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普遍W*-力学系と非有界微分

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1．序論

C*-力学系に対してある意味で普遍的なW*-力学系を導入し、その非有界微分の問題への応用を述べます。

(A,G,α) をC*-力学系とする、すなわちαは局所コンパクト群GからC*-代数Aの自己同型群Aut(A)への強連続な準同型である。πを$\pi\circ\alpha=\beta\circ\pi$なる$\pi(A)''$上の σ-弱連続な作用βが存在するような、すなわちW*-力学系 ($\pi(A)''$,G,β) があるようなAの表現とする。簡単のために、このようなπをG-共変表現と呼ぶ。(適当な表現空間上では共変ユニタリー表現を伴うがそれは本質的ではない。) G-共変表現に対する普遍問題を考えよう。このとき、普遍問題は解ける。すなわち、次のようなAのG-共変表現$\tilde{\pi}$が存在する。任意のG-共変表現πに対して$\Phi\circ\tilde{\pi}=\tilde{\pi}$であるような$\pi(A)''$から$\pi(A)''$上への*-準同型$\Phi$が存在する。このとき、$\tilde{\pi}(A)''$上には$\alpha$に伴うW*-力学系がある、これをC*-力学系 (A,G,α) に伴う普遍W*-力学系と呼ぼう。これらは、群作用を考慮しない場合と比較するならば、ちょうど普遍表現とenveloping von Neumann algebra に対応する。標準的な enveloping von Neumann algebra はＡの再双対Ａ** であるが、標準的な $\tilde{\pi}$(Ａ)″ はＡ** の商空間として構成される。

αの転置α* は双対Ａ* に作用する。α*(Ｌ¹(Ｇ))(Ａ*)のＡ* でのノルムについての閉包をM_*とすると、α* はM_*に強連続に作用する。したがって、再転置α** から自然に導かれる商フォン・ノイマン代数 Ａ**/$(M_*)^\circ$ (＝Ｍ)の作用はσ-弱連続であり、Ｍは$\tilde{\pi}$(Ａ)″ と*-同型になる。このようにして標準的な普遍Ｗ*-力学系が構成される。

普遍Ｗ*-力学系を具体的に捉えることは難しいと思われるが、簡単な例をふたつあげよう。

例１．(Ｃ(Ｈ),Ｇ,α) に伴う普遍Ｗ*-力学系は (Ｂ(Ｈ),Ｇ,α**) である。なぜなら、αはトレースクラスに強連続に作用するから。

例２．λをＧの左移動、μを左ハール測度とすると、(C_0(Ｇ),Ｇ,λ) に伴う普遍Ｗ*-力学系は (L^∞(Ｇ,μ),Ｇ,λ) である。なぜなら、α*(Ｌ¹(Ｇ,μ))(Ａ*)に属する任意の測度はμ-絶対連続であり、したがって、上述の先双対M_*はＬ¹(Ｇ,μ)に同型である。

普遍Ｗ*-力学系を導入した動機は、群作用と関連づけられたＣ*-代数Ａにおける非有界微分の問題に利用することである。技術的な問題として、このような非有界微分をＡ** において十分に拡張し、フォン・ノイマン代数における微分として扱いたいということがある。しかしながら、一般には非有界微分はＡ** においてはσ-弱閉化可能ではなく、また

α^{**} は σ-弱連続ではないためにいろいろな病理的なことが起こる。α^{**} が連続に作用するA** の要素すべてからなるC*-部分代数Cを考察することはG-共変表現との関連のもとでは適切ではないようである。eを上述のMにたいして $(M_*)^\circ = A^{**}e$ なる中心的射影とすると $e \in C$ であるが、eはMにおいては0であるから α^{**} が σ-弱連続でないかぎりCをMに埋め込むことはできない。しかし、Gが可換かコンパクトならスペクトル部分空間 $A^{\alpha}(K)^{**}$（Kは $\hat{G}$ のコンパクトあるいは有限な部分集合）で生成されるA** のC*-部分代数BはMに埋め込まれる。このようなBはA** において大き過ぎずかつ十分に大きなC*-部分代数である、そしてMはBを含むフォン・ノイマン代数として最適であろう。

Aにおける非有界微分はA** においてノルムに関する閉拡大を非常に多く持つであろう。作用 α を一径数群、δ をその生成作用素である閉*-微分としよう。α^{**} は σ-弱連続ではないとする。このとき δ はAにおいて σ-弱閉ではない、すなわち $A^{**} \oplus A^{**}$ における σ-弱閉包 $\bar{\delta}$ は many to many である。だが、$\{y \mid (0,y) \in \bar{\delta}\} = (M_*)^\circ$，また、$(x,y) \in \bar{\delta}$ かつ $x,y \in (M_*)^\circ$ ならば $x=0$ である。したがって、$\alpha^{**}|B$，$\alpha^{**}|C$ の生成作用素をそれぞれ δ_B，δ_C とおくと、$\delta_B \subset \delta_C$，$\delta_B \subset \bar{\delta}$ であるが、$(e,0) \in \delta_C$，$(e,0) \notin \bar{\delta}$ である。したがって、$\delta_C \not\subset \bar{\delta}$ である。

C*-代数Aにおける*-微分 δ はいかなる条件のもとで生成作用素になるかということを問題にしよう。Hille-Yoshida の定理によって次の2つの条件が十分条件である。

$\forall a \in D(\delta)$, $\forall \lambda \in R$, $\|(1+\lambda\delta)(a)\| \geq \|a\|$,

$(1+\delta)D(\delta)$ がAにおいてノルムに関して稠密である。

（ここで、$D(\delta)$ は δ の定義域である。）

共変表現 π をとって $\pi(A)$ においてノルムに関する稠密性を調べることが少なくない。もちろん、ノルムに関する稠密性よりも σ-弱稠密性を確かめる方が容易である。このような技術的な問題のためにもC*-代数におけるフォン・ノイマン代数における拡張、あるいは $1\pm\delta$ の像の σ-弱稠密性からノルムに関する稠密性を導くことなどが重要である。普遍W*-力学系を導入することによって、これらについてのいくつかの問題に統一的かつ容易に答えることができる。

2. Universal W*-dynamical systems

Let (A, G, α) be a C*-dynamical system. A representation π is said to be <u>G-covariant</u> if it induces a σ-weakly continuous action β on $\pi(A)''$ such that $\beta \circ \pi = \pi \circ \beta$.

<u>Theorem 2.1</u>([5, Theorem 1]). Let (A, G, α) be a C*-dynamical system where G is a locally compact group.

Then there exist a W*-dynamical system $(M, G, \hat{\alpha})$ and a *-isomorphism j of A into M satisfying the following statements, unique up to equivalence:

(i) $j(A)$ is weakly dense in M ;

(ii) $j \circ \alpha_t = \hat{\alpha}_t \circ j$ for all $t \in G$; and

(iii) if π is a G-covariant representation of A , then there exists a normal *-homomorphism $\bar{\pi}$ of M onto $\pi(A)''$ such that $\bar{\pi} \circ j = \pi$ and $\beta_t \circ \bar{\pi} = \bar{\pi} \circ \hat{\alpha}_t$ for all $t \in G$.

<u>Proof</u>. Let M_* denote the set of those $\phi \in A^*$ that $G \ni t \mapsto \phi \circ \alpha_t$ is uniformly continuous; then the polar $M_*{}^\circ$ of M_* in A^{**} is a closed α^{**}-invariant ideal of A^{**} and M_* is isomorphic to the predual of the von Neumann algebra $A^{**}/M_*{}^\circ$, denoted by M . Moreover there is an action $\bar{\alpha}$ on M such that $\hat{\alpha}_t \circ j = j \circ \alpha^{**}$ for all $t \in G$, where j denotes the canonical *-homomorphism of A^{**} onto M . By the definition of M_* , $\hat{\alpha}$ strongly acts on M_* and so σ-weakly acts on M . On the other hand, since $\{\phi \circ \alpha(f) \mid f \in L^1(G) , \phi \in A^*\}$ is norm-dense in M_* and $\sigma(A^*, A)$-dense in A^* , $j|A$ is injective and further

$\hat{\alpha}_t \circ (j|A) = (j|A) \circ \alpha_t$ for all $t \in G$.

Let π , N and β be as in the theorem; then, since β strongly acts on N_* , we have $N_* \circ \pi \subset M_*$ and so $\ker \pi^{**} \supset M_*^\circ$, where π^{**} denotes the canonical morphism of A^{**} onto N associated with π . There is therefore the normal $*$-homomorphism $\bar{\pi}$ of M onto N such that $\bar{\pi} \circ j = j \circ \pi^{**}$, so that $\beta_t \circ \bar{\pi} = \bar{\pi} \circ \hat{\alpha}_t$ clearly. We thus complete the proof of the theorem.

For brevity, we call the W*-dynamical system $(M, G, \bar{\alpha})$ constructed above the <u>universal W*-dynamical system</u> associated with (A, G, α) .

<u>Proposition 2.2</u>([5, Proposition 2]). Let (A, G, α) be a C*-dynamical system where G is locally compact abelian or compact. Let (M, G, α) be the universal W*-dynamical system associated with (A, G, α) .

If G is abelian, then $\overline{A^\alpha(K)} \simeq A^\alpha(K)^{**}$, as Banach spaces, and

$$M^\alpha(K) = \bigcap_V \overline{A^\alpha(K + V)}$$

for any compact subset K of the dual $\hat{G}$ of G , where the bar means σ-weak closure and V runs over all compact neighbourhoods of 0 in $\hat{G}$.

If G is compact, then for any finite subset K of $\hat{G}$

$$M^\alpha(K) = \overline{A^\alpha(K)} \simeq A^\alpha(K)^{**} .$$

<u>Proof</u>. Assume that G is abelian, and let K be a compact

subset of $\hat{G}$. For a compact neighbourhood V of 0 in $\hat{G}$, take a function $f \in L^1(G)$ such that the support of the Fourier transform $\hat{f}$ is contained in $K + V$ and $\hat{f}(\gamma) = 1$ on some neighbourhood of K. Then we have

$$M^\alpha(K) \subset \alpha(f)M \subset \overline{\alpha(f)A} \subset \overline{A^\alpha(K + V)} \subset M^\alpha(K + V) .$$

Since $\bigcap_V M^\alpha(K + V) = M^\alpha(K)$, we have $M^\alpha(K) = \bigcap_V \overline{A^\alpha(K + V)}$.

Next we show that $\overline{A^\alpha(K)} \simeq A^\alpha(K)^{**}$. Identify $A^\alpha(K)^{**}$ with the $\sigma(A^{**}, A^*)$-closure of $A^\alpha(K)$; then the norm-closure B of $\bigcup_{K'} A^\alpha(K')^{**}$ is a C*-subalgebra of A^{**}, where K' runs over all compact subsets of $\hat{G}$. The second adjoint action α^{**} uniformly continuously acts on $A^\alpha(K')^{**}$ and hence strongly continuously acts on B. Moreover $(\alpha^{**}|B)(f) = \alpha(f)^{**}|B$ for any $f \in L^1(G)$ with Supp $\hat{f}$ compact. Therefore j in Theorem 1 is injective on B. Since B is a C*-algebra, $j|B$ is isometric. Since the unit ball of $A^\alpha(K)^{**}$ is $\sigma(A^{**}, A^*)$-compact, its image under j is σ-weakly compact and hence σ-weakly closed. Therefore $j(A^\alpha(K)^{**})$ is σ-weakly closed and hence coincides with $\overline{A^\alpha(K)}$.

When G is compact, use the projection $\int \sum_{\gamma \in K} \dim\gamma \operatorname{Tr} \gamma_t^{-1} \alpha_t dt$ onto $M^\alpha(K)$, instead of $\alpha(f)$. Then we can obtain the conclusion in the proposition analogously.

When G is abelian, the following was proved by Bratteli and Kishimoto [1] in a different form, however its proof is much simpler than them.

<u>Lemma 2.3</u>([5, Lemma 3]). Let G be a locally compact group and α an action of G on a C*-algebra A. Let (M, G, α) be the universal W*-dynamical system associated with (A, G, α) and we may regard $A \subset M$.

Let δ be a *-derivation in A commuting with α.

Assume that $(1 \pm \delta)D(\delta)$ are σ-weakly dense in M and

$$\|(1+\lambda\delta)(a)\| \geqq \|a\| \quad \text{for any } a \in D(\delta) \text{ and } \lambda \in \mathbf{R} .$$

Then δ generates a strongly continuous one-parameter group of *-automorphisms of A.

In particular, if δ generates a σ-weakly continuous one-parameter group of *-automorphisms of M, then the conclusion holds.

<u>Proof</u>. It suffices to show that $(1 \pm \delta)D(\delta)$ are uniformly dense in A. Since δ is uniformly closable in A, we may assume that δ is uniformly closed in A and commutes with α. If $\phi \in A^*$ and $\phi \circ (1+\delta) = 0$, then we have

$$\phi \circ \alpha(f) \circ (1+\delta) = \phi \circ (1+\delta) \circ \alpha(f) = 0$$

for any $f \in L^1(G)$. Since $\phi \circ \alpha(f) \in M_*$ and $(1+\delta)D(\delta)$ is σ-weakly dense in M, we have $\phi \circ \alpha(f) = 0$ for any $f \in L^1(G)$, and hence $\phi = 0$. Thus $(1+\delta)D(\delta)$ is uniformly dense in A and also is $(1-\delta)D(\delta)$ similarly.

The following has been proved in [2] originally but is an immediate consequence of Lemma 2.3.

Corollary 2.4. ([2, Theorem 1.4]) Let G be a compact abelian group and α an action of G on a C*-algebra A. Let be a closed *-derivation in A commuting with α.

If $D(\delta) \supset A^{\alpha}$, then δ is a generator.

Proof. Let (M, G, α) be the universal W*-dynamical system associated with (A, G, α). Since $\delta | A^{\alpha}$ is an everywhere defined derivation in A^{α}, there is a self-adjoint element $h \in \overline{A^{\alpha}} = M^{\alpha}$ such that $\delta(a) = i[h, a]$ for all $a \in A^{\alpha}$. Denote by δ_{ih} the *-derivation on M implemented by ih; then $(\delta - \delta_{ih}) | A^{\alpha} = 0$ and $\delta - \delta_{ih}$ commutes with α. Hence $\delta - \delta_{ih}$ is σ-weakly closable because $\phi \circ (\delta - \delta_{ih}) = 0$ for all normal G-invariant state ϕ on M. Let δ' denote the σ-weak closure of $\delta - \delta_{ih}$, so that δ' commutes with α and $\delta' | M^{\alpha} = 0$. It therefore follows that δ' is a generator in M and the σ-weak closure of δ is a generator also. Consequently, by Lemma 2.3, δ is a generator in A.

Recall that, in a C*-dynamical system (A, G, α), a state ϕ on A is said to be G-centrally ergodic if $\pi_{\phi}(A)'' \cap \pi_{\phi}(A)' \cap U_G^{\phi}{}' = \mathbb{C}1$, where $\{\pi_{\phi}, U^{\phi}\}$ denotes the G-covariant representation associated with ϕ.

Lemma 2.5([5, Lemma 5]). Let α be an action of a compact group G on a separable C*-algebra A and let δ be a closed *-derivation in A commuting with α and satisfying $D(\delta) \supset A^{\alpha}$.

Assume that, for any G-centrally ergodic state on A , δ generates a σ-weakly continuous one-parameter group in the associated representation.

Then δ generates a strongly continuous one-parameter group of *-automorphisms of A .

Proof. Let (M, G, α) be the universal W*-dynamical system associated with (A, G, α) . Let Z^α be the fixed point algebra of the center of M , and e a nonzero projection of Z^α . Then there is a normal state ϕ on M with $\langle e, \phi \rangle \neq 0$. $\int \phi \circ \alpha_t dt$ is a G-invariant normal state on M and $\langle e, \int \phi \circ \alpha_t dt \rangle = \langle e, \phi \rangle \neq 0$. Therefore, by Zorn's lemma, there exists a family (ϕ_ι) of G-invariant states on A such that the sum of the support projections of π_{ϕ_ι} is the identity, so that $M \simeq \oplus_\iota \pi_{\phi_\iota}(A)''$. On the other hand, as seen in the proof of Corollary 4, $(\bar{\delta}-\delta_{ih})|M^\alpha = 0$ for some $h \in M$, and hence we have $\bar{\delta}|Z^\alpha = 0$, where $\bar{\delta}$ denotes the σ-weak closure of δ in M .

Let ϕ be a G-invariant state on A . We shall show that $\|\pi_\phi((1+\lambda\delta)(a))\| \geq \|\pi_\phi(a)\|$ for any $a \in D(\delta)$ and $\lambda \in \mathbf{R}$, and $\pi_\phi((1\pm\delta)D(\delta))$ are σ-weakly dense in $\pi_\phi(A)''$. Then it follows from the above discussion that $\|(1+\lambda\delta)(a)\| \geq \|a\|$ for any $a \in D(\delta)$ and $\lambda \in \mathbf{R}$, and $(1\pm\delta)D(\delta)$ are σ-weakly dense in M , which imply, by Lemma 2.3, that δ is a generator in A .

We may assume without loss of generality that A has an identity.

Let μ be a subcentral measure associated with $\pi_\phi(A)'' \cap \pi_\phi(A)' \cap U_G^{\phi}{}'$, where (π_ϕ, U^ϕ) denotes the G-covariant representation associated with ϕ . μ is supported by the set of G-invariant states on A and, since A is separable,

$$\pi_\phi = \int^{\oplus} d\mu(\psi)\pi_\psi ,$$

where $(\pi_\psi, U^\psi, \mathcal{H}_\psi)$ denotes the G-covariant representation associated with G-invariant state ψ . Moreover, μ-almost all ψ are G-centrally ergodic. For,

$$(\pi_\phi(A)''\cap\pi_\phi(A)'\cap U_G^{\phi}{}')' = (\pi_\phi(A)'\cup\pi_\phi(A)\cup U_G^\phi)'' = \int^{\oplus} d\mu(\psi)B(\mathcal{H}_\psi) ,$$

$$\pi_\phi(A)' = \int^{\oplus} d\mu(\psi)\pi_\psi(A)' \quad \text{and} \quad U_t^\phi = \int^{\oplus} d\mu(\psi)U_t^\psi ,$$

and hence $(\pi_\psi(A)'\cup\pi_\psi(A)\cup U_G^\psi)'' = B(\mathcal{H}_\psi)$ for μ-almost all ψ , that is, $\pi_\psi(A)''\cap\pi_\psi(A)'\cap U_G^{\psi}{}' = \mathbf{C}1$ for μ-almost all ψ .

By the assumption, for μ-almost all ψ ,

$$\|\pi_\psi((1+\lambda\delta)(a))\| \geqq \|\pi_\psi(a)\| \quad \text{for all } a \in D(\delta) \text{ and } \lambda \in \mathbf{R}$$

and $\pi_\psi((1\pm\delta)D(\delta))$ are σ-weakly dense in $\pi_\psi(A)''$. Therefore, for all $a \in D(\delta)$ and $\lambda \in \mathbf{R}$, we have

$$\|\pi_\phi((1+\lambda\delta)(a))\| = \text{ess sup } \|\pi_\psi((1+\lambda\delta)(a))\| \geqq \text{ess sup } \|\pi_\psi(a)\|$$

$$= \|\pi_\phi(a)\| .$$

Assume that $\langle\pi_\phi((1+\bar{\delta})D(\bar{\delta})), f\rangle = 0$ for a normal form f on $\pi_\phi(A)''$. f is decomposable with $f = \int^{\oplus} d\mu(\psi)f(\psi)$. For any $a \in D(\delta)$ and $z \in Z^\alpha$, we have $z(1+\delta)(a) = (1+\bar{\delta})(za)$. Therefore, for any $a \in D(\delta)$ and $z = (z(\psi))_\psi \in \pi_\phi(Z^\alpha)$,

$$\int d\mu(\psi)z(\psi)\langle\pi_\psi((1+\delta)(a)), f(\psi)\rangle = \langle z\pi_\phi((1+\delta)(a)), f\rangle = 0 .$$

Since $\pi_\phi(Z^\alpha) = \pi_\phi(A)''\cap\pi_\phi(A)'\cap U_G^{\phi}{}' \simeq L^\infty(\mu)$,

$\langle \pi_\psi((1+\delta)(a)), f(\psi)\rangle = 0$ for μ-almost all ψ, but, since δ has a separable core, $\langle \pi_\psi((1+\delta)D(\delta)), f(\psi)\rangle = 0$ for μ-almost all ψ. Hence $f(\psi) = 0$ for μ-almost all ψ, and so $f = 0$. Therefore it follows that $\pi_\phi((1+\delta)D(\delta))$ is σ-weakly dense in $\pi_\phi(A)''$ and also is $\pi_\phi((1-\delta)D(\delta))$. We thus complete the proof of the theorem.

Let ϕ be a G-centrally ergodic state on A. Let Ω be a compact space such that $C(\Omega)$ is isomorphic to the C*-algebra of those elements x of the center of $\pi_\phi(A)''$ such that $t \mapsto \alpha_t x$ is norm-continuous; then, since ϕ induces the measure on Ω whose support is Ω, the center of $\pi_\phi(A)''$ is isomorphic to $L^\infty(\Omega)$. Moreover, G acts on Ω continuously and ergodically, and hence an orbit $G\omega$ is compact, so that $G\omega = \Omega$. If H is the stabilizer of ω, then Ω is homeomorphic to G/H as left translations. Therefore it follows from the imprimitivity theorem [7, Theorem 10.5] that the W*-dynamical system associated with ϕ is equivalent to an induced system.

This discussion and Lemma 2.5 imply the following.

<u>Theorem 2.6</u>([5, Theorem 6]). Let α be an action of a compact group G on a C*-algebra A and let δ be a closed *-derivation in A commuting with α.

If A is of type I and separable, and $D(\delta)\supset A^\alpha$, then δ is a generator.

<u>Proof</u>. By Lemma 5 it suffices to show that any σ-weakly

closed *-derivation δ' in a von Neumann algebra M commuting with α is a generator in M if M is of type I, α ergodically acts on the center Z of M and $\delta'|M^{\alpha} = 0$.

Since α ergodically acts on Z , it follows that $Z \simeq L^{\infty}(G/H)$ and α acts on $L^{\infty}(G/H)$ as left translations for some closed subgroup H of G . By the imprimitivity theorem [7, Theorem 10.5] we have $\{M, \alpha\} \simeq \mathrm{Ind}_H^G\{N, \beta\}$ for some type I factor N and action β of H on N , and so we may assume that $M = (N \otimes L^{\infty}(G))^{\beta\otimes\rho}$ and $\alpha = \iota\otimes\lambda$, where $\rho_s f(u) = f(us)$ and $\lambda_t f(u) = f(t^{-1}s)$ for $f \in L^{\infty}(G)$, $s \in H$ and $t, u \in G$, and ι denotes the trivial action of G on N .

We shall show that there exists an increasing directed family (e_ι) of β-invariant finite dimensional projections of N which strongly converges to the identity. Denote by U the set of those $u \in N$ that there is an element $s \in H$ such that $uxu^* = \beta_s(x)$ for all $x \in N$; then U is a group of unitaries, because $uu^* = \beta_s(1) = 1$ and $1 = uu^*uu^* = \beta_s(u^*u)$. Moreover U is strongly compact and hence is a topological group. In fact, the function $u \mapsto uxu^*$ is weakly continuous on the unit ball of N for any x of rank 1 and hence for any compact operator x , i.e., the function $u \mapsto u \cdot u^*$ from the unit ball of N into the space of weakly continuous linear mappings in N is continuous with respect to the weak topology on the unit ball of N and the topology of pointwise weak convergence in the *-subalgebra of all compact operators. β_H is compact with respect to the topology of poinwise weak convergence in N and the topology of pointwise weak convergence in the *-subalgebra of all compact operators. Therefore $U = \{u \mid u \cdot u^* \in \beta_H\}$ is weakly compact and hence strongly compact, because the weak topology and strong topology on the group of unitaries coincide. The strongly continuous unitary representation $u \mapsto u$ of the compact group U can be decomposed in irreducible (finite dimensional) representations and therefore there is an orthogonal family (e'_K) of finite dimensional projections of N such that $u = \sum ue'_K$ and $ue'_K u^* = e'_K$ for all $u \in U$. There is thus an increasing directed family (e_ι) of β-invariant finite dimensional projections of N which strongly converges to the identity.

Now, since $e_\iota \otimes 1 \in M^\alpha$, the restriction $\delta' | (e_\iota \otimes 1)M(e_\iota \otimes 1)$ is a σ-weakly closed $*$-derivation in $(e_\iota \otimes 1)M(e_\iota \otimes 1)$ satisfying the same properties as δ'. Since $\beta_s \otimes \rho_s$ and α_t commute, $((e_\iota \otimes 1)M(e_\iota \otimes 1))^\alpha(\gamma) = (e_\iota N e_\iota \otimes L^\infty(G)^\lambda(\gamma))$ M for any $\gamma \in \hat{G}$ and so they are finite dimensional. It therefore follows that $\delta' | (e_\iota \otimes 1)M(e_\iota \otimes 1)$ is a generator in $(e_\iota \otimes 1)M(e_\iota \otimes 1)$, so that $(1+\lambda\delta')((e_\iota \otimes 1)D(\delta')(e_\iota \otimes 1)) = (e_\iota \otimes 1)M(e_\iota \otimes 1)$ and $\|(1+\lambda\delta')(x)\| \geq \|x\|$ for any $x \in (e_\iota \otimes 1)D(\delta')(e_\iota \otimes 1)$ and $\lambda \in \mathbf{R}\backslash\{0\}$. Since $\bigcup_\iota (e_\iota \otimes 1)M(e_\iota \otimes 1)$ is a weakly dense $*$-subalgebra of M, it follows from the σ-weak closability of δ' and the Kaplansky's density theorem that $(1+\lambda\delta')D(\delta') = M$ and $\|(1+\lambda\delta')(x)\| \geq \|x\|$ for any $x \in D(\delta')$ and $\lambda \in \mathbf{R}\backslash\{0\}$, so that δ' is a generator in M. We thus complete the proof of the theorem.

Let (A, G, α) be a C*-dynamical system, provided that G is abelian or compact. Let π be a G-covariant representation of A such that $\beta \circ \pi = \pi \circ \alpha$, and denote $\pi(A)''$ by N . Let A_F and N_F denote the union of spectral subspaces $A^{\alpha}(K)$ and $N^{\beta}(K)$, respectively, where K is a compact subset of $\hat{G}$ if G is abelian, and a finite subset of $\hat{G}$ if G is compact. The following theorem is an immediate consequence of Theorem 2.1 and Proposition 2.2.

<u>Theorem 2.7</u>([4, Theorem 1]). Let (A, G, α) be a C*-dynamical system, provided that G is abelian or comact. Let π be a G-covariant representation.

Let δ be a *-derivation in A which is bounded on each spectral subspace $A^{\alpha}(K)$,where K is comapct or finite.

If G is abelian, then there exists a unique *-derivation $\tilde{\delta}$ in $N = \pi(A)''$ such that $\tilde{\delta}$ is defined on N_F , $\tilde{\delta} \circ \pi \supset \pi \circ \delta$ and $\tilde{\delta}$ is σ-weakly continuous on $N^{\beta}(K)$ for any compact subset K of $\hat{G}$. Furthermore we have

$$\| \tilde{\delta} | N^{\beta}(K) \| \leq \inf_{V} \| \delta | A^{\alpha}(K + V) \| ,$$

where V runs over all compact neighbourhoods of 0 in $\hat{G}$.

Even if G is compact, the above consequences hold, provided that $A^{\alpha}(K)$ and $N^{\beta}(K)$ should be replaced with $A^{\alpha}(\gamma)$ and $N^{\beta}(\gamma)$ corresponding to $\gamma \in \hat{G}$ respectively, and the inequality becomes as follows:

$$\| \tilde{\delta} | N^{\beta}(\gamma) \| \leq \| \delta | A^{\alpha}(\gamma) \| .$$

<u>Proof</u>. Let (M, G, α) be the universal W*-dynamical

system. Defininig $\hat{\delta}$ by $\rho\circ\bigcup_K(\delta|A^\alpha(K))^{**}$, where ρ is the canonical *-homomorphism of A^{**} onto M , $\hat{\delta}$ is clearly a *-derivation in M defined on M_F which is σ-weakly continuous on $M^\alpha(K)$, in virtue of Proposition 2.2.

In a general case, denoting by p the support projection of the kernel of the canonical *-homomorphism of M onto N , p is a central projection in M^α . Therefore $\hat{\delta}(p) = 0$ and $\hat{\delta}(x(1-p)) = \hat{\delta}(x)(1-p)$ for any $x \in M_F$. Since N_F is isomorphic to $M_F(1-p)$, we obtain a desired *-derivation $\tilde{\delta}$ in N_F from $\hat{\delta}$.

The following corollary is an immediate consequence of a series of lemmas in [6] and Theorem because u_t^l and u_s^l as below commute.

<u>Corollary 2.8</u>([4, Corollary 3]). Suppose that G is abelian. Suppose that there exist a faithful family (π_l) of representations of A and a family (α^l) such that α^l is an action of G on $\overline{\pi_l(A)}$, $\alpha_t^l\circ\pi_l = \pi_l\circ\alpha_t$ and each α_t^l is implemented by a unitary u_t^l fixed by α^l.

Then δ is closable and its closure is a generator. Furthermore, for any finite measure μ on G with $\hat{\mu}(0) = 0$, the *-derivation δ_μ on A_F , defined by

$$\delta_\mu = \int\alpha_t\circ\delta\circ\alpha_{-t}d\mu(t) ,$$

is bounded and $\|\delta_\mu\| \leq \inf_V \|\delta|A^\alpha(K + V)\|\|\mu\|$, where V runs over all compact neighbourhoods of 0 in $\hat{G}$.

Let A be a C*-algebra, α a strongly continuous one-parameter group of *-automorphisms of A and δ_0 the generator of α . In general, δ_0 is not σ-weakly closable in A^{**} . Let $\overline{\delta_0}$ denote the σ-weak closure of δ_0 in $A^{**}\oplus A^{**}$, ρ the canonical *-homomorphism of A^{**} onto M , and I the kernel of ρ . Then we have

$$\{(0, x) \mid x \in A^{**}\} \cap \overline{\delta_0} = \{(0, x) \mid x \in I\}$$

and

$$\{(x, y) \mid x \in I , y \in I\} \cap \overline{\delta_0} = \{(0, y) \mid y \in I\} .$$

If $(0, x) \in \overline{\delta_0}$, then $(1 - \delta_0)^{-1**}x = 0$, and hence $x \in I$, because $\rho \circ (1 - \delta_0)^{-1**} = (1 - \hat{\delta}_0)^{-1} \circ \rho$. Since $(1 - \delta_0)^{-1*}\phi = \int_0^\infty e^{-t}\alpha_t^*\phi dt \in M_*$ for all $\phi \in A^*$, we have for any $x \in I$ and $\phi \in A^*$

$$\langle (1 - \delta_0)^{-1**}x, \phi\rangle = \langle x, (1 - \delta_0)^{-1*}\phi\rangle = 0 ,$$

and hence $(1 - \delta_0)^{-1**}x = 0$ and $(0, x) \in \overline{\delta_0}$. We have thus $0\oplus A^{**} \cap \delta_0^{**} = 0\oplus I$. If $x, y \in I$ and $(x, y) \in \overline{\delta_0}$, then, since $(0, x-y) \in \overline{\delta_0}$, $(x, x) \in \overline{\delta_0}$ and $x = (1-\delta_0)^{-1**}(x-x) = 0$. Thus we have $I\oplus I \cap \overline{\delta_0} = 0\oplus I$.

By these facts we obtain the following proposition, which is available for proving the stability of ground states under perturbation, or that of KMS-states for a C*-algebra of type I (see [3]). Even though, by Theorem 2.7, $\tilde{\delta}_0$ is σ-weakly continuous on each spectral subspace corresponding to compact sets , the proposition is not trivial .

Proposition 2.9([4, Proposition 4]). Let A be a C*-algebra, α a strongly continuous oneparameter group of *-automorphisms of A and δ_0 be the generator of α. Let π be a G-covarinat representation of A and $\tilde{\delta}_0$ the generator in $\pi(A)''$ such that $\tilde{\delta}_0 \circ \pi = \pi \circ \delta_0$. Let δ be a *-derivarion in A defined on A_F.

Suppose that $\|\delta(x)\| \leq a\|x\| + b\|\delta_0(x)\|$ on A_F for real numbers $a, b \geq 0$.

Then there exists a unique *-derivation $\tilde{\delta}$ in $\pi(A)''$ defined on $D(\tilde{\delta}_0)$ such that the mapping $(x, \tilde{\delta}_0(x)) \mapsto \tilde{\delta}(x)$ $(x \in D(\tilde{\delta}_0))$ is σ-weakly continuous and $\|\tilde{\delta}(x)\| \leq a\|x\| + b\|\tilde{\delta}_0(x)\|$ on $D(\tilde{\delta}_0)$.

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