \subset U_\lambda$ for each $\lambda$, there exists an interior-preserving family $\mathcal{V} = \{ V_\lambda : \lambda \in \Lambda \}$ of open subsets of $X$ such that $F_\lambda \subset V_\lambda \subset U_\lambda$ for each $\lambda$.

Definition 2. We call a space $X$ IP-expandable if for a closure-preserving family $\mathcal{F} = \{ F_\lambda : \lambda \in \Lambda \}$ of closed subsets of $X$ and a family $\mathcal{U} = \{ U_\lambda : \lambda \in \Lambda \}$ of open subsets of $X$ such that $F_\lambda \subset U_\lambda$ for each $\lambda$, there exists a family $\mathcal{V} = \{ V_\lambda : \lambda \in \Lambda \}$ of open subsets of $X$ such that $F_\lambda \subset V_\lambda \subset U_\lambda$.
for each $\lambda$ and $\{ V_\lambda - F_\lambda : \lambda \in \Lambda \}$ is interior-preserving in $X$.

These spaces have the following properties:

Proposition 3. If a space $X$ is collectionwise normal, then $X$ is d-IP-expandable.

Proposition 4. If a space $X$ is orthocompact, then $X$ is d-IP-expandable.

Theorem 5. If a space $X$ is submetacompact and d-IP-expandable, then $X$ is orthocompact.

Corollary 6. Let $X$ be a developable space. Then $X$ is d-IP-expandable if and only if $X$ is non-archimedean quasi-metrizable.

Theorem 7. For a space $X$, the following are equivalent:

(1) $X$ is an orthocompact developable space.

(2) $X$ has a development $\{ \bigcup_n : n \in N \}$ such that each $\bigcup_n$ is interior-preserving in $X$.

(3) $X$ is a d-IP-expandable developable space.

(4) $X$ is a semi-stratifiable, non-archimedean quasi-metrizable space.
Corollary 8. If for each \( n \in \mathbb{N} \), \( X_n \) is an orthocompact developable space, then so is \( \prod_{n=1}^{\infty} X_n \).

Theorem 9. If a space \( X \) is non-archimedean quasi-metrizable, then \( X \) has the property (P):

(P) For a closed \( G_\delta \)-set \( F \) of \( X \), there exists a family \( \mathcal{U} \) of open subsets of \( X \) satisfying the following:

1. \( \mathcal{U} / (X - F) \) is interior-preserving in \( X - F \).
2. For each open set \( V \) of \( X \), there exists \( U \in \mathcal{U} \) such that
   \[ V \cap F = U \cap F \subseteq U \subseteq V. \]

Corollary 10. If a space \( X \) is perfect and non-archimedean quasi-metrizable, then \( X \) is d-IP-expandable.

Corollary 11. Under the same hypothesis as above, every closed subset \( F \) of \( X \) has an outer base \( \mathcal{U} \) in \( X \) such that \( \mathcal{U} \) is interior-preserving in \( X - F \).

Theorem 12. Let \( X \) be a developable space. Then \( X \) is IP-expandable if and only if \( X \) is d-IP-expandable.

Theorem 13. Let \( X \) be a stratifiable space. Then \( X \) is an L-space if and only if \( X \) is IP-expandable.
Theorem 14. If a space $X$ is semi-stratifiable, then d-IP-\(\star\) expandability implies D-expandability.

Finally, we propose the following question:

Question. If a space $X$ is developable and quasi-metrizable, then is $X$ d-IP-expandable?

This is equivalent to the well-known problem due to Junnila whether every developable quasi-metrizable space is non-archimedean quasi-metrizable. For the further details, see [1].