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On a question of expansion of closure-preserving families

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Through this paper, all spaces are assumed to be T_1 topological spaces.

We introduce the d-IP-expandability and IP-expandability as follows:

Definition 1. We call a space X <u>d-IP-expandable</u> if for a discrete family Ⓕ $= \{ F_\lambda : \lambda \in \Lambda \}$ of closed subsets of X and a family Ⓤ $= \{ U_\lambda : \lambda \in \Lambda \}$ of open subsets of X such that $F_\lambda \subset U_\lambda$ for each λ, there exists an interior-preserving family Ⓥ $= \{ V_\lambda : \lambda \in \Lambda \}$ of open subsets of X such that $F_\lambda \subset V_\lambda \subset U_\lambda$ for each λ.

Definition 2. We call a space X <u>IP-expandable</u> if for a closure-preserving family Ⓕ $= \{ F_\lambda : \lambda \in \Lambda \}$ of closed subsets of X and a family Ⓤ $= \{ U_\lambda : \lambda \in \Lambda \}$ of open subsets of X such that $F_\lambda \subset U_\lambda$ for each λ, there exists a family Ⓥ $= \{ V_\lambda : \lambda \in \Lambda \}$ of open subsets of X such that $F_\lambda \subset V_\lambda \subset U_\lambda$

for each λ and $\{ V_\lambda - F_\lambda : \lambda \varepsilon \Lambda \}$ is interior-preserving in X.

These spaces have the following properties:

Proposition 3. If a space X is collectionwise normal, then X is d-IP-expandable.

Proposition 4. If a space X is orthocompact, then X is d-IP-expandable.

Theorem 5. If a space X is submetacompact and d-IP--expandable, then X is orthocompact.

Corollary 6. Let X be a developable space. Then X is d-IP-expandable if and only if X is non-archimedean quasi--metrizable.

Theorem 7. For a space X, the following are equivalent:

(1) X is an orthocompact developable space.

(2) X has a development $\{ \mathcal{U}_n : n \varepsilon N \}$ such that each $\mathcal{U}_n$ is interior-preserving in X.

(3) X is a d-IP-expandable developable space.

(4) X is a semi-stratifiable, non-archimedean quasi--metrizable space.

Corollary 8. If for each $n \in N$, X_n is an orthocompact developable space, then so is $\Pi_{n=1}^{\infty} X_n$.

Theorem 9. If a space X is non-archimedean quasi-metrizable, then X has the property (P):

(P) For a closed G_δ-set F of X, there exists a family $\mathcal{U}$ of open subsets of X satisfying the following:

(1) $\mathcal{U} / (X - F)$ is interior-preserving in $X - F$.

(2) For each open set V of X, there exists $U \in \mathcal{U}$ such that

$$V \cap F = U \cap F \subset U \subset V.$$

Corollary 10. If a space X is perfect and non-archimedean quasi-metrizable, then X is d-IP-expandable.

Corollary 11. Under the same hypothesis as above, every closed subset F of X has an outer base $\mathcal{U}$ in X such that $\mathcal{U}$ is interior-preserving in $X - F$.

Theorem 12. Let X be a developable space. Then X is IP-expandable if and only if X is d-IP-expandable.

Theorem 13. Let X be a stratifiable space. Then X is an L-space if and only if X is IP-expandable.

Theorem 14. If a space X is semi-stratifiable, then d-IP--expandability implies D-expandability.

Finally, we propose the following question:

Question. If a space X is developable and quasi--metrizable, then is X d-IP-expandable ?

This is equivalent to the well-known problem due to Junnila whether every developable quasi-metrizable space is non--archimedean quasi-metrizable. For the further details, see [1].

[1] T.Mizokami: Expansion of discrete and closure-preserving families, forthcoming in Proc. Amer. Math. Soc.