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線形ダイナミカルシステムのモデル,微分作用素,可制御性

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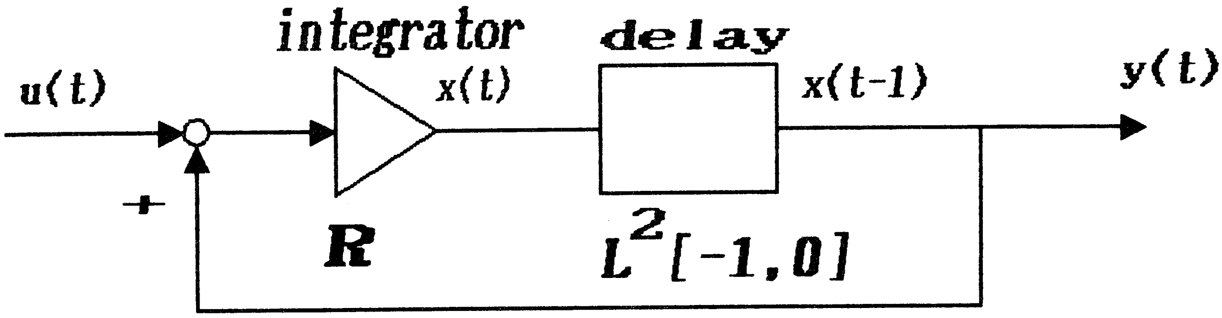
1. Introduction

Consider the following delay-differential system:

$$\dot{x}(t) = x(t-1) + u(t), \qquad u(t) : \text{input vector}$$

$$y(t) = x(t-1). \qquad y(t) : \text{output vector.} \qquad (1)$$



Block diagram for the above system

Clearly , we need to have a function space on [0, 1] (or [-1, 0]) to store the last one second behavior for the state-space model. (Hale [6] and others.)

A well-known standard choice is:

$X = R \times L^2[-1,0]$ (called an M_2 space)

by Delfour, Mitter, and others ([3,4]). It induces the following functional differential equation:

$$\frac{d}{dt}\begin{bmatrix} x_t \\ z_t \end{bmatrix} = \begin{bmatrix} z_t(-1) \\ (\partial/\partial\theta)z_t(\theta) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \qquad z_t(0) = x_t$$

$$y = z_t(-1). \qquad (2)$$

This model has been effectively used for many purposes, say, optimal control, feedback stabilization, etc. Recently, there is even a control scheme by actively using a delay element in the compensator (called repetitive control: [10], [18]).

Question: Where does this function space (and the model (2)) come from?

2. Spectrum, Eigenfunction Completeness and Reachability.

Standard Realization Procedure ([15, 16, 17]):

Basic Idea: Use the left shift σ_t in $L^2_{loc}[0,\infty)$ as a universal model.

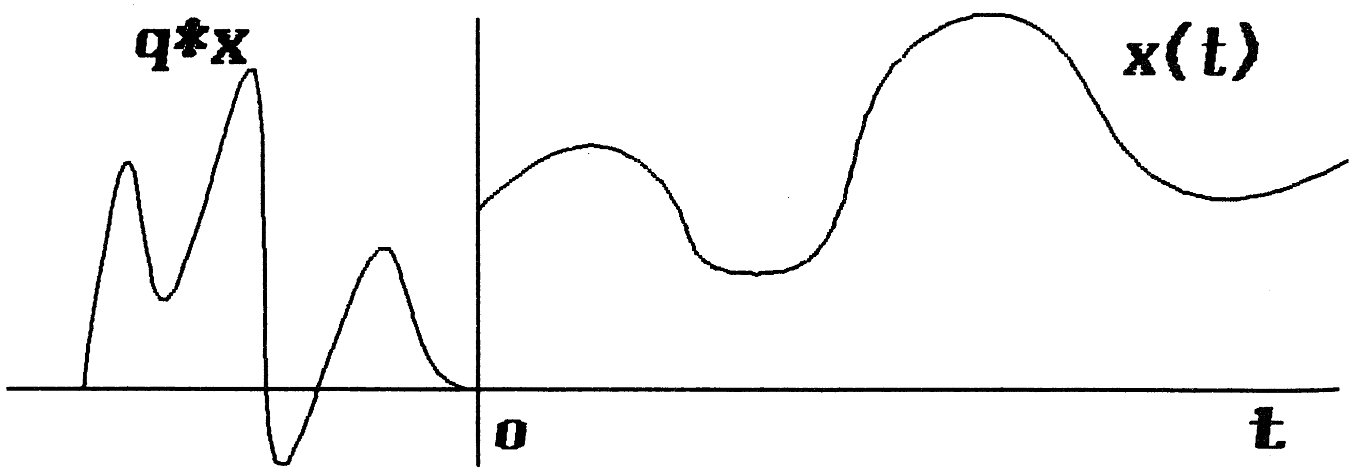
1) Express the input/output relationship of system (1) as $y = A*u$ where A is the impulse response of (1).

2) Express A as the ratio $q^{-1}*p$ of distributions with compact support in $(-\infty,0]$. In (1), $A = \delta/\{\delta_{-1}'-\delta\}$. (If this is possible, A is called **pseudo-rational**.)

3) Take the closed subspace

$$X^q := \{x \in L^2_{loc}[0,\infty);\ \text{supp}\,(q*x) \text{ in } (-\infty,0]\} \qquad (3)$$

as the state space and σ_t in X^q as the generating semigroup for state transition.



In the above example, X^q is given by the closure, taken in $L^2_{loc}[0,\infty)$, of the space of solutions of the equation

$(d/dt)x(t+1) = x(t)$, for $t \geq 0$.

It is readily seen that this space is isomorphic to

$R \times L^2[0,-1]$.

4) The desired functional differential equation model is then given by

$$(d/dt)x_t(\cdot) = Fx_t(\cdot) + A(\cdot)u(t) \qquad (4)$$

where F : infinitesimal generator of σ_t.

Questions on the above construction:

a) What is the meaning of supp(q*x) in $(-\infty,0]$?

b) When does q have compact support in $(-\infty,0]$?

c) What is F ?

d) What is $\sigma(F)$?

e) What is the space M of (generalized) eigenvectors of F?

f) When is M dense in X^q?

h) When is system (4) reachable?

Some Remarks and Answers:

On a),b): Paley-Wiener Theorem:

Theorem (Paley-Wiener-Schwartz [14]) q is a distribution with compact support contained in $(-\infty,0]$ iff

$\hat{q}(s)$ is an entire function of s such that

$|\hat{q}(s)| \leq C(1+|s|)^m \exp(a \cdot \mathrm{Re}\ s)$, $\mathrm{Re}\ s \geq 0$

$$\leq C(1+|s|)^m, \quad \text{Re } s \leq 0 \tag{5}$$

This implies

$$x \in X^q \iff \text{supp }(q*x) \text{ in } (-\infty,0] \tag{6}$$

(Note supp (q*x) is always compact.)

=> **all singularities of x^(s) are cancelled by** q^(s).

On c): F = d/dt (better to write d/dτ by change of variable). **Remark:** The model (2) is actually obtained by the above realization procedure. Somewhat surprisingly, the right-hand side operator in (2) is actually the differential operator d/dτ represented in the space **R** $\times$ $L^2[0,1]$, which is isomorphic to X^q. For details, see [17].

On d): Spectrum of F.

Let us compute the point spectrum only.

$(\lambda I-F)x=0 \iff dx/dt = \lambda x,\ x \in X^q$

$\iff H(t)\exp(\lambda t) \in X^q$

$\iff$ q^(s)·1/{s-λ} satisfies the Paley-Wiener estimate (5).

$\iff$ q^(s)·1/{s-λ} is an entire function.

$\iff$ q^(λ) = 0.

Actually, we can prove that ([15])

i) if q^(λ) $\neq$ 0 then $\lambda \in \rho(F)$.

Therefore,

ii) every $\lambda \in \sigma(F)$ is an eigenvalue (with finite multiplicity).

On e): Let m := order of λ as a zero of q^(s).

Then the generalized eigenspace M_λ corresponding to λ is span $\{\exp(\lambda t), t\exp(\lambda t), \ldots, t^m\exp(\lambda t)\}$.

$$\Rightarrow M = \operatorname*{span}_{\lambda,m} \{\exp(\lambda t), t\exp(\lambda t), \ldots, t^m\exp(\lambda t)\}.$$

On f): M is dense in X^q

$$\Leftrightarrow x^* \in (X^q)', \langle x^*, x\rangle = 0 \text{ for all } x \in M \Rightarrow x^* = 0 \qquad (7)$$

[REMARK] This question is closely related to the question of reachability, feedback stabilization, etc., and has been studied via the state space representation as in (2) by a number of authors: [7], [8], [9], [10], [12], [13], etc. (some of them only study reachability). However, a concrete algebraic criterion is fairly difficult to obtain, and has been obtained via somewhat ad hoc methods for delay-differential systems (e.g., [9], [11], [13]). We here attempt to pursue a more unified and systematic approach for pseudo-rational systems, which are known to include the class of delay-differential systems.

Our question is then: What is $(X^q)'$?

[LEMMA 1] $(X^q)' \simeq \cup L^2[-n,0]/q*(\cup L^2[-n,0])$

$$= \varinjlim L^2[-n,0]/q*(\varinjlim L^2[-n,0]) \qquad (8)$$

Proof. Omitted. A standard fact from locally convex duality, and the fact that $L^2_{loc}[0, \infty)$ is the projective limit of $\{L^2[0, n]\}$. □

[LEMMA 2] $\langle x^*, x\rangle = 0$ for all $x \in M \Leftrightarrow$

$$x^{*\wedge}(s)/q^\wedge(s) = \text{entire function of } s.$$

Proof. For simplicity, assume $q^\wedge(s)$ has simple roots only. The duality in Lemma 1 is ([16])

$$\langle \varphi,x\rangle := \int \varphi(t)x(-t)dt = \int \varphi(-t)x(t)dt,$$

$$\varphi \in \lim L^2[-n,0],\ x \in X^q.$$

Then $\langle x^\cdot,x\rangle = 0$ for all $x \in M$ <=>

$$\langle x^\cdot,\exp(\lambda t)\rangle = x^{\cdot\wedge}(\lambda)=0, \text{ any } \lambda \text{ such that } q^\wedge(\lambda)=0.$$

<=> $x^{\cdot\wedge}(s)/q^\wedge(s)$ is entire. □

Therefore, we have proved

$$x^\cdot \perp M \iff x^{\cdot\wedge}(s) = q^\wedge(s)\varphi(s) \text{ for some entire function } \varphi(s). \quad (8)$$

If any such φ were the Laplace transform of a distribution with compact support in $(-\infty,0]$, then M would be dense in X^q, i.e. this system is eigenfunction complete.

Let us first prove that φ is always the Laplace transform of a distribution with compact support not necessarily contained in $(-\infty,0]$.

To this end, we need to prove, in view of the Paley-Wiener theorem (5), that

i) $\varphi(s)$ is an entire function of exponential type;

ii) it has polynomial growth on the imaginary axis.

We give a proof for i) only (for details, see [19]).

Proof of i) By the well-known Hadamard factorization theorem ([2]) for entire functions, it is clear that φ is of order 1, i.e.,

for any $\varepsilon > 0$, there exists $R > 0$ such that

$$|\varphi(s)| < \exp(|s|^{1+\varepsilon})$$

for $|s| > R$.

We must quote the following deep result by Lindelöf from complex analysis:

[Lindelöf's theorem] ([2]) Let f be an entire function of order 1. Let $\lambda_1, \ldots, \lambda_n, \ldots$ be the zeros of f(s), counted according to multiplicity. Define

$n(r) :=$ no. of zeros of f in $|s| < r$

$$S(r) := \sum_{|\lambda_n| \le r} 1/\lambda_n$$

Then f(s) is of exponential type, i.e., $|f(s)| \le C\exp(K|s|)$ iff

i) $n(r) = O(r)$;

ii) $S(r)$ is bounded.

Proof of φ = exponential type.

Let $\{\lambda_1, \ldots, \lambda_n, \ldots\}$ be the zeros of $q^\wedge(s)$, and $\{\mu 1, \ldots, \mu n, \ldots\}$ the zeros of $\varphi(s)$. Then the zeros of $x\cdot^\wedge(s) = \{\lambda 1, \ldots, \lambda n, \ldots\} \cup \{\mu 1, \ldots, \mu n, \ldots\}$.

i) $n_\varphi(r) = O(r)$ is obvious since $x\cdot^\wedge(s)$ satifies this property

ii) $|S_\varphi(r)| = |S_{x\cdot^\wedge}(r) - S_{q^\wedge}(r)|$

$\le |S_{x\cdot^\wedge}(r)| + |S_{q^\wedge}(r)|$,

so that $S_\varphi(r)$ is also bounded. □

Suppose now that we have agreed that φ is indeed the Laplace transform of a distribution with compact support. (To show this we need a little more work to ensure that $\varphi(s)$ is of polynomial growth on the imaginary axis; see [19].) In view of the fact (8),

$$\textbf{eigenfunction completeness} \Leftrightarrow \textbf{supp } \varphi \subset (-\infty, 0] \textbf{ for all such } \varphi. \tag{9}$$

Question: When is supp $\varphi \subset (-\infty, 0]$?

Define

$$r(\varphi) := \sup \{t;\ t \in \operatorname{supp} \varphi\}.$$

[LEMMA 3] Suppose $r(\varphi)$, $r(\psi) < \infty$. Then

$$r(\varphi * \psi) = r(\varphi) + r(\psi).$$

Indication of Proof.

$r(\varphi * \psi) \le r(\varphi) + r(\psi)$ is obvious.

To prove the reverse inequality, we need to show

φ, ψ do not vanish in a neighborhood of endpoints a, b

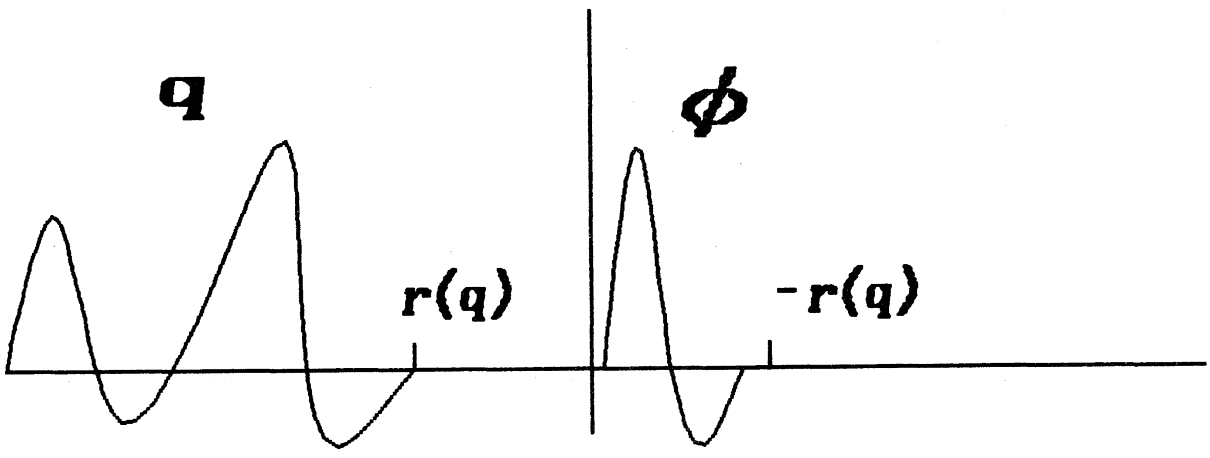
$\Rightarrow$ $\varphi * \psi$ does not vanish in a neighborhood of a+b.

This follows from the local version of the Titchmarsh convolution theorem ([5]). (Need to go back to the original proof, or a proof by Miksinski; the usual proof ([20]) covers only the global version.) □

[THEOREM 1] The system (4) is eigenfunction complete (i.e., M is dense in X^q) iff $r(q) = 0$.

Proof. Observe that $r(\varphi * q) = r(x^{\cdot}) \le 0$ and $r(\varphi * q) = r(\varphi) + r(q)$.

If $r(q)=0$ then $r(\varphi)=r(x^{\cdot})\le 0$.



Conversely, if $r(q) < 0$, then any φ with supp $\varphi \subset (0,-r(q))$ gives rise to an $x^{\cdot}$ such that $\varphi := x^{\cdot} * q^{-1}$
has the property

i) $\varphi\hat{}(s)$ is entire, and $r(\varphi) > 0$.

This contradicts statement (9), whence the eigenfunction completeness.

□

Let us now consider the reachability (controllability) question.

[DEFINITION] The system (4) is said to be <u>quasi-reachable</u> if the set of all elements in X^q that can be driven from 0 by a suitable application of an input is dense in X^q. It is said to be <u>spectrally reachable</u> if any element in M is reachable from 0 by an action of an input.

[LEMMA 4] The above system is spectrally reachable iff

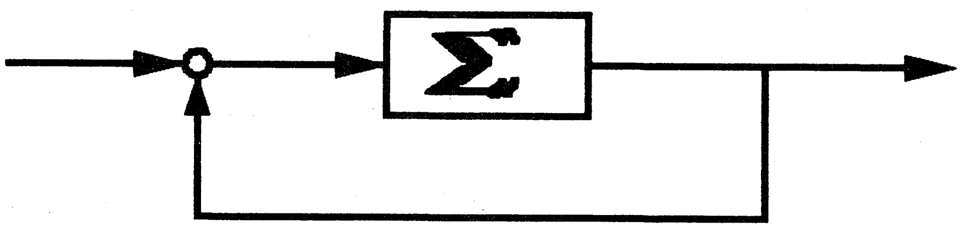
$$\text{rank } [q\hat{}(\lambda) \mid p\hat{}(\lambda)] = \text{full for any } \lambda \in \mathbf{C}.$$

In the present scalar case, this is equivalent to:

no common zero between $q\hat{}(s)$ and $p\hat{}(s)$.

Proof. Omitted. □

[LEMMA 5] Let Σ be a system. Σ is quasi-reachable iff the following system is quasi-reachable.



Combining the above lemmas together, we have

[THEOREM 2] The system (4) defined via X^q is quasi-reachable iff

i) rank $[q^{\wedge}(\lambda) \mid p^{\wedge}(\lambda)]$ = full for any $\lambda \in \mathbf{C}$; and

ii) max $\{r(q), r(p)\} = 0$.

Sketch of Proof. We only prove the sufficiency. For details, see [19].

Case I) $r(q) = 0$. In this case, the space M of eigenfunctions is already dense. Since by i) the system is spectrally reachable, i.e., every element in M is reachable, we must have quasi-reachability.

Case II) $r(q) < 0$ but $r(p) = 0$. In this case, form the feedback system in the above diagram. Then the new system has the impulse response $(q+p)^{-1}*p$, i.e., we have a new denominator $(q+p)$. Clearly, $r(q+p) = 0$. Then by Lemma 5 and the above argument in Case I), the result follows. □

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