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Generalized foliations and algebraic representations of self-covering maps

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The strongest useful equivalence for study of the orbit structures of covering maps of a compact metric space M will be topological conjugacy. A continuous map $f : M \to M$ is called a *covering map* if for $x \in M$ there exists an open neighborhood U_x of x in M such that $f^{-1}(U_x)$ is the disjoint union of open subsets of M each of which is mapped homeomorphically onto U_x by f. We say two continuous maps $f : M \to M$ and $g : M \to M$ are *topologically conjugate* if there exists a homeomorphism $h : M \to M$ such that $f \circ h = h \circ g$. In this case any orbit of g is mapped by h to an isomorphic orbit of f.

Interesting examples in our study are among matrices of the following types

$$(\mathrm{I})\ \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad (\mathrm{II})\ \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \text{ and } (\mathrm{III})\ \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}.$$

Type (I) has eigenvalues of modulus < 1 and modulus > 1 (i.e. hyperbolic), eigenvalues of type (II) are larger than one (i.e. expanding) and those of type (III) are in the same condition as type (I) (i.e. hyperbolic). They can be though of as linear maps of the plane $\mathbb{R}^2$ which preserves the lattice $\mathbb{Z}^2$ of points with integer coordinates. The type (I) induces automorphisms of the factor group $\mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$, the 2-torus and the types (II) and (III) induce endomorphisms of $\mathbb{T}^2$. The automorphisms of the type (I) and the endomorphisms of the type (II) satisfy the much stronger property of being structurally stable. However the endomorphism of the type (III) does not have such the property (see a result of Przytycki [16] stated below). A differentiable map f is said to be *structurally stable* if there is a open neighborhood $N(f)$ of f in the C^1 topology on the set of all differentiable maps of a closed C^∞ manifold M such that $g \in N(f)$ implies f and g are topologically conjugate.

Let M be a closed C^∞ manifold. A diffeomorphism $f : M \to M$ is called an *Anosov diffeomorphism* if there is a continuous splitting of the tangent bundle $TM = E^s \oplus E^u$ which is preserved by the derivative Df and if there are constants $c > 0$, $0 < \mu < 1$ and a Riemannian metric $\| \ \|$ on TM such that for $n \geq 0$ we have

$$(1) \quad \|Df^n(v)\| \leq c\mu^n\|v\| \qquad (v \in E^s),$$

$$(2) \quad \|Df^n(v)\| \geq c^{-1}\mu^{-n}\|v\| \qquad (v \in E^u).$$

For the main properties and topological conjugacy of Anosov diffeomorphisms, see Smale [14], Franks [7, 8] and Manning [11].

A map $f \in C^1(M, M)$ is called *expanding* if there are constants $c > 0$, $0 < \mu < 1$ and a Riemannian metric on TM such that for $n \geq 0$

$$\|Df^n(v)\| \geq c\mu^{-n}\|v\|.$$

See [8], Gromov [4] and Shub [13] for the properties of expanding maps. The above notions were extended by Przytycki [16] as follows.

A map $f \in C^1(M, M)$ is called an *Anosov map* if f is a regular covering map and if there exist constants $c > 0$, $0 < \mu < 1$ and a Riemannian metric on TM such that for a sequence of points in M satisfying $f(x_n) = x_{n+1}$ for every integer n, there is a splitting of $\bigcup_{n=-\infty}^{\infty} T_{x_n}M = E^s \oplus E^u = \bigcup_{n=-\infty}^{\infty} E^s_{x_n} \oplus E^u_{x_n}$

which is preserved by the derivative Df and conditions (1), (2) are satisfied.

It does not follows there is a splitting of the tangent bundle $TM = E^s \oplus E^u$. Note $E^u_{x_0}$ depends on the sequence (x_n). Thus it may happen $E^u_{x_0} \neq E^u_{y_0}$ if $x_0 = y_0$ but $(x_n) \neq (y_n)$. Such a phenomenon is impossible for $E^s_{x_0}$, that is, it depends only on x_0.

A map $f \in C^1(M, M)$ is called a *special Anosov map* if f is a Anosov map and E^u_x does not depend on the sequence (x_n) containing x. Thus special Anosov maps satisfy the conditions (1) and (2) as well as diffeomorphisms. A typical example of such the map is the type (Ⅲ). The class of special Anosov maps contains Anosov diffeomorphisms and expanding differentiable maps.

The type (Ⅰ) is an example of Anosov diffeomorphisms and the type (Ⅱ) is that of expanding differentiable maps.

Przytycki proved in [16] that Anosov maps do not be structurally stable ; that is, every non-empty open in the C^1-topology subset of the set of all Anosov maps of class C^1, which are not diffeomorphisms nor expandings, contains an uncountable subset such that, if $f \neq g$ are any elements of its, then there exists no continuous surjection $\varphi : M \to M$ which makes the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & M \\ \varphi \downarrow & & \downarrow \varphi \\ M & \xrightarrow[\ g\ ]{} & M \end{array} \qquad \text{commutes.}$$

If the manifold M is the torus then we know every expanding differentiable map is topologically conjugate to some expanding toral endomorphism (Shub [13]), and every Anosov diffeomorphism is topologically conjugate to some hyperbolic toral automorphism (Manning [11]). It is unknown, however, whether every special Anosov maps of the torus are topologically conjugate to a toral homomorphism.

Our investigation will be within the context of the conjugacy problem for Anosov maps, which can be stated in its more general setting, topological setting.

Let X be a metric space with metric d. A homeomorphism $f : X \to X$ is called *expansive* if there is a constant $e > 0$ such that $x, y \in X$ $(x \neq y)$ implies $d(f^n(x), f^n(y)) > e$ for some integer n. Such an constant e is called an *expansive constant* of f. A continuous map $f : X \to X$ is *positively expansive* if there is a constant $e > 0$ such that if $x \neq y$ then $d(f^n(x), f^n(y)) > e$ for some positive integer n (e is called an *expansive constant* of f). For compact spaces, these notions are independent of the compatible metric used, although not the expansive constants.

If X is compact and $f : X \to X$ is a positively expansive map, then there exist a compatible metric D and constants $\varepsilon > 0$, $\lambda > 1$ such that $D(x,y) \leq \varepsilon$ $(x,y \in X)$ implies $D(f(x),f(y)) \geq \lambda D(x,y)$ (Reddy [17]). Define a local stable set $W^s_\varepsilon(x,d)$ and a local unstable set $W^s_\varepsilon(x,d)$ by

$$W^s_\varepsilon(x,d) = \{y \in X : d(f^n(x), f^n(y)) \leq \varepsilon,\ n \geq 0\},$$

$$W^u_\varepsilon(x,d) = \{y \in X : d(f^{-n}(x), f^{-n}(y)) \leq \varepsilon,\ n \geq 0\}.$$

If $f : X \to X$ is an expansive homeomorphism, then there exist an equivalence metric D and constants $\varepsilon > 0$, $0 < \lambda < 1$ and $a \geq 1$ such that if $y \in W^s_\varepsilon(x,D)$ then $D(f^n(x), f^n(y)) \leq a\lambda^n D(x, y)$ and if $y \in W^u_\varepsilon(x, D)$ then $D(f^{-n}(x), f^{-n}(y)) \leq a\lambda^n D(x, y)$ for all n

≥ 0 (Reddy [17]).

If a homeomorphism of a compact metric space is positively expansive then the space is a space consisting of finite points. Since f is positively expansive, there is a compatible metric D, constants $\delta > 0$ and $0 < \lambda < 1$ such that $D(x,y) \leq \delta$ implies $D(f^{-1}(x), f^{-1}(y)) < \lambda D(x,y)$. Thus $\Phi^- = \{f^{-i} : i \geq 0\}$ is equicontinuous. Put $\Phi = \Phi^- \cup \Phi^+$ where $\Phi^+ = \{f^i : i \geq 0\}$. By a metric ρ defined $\rho(f,g) = \max\{D(f(x), g(x)) : x \in X\}$, Φ^- is totally bounded. Define a map $G : \Phi^- \to \Phi^+$ by $G(f^{-i}) = f^i$ for $i \geq 0$. Then G is ρ-isometric. Therefore Φ^+ is totally bounded. Since X is compact, Φ^+ is uniformly equicontinuous and so there is $\nu > 0$ such that $D(x,y) < \nu$ implies $D(f^i(x), f^i(y)) < e$ for all $i \geq 0$ (e is an expansive constant). This shows $x = y$ and therefore x is an isolated.

A continuous map f is called *expanding* if f is positively expansive and it is a covering map. Thus every positively expansive map of a closed topological manifold onto itself is expanding in our sense. Expanding (differentiable) map of a compact manifold onto itself is positively expansive.

A sequence of points $\{x_i : a < i < b\}$ of a metric space X is called a *δ-pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for $i \in (a, b-1)$. Given $\varepsilon > 0$ a δ-pseudo orbit $\{x_i\}$ is called to be *ε-traced* by a point $x \in X$ if $d(f^i(y), x_i) < \varepsilon$ for every $i \in (a, b)$. Hence the symbols a and b are taken as $-\infty \leq a < b \leq \infty$ if f is bijective and as $0 \leq a < b \leq \infty$ if f is not bijective. We call f to have the *pseudo-orbit tracing property* (abbrev. POTP) if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit of f can be ε-traced by some point of X. For compact spaces this notion is independent of a compatible metric used. It is well known every expanding (differentiable) map satisfies POTP and every Anosov diffeomorphism obeys expansivity and POTP.

Henceforth X will be a campact metric space. For $f : X \to X$ a continuou surjection, we let

$$(1.0)\quad \begin{cases} X_f = \{(x_i) : x_i \in X \text{ and } f(x_i) = x_{i+1},\ i \in \mathbb{Z}\}, \\ \sigma((x_i)) = (f(x_i)). \end{cases}$$

Then X_f is a subset of the product space $\mathbb{X} = \prod_{i \in \mathbb{Z}} X_i$ where each X_i is a replica of X and $\sigma : X_f \to X_f$ is called the *shift* determined by f. If $P_i : X_f \to X$ is a projection defined by $(x_i) \to x_i$ $(i \in \mathbb{Z})$, then $P_i \circ \sigma = f \circ P_i$ holds. We give a metric d for $\mathbb{X}$ by

$$(1.1)\qquad \tilde{d}((x_i), (y_i)) = \sum_{i=-\infty}^{\infty} \frac{1}{2^{|i|}} d(x_i, y_i) .$$

Then X_f is closed in $\mathbb{X}$. We say (X_f, σ) is the *inverse limit* of (X, f). If in particular X is a torus and $f : X \to X$ is a toral endomorphism (surjective but not one-to-one), then X_f is the topological group which is called the *solenoidal group*. The notion of solenoidal group will be later defined in other words.

A continuous surjection $f : X \to X$ is called c-*expansive* if there is a constant $e > 0$ (called an *expansive constant*) such that for $(x_i), (y_i) \in X_f$ if $d(x_i, y_i) < e$ for $i \in \mathbb{Z}$ then $(x_i) = (y_i)$. The notion of c-expansivity for continuous surjection is weaker than that of positive expansivity. For homeomorphisms the notion of c-expansivity implies that of expansivity. It is easily checked if $f : X \to X$ is c-expansive then so is f^k for $k > 0$.

A continuous surjection $f : X \to X$ is c-expansive if and only if $\sigma : X_f \to X_f$ is expansive. For, if $(x_i) \neq (y_i)$ then there is $k \in \mathbb{Z}$ such that $d(x_k, y_k) > e$ by c-expansivity.

Then $\tilde{d}(\sigma^k(x_i), \sigma^k(y_i)) \geq d(x_k, y_k) > e$. Therefore σ is expansive.

Conversely let $c > 0$ be an expansive constant of σ and let $d(x_i, y_i) < c/4$ for $(x_i), (y_i) \in X_f$. Then we have $\tilde{d}(\sigma^n(x_i), \sigma^n(y_i)) \leq c$ for $n \in \mathbb{Z}$, and hence $(x_i) = (y_i)$ by expansivity of σ. Therefore $c/4$ is an expansive constant of $f : X \to X$.

A continuous surjection $f : X \to X$ obeys POTP if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every δ-pseudo orbit $(x_i) \in X$ there exists a point $(y_i) \in X_f$ so that $d(y_i, x_i) < \varepsilon$ for $i \in \mathbb{Z}$. This is easily checked and so we omit the proof.

If $f : X \to X$ has POTP then $\sigma : X_f \to X_f$ obeys POTP. This is checked as follows. Let $\alpha = \mathrm{diam}(X)$ and $\varepsilon > 0$. Choose $N > 0$ with $\alpha/2^{N-2} < \varepsilon$. Let $\gamma > 0$ be a number such that

$$d(x, y) \leq \gamma \implies \max_{0 \leq i \leq 2N} d(f^i(x), f^i(y)) \leq \varepsilon/8,$$

and $\delta' > 0$ be a number with property in the definition of POTP of f. Choose δ such that $0 < 2^N\delta < \delta'$. Let $k > 0$ and $\{(x_i^n) : 0 \leq n \leq k\}$ be a δ-finite pseudo orbit of σ in X_f. Then we have

$$\delta > \tilde{d}(\sigma(x_i^n), (x_i^{n+1})) \geq d(f(x_{-N}^n), x_{-N}^{n+1})/2^N \qquad (0 \leq n \leq k-1)$$

and hence $\{x_{-N}^n : 0 \leq n \leq k\}$ is a δ'-pseudo orbit of f. Since f has POTP, there is $y \in X$ such that

$$d(f^n(y), x_{-N}^n) \leq \gamma \qquad (0 \leq n \leq k).$$

And so put $y_{i-N} = f^i(y)$ for $i \geq 0$ and take $y_{i-N} \in f^{-1}(y_{i+1})$ for $i < 0$. Then $(y_i) \in X_f$ and

$$\tilde{d}(\sigma^n(y_i), (x_i^n)) = \sum_{j=-\infty}^{\infty} d(f^n(y_i), x_i^n)/2^{|j|}$$

$$= \sum_{j=-N}^{N} d(f^n(y_j),\ x_j^n)/2^{|j|} + \sum_{j=-\infty}^{-N-1} d(f^n(y_j),\ x_j^n)/2^{|j|}$$

$$+ \sum_{j=N+1}^{\infty} d(f^n(y_j),\ x_j^n)/2^{|j|} \le \varepsilon/4 + \varepsilon/4 + \varepsilon/4 < \varepsilon$$

because $d(f^n(y_j),\ x_j^n) \le \varepsilon/8$ ($|j| \le N$) by the fact that $d(f^n(y_{-N}),\ x_{-N}^n) \le \gamma$.

If $\sigma : X_f \to X_f$ has POTP and $f : X \to X$ is a local homeomorphism, then f has POTP. For $\varepsilon > 0$ let $\delta > 0$ be a number with property in the definition of POTP of σ. Choose α such that $0 < \alpha/2^{N-2} < \delta$. Then we can find $\gamma > 0$ such that if $d(x, y) \le \gamma$ then there are $\{x_i : 0 \le i \le N\}$ and $\{y_i : 0 \le i \le N\}$ such that $x_0 = x$, $f(x_i) = x_{i+1}$ and $y_0 = y$, $f(y_i) = y_{i+1}$ and $d(x_i,\ y_i) < \delta/8$ for $|i| \le N$. Let $\{z_i : 0 \le i < \infty\}$ be a γ-pseudo orbit of f. Then for $i \ge 0$ we can find $(z_n^i) \in X_f$ such that $z_0^i = z_i$ and $\{(z_n^i)\}$ is a δ-pseudo orbit of σ. Since σ has POTP, there is $(x_n) \in X_f$ such that $\tilde{d}(\sigma^i(x_n),\ (z_n^i)) \le \varepsilon$ for $0 \le i < \infty$, and then $\varepsilon \ge \tilde{d}(\sigma^i(x_n),\ (z_n^i)) \ge d(f^i(x_0),\ z_0^i) = d(f^i(x_0),\ z_i)$.

We know (cf. [16]) every Anosov map of a closed manifold obeys c-expansivity and POTP. Thus we can give the notion belonging to Anosov property for continuous surjections, in more general setting. We say a continuous surjection $f : X \to X$ is a *c-map* if f is c-expansive and satisfies POTP and if f is a covering map. We say a continuous map $f : X \to X$ is a *special c-map* if f satisfies the following conditions;

(i) f is a c-map,

(ii) for every $(x_i),\ (y_i) \in X_f$ with $x_0 = y_0$

$$W^u((x_i)) = W^u((y_i))$$

where

(1.2) $W^u((x_i)) = \{y_0 \in X : \exists(y_i) \in X_f,\ d(y_i, x_i) \to 0 \text{ as } i \to -\infty\}$.

The notion of special c-maps is a generalization of that of special Anosov maps.

The class of special c-maps is a much larger class of continuous maps, in which contains homeomorphisms with expansivity and POTP, and positively expansive maps.

Let $\mathcal{CM}$ denote the set of all c-maps of X and $\mathcal{SCM}$ denote the set of all special c-maps of X. In general the set $\mathcal{C} = \mathcal{CM} - \mathcal{SCM}$ is non-empty and the union of the set $\mathcal{EHP}$ of all expansive homeomorphisms with POTP and the set $\mathcal{PEM}$ of all positively expansive maps is properly contained in $\mathcal{SCM}$.

The orbit structures of the types (I), (II) and (III) would be characterized, in a category of C^0-topology, as follows:

(a) Type (I) is expansive,

(b) Type (II) is positively expansive,

(c) Type (III) is c-expansive,

and (a), (b) and (c) derives POTP.

In general setting, every positively expansive covering map derives POTP. However we know the existance of examples of a homeomorphism whose expansivity does not derive POTP and a continuous map whose c-expansivity does not derive POTP.

Conversely we can easily checked

(d) if every toral automorphism is expansive then it is hyperbolic (i.e. type (I)),

(e) if every toral endomorphism is positively expansive then it is an endomorphism of type (II),

(f) if every toral endomorphism is c-expansive but not positively expansive then it is an endomorphism of type (III).

We now restrict ourself to the n-torus. Then the topological conjugacy problem raised by Franks [8] will be solved for c-maps as follows :

(i) every homeomorphism with both expansivity and POTP is topologically conjugate to a toral automorphism of type (I),

(ii) every positively expansive map is topologically conjugate to a toral endomorphism of type (II),

(iii) every special c-map, which is not bijective nor positively expansive, is topologically conjugate to a toral endomorphism of type (III),

(iv) for every c-map which is not special, its inverse limit is topologically conjugate to a solenoidal automorphism.

Although we do not settle here, to avoid complication, the case of c-maps of infra-nil-manifolds, it seems likely that the methods and the results presented here could be pushed to give that case.

More precisely we describe our results as follows.

Theorem 1. *Let* $\mathbb{T}^n$ *be the* n-*torus and* $g : \mathbb{T}^n \to \mathbb{T}^n$ *be a covering map. Then there exist a finite cover* $\tilde{\mathbb{T}}^n$ *of* $\mathbb{T}^n$ *and a covering map* $g' : \tilde{\mathbb{T}}^n \to \tilde{\mathbb{T}}^n$ *for which*

$$\begin{array}{ccc} \tilde{\mathbb{T}}^n & \xrightarrow{\quad g' \quad} & \tilde{\mathbb{T}}^n \\ \pi' \Big\downarrow & & \Big\downarrow \pi' \\ \mathbb{T}^n & \xrightarrow[\quad g \quad]{} & \mathbb{T}^n \end{array} \qquad \textit{commutes,}$$

such that there is an n-*solenoidal group* S *such that for* f *a covering map homotopic to* g *there is a homeomorphism* $\tilde{f} : S \to S$ *so that* $(S, \tilde{f})$ *is homeomorphically conjugate to the inverse limit of a finite cover* $(\tilde{\mathbb{T}}^n, f')$ *of* $(\mathbb{T}^n, f)$.

Theorem 2. *Under the notations of Theorem 1. If* $f : \mathbb{T}^n$

$\to \mathbb{T}^n$ *is a* c-*map, then there exists a solenoidal automorphism* $\tilde{A}$: $S \to S$ *and a homeomorphism* $\tilde{h} : S \to S$ *such that the diagram*

$$\begin{array}{ccc} S & \xrightarrow{\tilde{A}} & S \\ \tilde{h}\downarrow & & \downarrow\tilde{h} \\ S & \xrightarrow[\tilde{f}]{} & S \end{array}$$

commutes.

Theorem 3. *If* $f : \mathbb{T}^n \to \mathbb{T}^n$ *is a special* c-*map, then* f *is topologically conjugate to a toral endomorphism.*

Theorem 4. *If* $f : \mathbb{T}^n \to \mathbb{T}^n$ *is a special* c-*map, then there exist a finite cover* $\tilde{\mathbb{T}}^n$ *of* $\mathbb{T}^n$ *and a special* c-*map* $f' : \tilde{\mathbb{T}}^n \to \tilde{\mathbb{T}}^n$ *such that*

$$\begin{array}{ccc} \tilde{\mathbb{T}}^n & \xrightarrow{f'} & \tilde{\mathbb{T}}^n \\ \pi'\downarrow & & \downarrow\pi' \\ \mathbb{T}^n & \xrightarrow[f]{} & \mathbb{T}^n \end{array}$$

commutes

(π' *is a finite covering map) which satisfy the following conditions : there exist the maximum* n_i-*torus subgroups* $\mathbb{T}^{n_i}$ (i = 1,2) *and the* n_3-*torus subgroup* $\mathbb{T}^{n_3}$ *of* $\tilde{\mathbb{T}}^n$ *and*

(a) *an expansive automorphism* $A_1 : \mathbb{T}^{n_1} \to \mathbb{T}^{n_1}$,

(b) *a positively expansive endomorphism* $A_2 : \mathbb{T}^{n_2} \to \mathbb{T}^{n_2}$,

(c) *a* c-*expansive endomorphism* $A_3 : \mathbb{T}^{n_3} \to \mathbb{T}^{n_3}$,

(d) *a homeomorphism* $h : \mathbb{T}^{n_1} \times \mathbb{T}^{n_2} \times \mathbb{T}^{n_3} \to \tilde{\mathbb{T}}^n$

such that the diagram

$$\mathbb{T}^{n_1} \times \mathbb{T}^{n_2} \times \mathbb{T}^{n_3} \xrightarrow{A_1 \times A_2 \times A_3} \mathbb{T}^{n_1} \times \mathbb{T}^{n_2} \times \mathbb{T}^{n_3}$$

$$\begin{array}{ccc} \Big\downarrow h & & \Big\downarrow h \\ \tilde{\mathbb{T}}^n & \xrightarrow{\quad f' \quad} & \tilde{\mathbb{T}}^n \end{array}$$

commutes.

As an easy corollary we have

Corollary 5. *(a) If* $f : \mathbb{T}^n \to \mathbb{T}^n$ *is a homeomorphism with expansivity and* POTP, *then* f *is topologically conjugate to a toral automorphism of the type* (I),

(b) If $f : \mathbb{T}^n \to \mathbb{T}^n$ *is positively expansive, then* f *is topologically conjugate to a toral endomorphism of the type* (II),

(c) If $f : \mathbb{T}^n \to \mathbb{T}^n$ *is a special* c*-map which is not bijective nor positively expansive, then* f *is topologically conjugate to a toral endomorphism of the type* (III).

From the definition of c-maps we have

Corollary 6. *If* $f : \mathbb{T}^n \to \mathbb{T}^n$ *is a* c*-map but not special, then there do not exist toral endomorphisms to which* f *is topologically conjugate.*

Let $\mathfrak{g}(\mathbb{T}^n)$ be the set of all continuous surjections of $\mathbb{T}^n$. Then $\mathfrak{g}(\mathbb{T}^n)$ becomes a complete metric space with respect to d defined by $d(f, g) = \max\{d(f(x), g(x)) : x \in \mathbb{T}^n\}$ for $f, g \in \mathfrak{g}(\mathbb{T}^n)$ (here d denotes the metric for $\mathbb{T}^n$ induced by euclidean metric $\bar{d}$ for $\mathbb{R}^n$).

Corollary 7. *Let* f_i $(\mathbb{T}^n \to \mathbb{T}^n)$ *converge to a covering map* f $(\mathbb{T}^n \to \mathbb{T}^n)$ *as* $i \to \infty$ *and* A *denote the toral endomorphism of*

$\mathbb{T}^n$ *homotopic to* f. *Then the following holds* :

(a) if each f_i *is a special* c-*map then there exists a continuous surjection* $h : \mathbb{T}^n \to \mathbb{T}^n$ *such that the diagram*

$$\begin{array}{ccc} \mathbb{T}^n & \xrightarrow{\ f\ } & \mathbb{T}^n \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \mathbb{T}^n & \xrightarrow[\ A\ ]{} & \mathbb{T}^n \end{array}$$

commutes.

(b) let $(S, \tilde{f})$ *be homeomorphic to the inverse limit of* $(\mathbb{T}^n, f)$ *as in Theorem 1 and* $\tilde{A}$ *be a solenoidal automorphism induced by* A. *Then there exists a continuous surjection* $\tilde{h} : S \to S$ *such that the diagram*

$$\begin{array}{ccc} S & \xrightarrow{\ \tilde{f}\ } & S \\ {\scriptstyle \tilde{h}}\downarrow & & \downarrow{\scriptstyle \tilde{h}} \\ S & \xrightarrow[\ \tilde{A}\ ]{} & S \end{array}$$

commutes.

In order to capture orbit structures of c-maps, we shall prepare the following two key points. The first point is to establish "uniformly local decomposition theorem" which corresponds to the differential structures for Anosov maps, and to use solenoidal groups which is the natural extension of the n-torus (Theorem 1).

It seems likely that this point is useful in studying dynamics without differential structure.

The second point is to introduce a notion of generalized foliations, by using c-maps of closed topological manifolds, which is an extension of that of foliations in usual sense, and define orientability for such the foliations. This is important in minutely calculating fixed point indices of c-maps.

References

[1] N.Aoki, Topological stability of solenoidal automorphisms, Nagoya Math. J. 90 (1983), 119-135 (correction to : Topological stability of solenoidal automorphisms, ibid 95 (1984), 103).

[2] G.Bredon, Sheaf Theory, McGraw-Hill, New York, 1967.

[3] M.Brin and A.Manning, Anosov diffeomorphisms with pinched spectrum, Lecture Note in Math., 898 (1981), Springer, 48-53.

[4] M.Gromov, Groups of polynomial growth and expanding maps, IHES Publ. Math. 53 (1981), 53-78.

[5] A.Dold, Fixed point index and fixed point theorem for Euclidean neigh- borhood retracts, Topology 4 (1965), 1-8.

[6] P.Duvall and L.Husch, Analysis on topological manifolds, Fund. Math. 77 (1972), 75-90.

[7] J.Franks, Anosov diffeomorphisms on tori, Traus. Awer. Math. Soc. 145 (1969), 117-124.

[8] J.Franks, Anosov diffeomorphisms, Global analysis, Proc. Sympos. Pure Math. 14 (1970), 61-93.

[9] G.Hector and U.Hirsch, Introduction to the geometry of foliations, Part A, Braunschweig / Wiesbaden, 1981.

[10] K.Hiraide, Expansive homeomorphisms of compact surfaces are pseudo Anosov, preprint (1987).

[11] A.Manning, There sre no new Anosov diffeomorphisms on tori, Amer. J. Math. 96 (1974), 422-429.

[12] C.Robinson, Stability theorems and hyperbolicity in dynamical systems, Rocky Mountain J. Math. 7, 425-437.

[13] M.Shub, Endomorphisms of compact differentiable manifolds,

Amer. J. Math. 91 (1969), 175-199.

[14] S.Smale, Differentiable dynamical systems, Bull. Amer. Math. Soc. 73 (1967), 747-817.

[15] E.Spannier, Algebraic Topology, McGraw-Hill, New York, 1966.

[16] F. Przytycki, Anosov endomorphisms, Studia Math. 58 (1976), 249-285.

[17] W.Reddy, Expansive maps on compact metric spaces, Topology and its Appl. 13 (1982), 327-334.