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<td>Author(s)</td>
<td>SEKI, Hiroyuki; TANIGUCHI, Kenichi; KASAMI, Tadao</td>
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Kyoto University
On Operational Semantics of Congruence Relation Defined in Algebraic Language ASL/*

Hiroyuki SEKI, Kenichi TANIGUCHI and Tadao KASAMI

Department of Information and Computer Sciences
Faculty of Engineering Science
Osaka University
Toyonaka, Osaka 560, JAPAN

Abstract An algebraic specification (or text) specifies a congruence relation on a set of expressions. In algebraic language ASL/*, a pair \((G, AX)\) is called a text, where \(G\) is a context-free grammar and \(AX\) is a set of axioms. A text \(t=(G, AX)\) specifies the set \(E_G\) of expressions generated by \(G\) and the least congruence relation on \(E_G\) satisfying all the axioms in \(AX\), where 'congruency' is defined based on the syntax (phrase structure) of the expressions. In general, for a text \(t\) in ASL/*, the condition, (A) \(e\) is congruent with \(e'\) in \(t\), is not logically equivalent to the condition, (B) \(e'\) is obtained from \(e\) by rewriting \(e\) when the axioms in \(t\) are regarded as 'bidirectional' rewrite rules. We present a sufficient condition for a text \(t\) under which (A) and (B) are equivalent for any pair of expressions \(e\) and \(e'\), which means that the congruence relation specified by \(t\) is simply defined operationally.

1. Introduction

In a stepwise refinement process of a software development activity, it is desirable that the semantics of programs and their specifications should be defined formally. Algebraic semantics is one of the most promising methods to define the semantics of programs and specifications, and has the following advantages [1]:

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**Abstract** An algebraic specification (or text) specifies a congruence relation on a set of expressions. In algebraic language ASL/*, a pair \((G, AX)\) is called a text, where \(G\) is a context-free grammar and \(AX\) is a set of axioms. A text \(t=(G, AX)\) specifies the set \(E_G\) of expressions generated by \(G\) and the least congruence relation on \(E_G\) satisfying all the axioms in \(AX\), where 'congruency' is defined based on the syntax (phrase structure) of the expressions. In general, for a text \(t\) in ASL/*, the condition, (A) \(e\) is congruent with \(e'\) in \(t\), is not logically equivalent to the condition, (B) \(e'\) is obtained from \(e\) by rewriting \(e\) when the axioms in \(t\) are regarded as 'bidirectional' rewrite rules. We present a sufficient condition for a text \(t\) under which (A) and (B) are equivalent for any pair of expressions \(e\) and \(e'\), which means that the congruence relation specified by \(t\) is simply defined operationally.

1. Introduction

In a stepwise refinement process of a software development activity, it is desirable that the semantics of programs and their specifications should be defined formally. Algebraic semantics is one of the most promising methods to define the semantics of programs and specifications, and has the following advantages [1]:
(1) Since the semantics of both of the programs and their specifications (called texts, in this paper) are defined simply and clearly by using congruence relation, the correctness of a text can be verified relatively easily.

(2) A text can be written at an arbitrary level of abstraction; a text corresponding to a specification written in a natural language can be refined stepwise to obtain an efficient program satisfying the given specification in the single semantical framework. Especially, a class of functional programs can be regarded as a special subclass of texts [5].

On the other hand, it has been pointed out from the practical point of view that several problems arise when defining a fairly large specification in algebraic methods. They are summarized as follows:

(a) The syntax of the terms (or expressions in this paper) is usually restricted to so called prefix notation such as $p(g(c_1, c_2), c_3)$, and it is not allowed for the writers to define arbitrary syntax, such as infix notation, at their own discretion.

(b) Different data types (or sorts) cannot share common syntax. Hence, if there is an inclusion relation among sorts (or data types, e.g., integer and real), (1) the definition of each operation on each type must be given separately even if they are homomorphic (e.g., the addition '+' on integer and '+' on real), and, (2) the type transformation must be specified explicitly ('1' on integer vs. '1.0' on real).

(c) When error handling and/or exception handling are considered (e.g., popping the empty stack results in the error state and no operation is defined on that error state), the text tends to be complicated [1].

In order to resolve these problems, several extensions of algebraic semantics such as error algebra[2] have been proposed. However, the semantics can no longer be defined simply in these approaches.

In ASL/*, the syntax of expressions can be described by using context-free grammar (abbreviated as cfg). An inclusion relation among sorts can be represented by an inclusion relation among the sets of expressions which are derived from specific non-terminal symbols corresponding to the sort names. Error handling can be also simply specified as follows: First, let $G$ be a cfg which has two specific non-terminal symbols; (a) a non-terminal symbol from which all the states including both legal and error states are derived, and, (b) a non-terminal symbol, say state, from which only the legal states are derived. Then, a 'strict' operation should be defined only on the expressions derived from the non-terminal symbol state. Details of sort inclusion and error handling are described in [6, 8].
Since the set of expressions is defined by a cfg in ASL/*, the concept 'congruency' must be generalized so that a congruence relation can be defined on a context-free language (cfg). Remember that a binary relation $R$ is called a congruence relation if (1) $R$ is an equality relation, i.e., $R$ is reflexive, symmetric and transitive, and (2) $R$ is closed under each operation, i.e., for each $(n$-ary$)$ operation $f$, whenever $e_1 R e_1', e_2 R e_2', \ldots, e_n R e_n'$ hold then $f(e_1, e_2, \ldots, e_n) R f(e_1', e_2', \ldots, e_n')$ also holds. In ASL/*, for each expression $e$, several substrings of $e$ are specified as subexpressions, which are considered to be arguments of the operation represented by $e$. A congruence relation is defined to be an equality relation which is closed under replacement of subexpression.

In ASL/*, a pair $(G, AX)$ is called a text, where $G$ is a cfg without a start symbol, and $AX$ is a finite set of axioms. A text $t=(G, AX)$ specifies the least congruence relation on $E_G$ which satisfies all the axioms in $AX$, where $E_G$ is the set of all the expressions derived from some non-terminal symbol in $G$. A string in $E_G$ is called an expression. More formally, for an axiom $l \Rightarrow r$ in $AX$, let $Q_t(ax)$ denote $(l\theta, r\theta) | l\theta (or r\theta)$ is an arbitrary expression obtained from $l$ (or $r$) by substituting an expression for each variable in $l$ (or $r$) and $Q_t$ denote $ax \in AX Q_t(ax)$. The congruence relation $\hat{Q}_t$ specified by the text $t$ is defined to be the least congruence relation on $E_G$ containing $Q_t$. For expressions $e$ and $e'$, if $e \hat{Q}_t e'$ holds then $e$ is said to be congruent with $e'$ (in $t$).

In the case that the syntax of expressions is restricted to the prefix notation, the close relation between algebraic axioms and rewriting systems is well-known [4], i.e., the following property holds in our terminology.

[Operational Completeness (text version)] For a given text $t=(G, AX), e$ is congruent with $e'$ if and only if $e'$ is obtained from $e$ by rewriting $e$ finitely many times when $Q_t$ is regarded as a set of 'bidirectional' rewrite rules.

Unfortunately, in ASL/*, operational completeness does not hold in general. In this paper, a sufficient condition for a given text $t$ in ASL/* is presented under which operational completeness stated above holds so that the semantics of the congruence relation specified by $t$ is simply defined operationally.

In section 2, for a set $E$ of expressions and a relation $R$ on $E$ ($R$ is called a set of equations), the congruence relation $\hat{R}$ on $E$ generated by $R$ is defined in an abstract way; $\hat{R}$ is defined independently of the mechanisms which specify the set $E$ of expressions and the set $R$ of equations. Then, we give a sufficient condition for a set $E$ of expressions and a set $R$ of equations to satisfy operational completeness:
[Operational Completeness (abstract version)] For any expressions $e$ and $e'$ in $E$, $e \overline{R} e'$ holds if and only if $e'$ is obtained from $e$ by rewriting $e$ finitely many times when $R$ is regarded as a set of rewrite rules. ■

For a relation $R$, let $R^{-1}$ be the inverse of $R$, i.e., $e R^{-1} e'$ if and only if $e' R e$. The sufficient condition for $E$ and $R$ is as follows.

[1] E satisfies the following syntactic transitivity.

[Syntactic Transitivity] For any expressions $e_1$, $e_2$ and $e_3$ in $E$, if $e_1$ is a subexpression of $e_2$ and $e_2$ is a subexpression of $e_3$, then $e_1$ is also a subexpression of $e_3$.

(2) $E$ and $R$ satisfy the following local syntax compatibility.

[Local Syntax Compatibility] For any pair $(e_1, e_1')$ in $R \cup R^{-1}$ and any expression $e_2$ in $E$ containing $e_1$ as a subexpression, let $e_2'$ be the string obtained from $e_2$ by replacing $e_1$ with $e_1'$. Then $e_2'$ is also an expression in $E$ and $e_1'$ is also a subexpression of $e_2'$.

In sections 3 and 4, for a text $t=(G, AX)$, we formally define the set of expressions $E_G$ and the set $Q_t$ of equations. Next, we assume that a cfg $G$ satisfies a sufficient condition (having unambiguous structure, see 3.2) for $G$ to satisfy syntactic transitivity. Then we give a decidable sufficient condition (called global syntax compatibility, see 3.3) for $E_G$ and $Q_t$ to satisfy local syntax compatibility. To summarize, for a given text $t=(G, AX)$, if (1) $G$ has unambiguous structure and (2) $Q_t$ satisfies global syntax compatibility (which is decidable under the assumption that (1) holds) then operational completeness (text version) holds.

2. A Congruence Relation on a Set of Expressions

2.1 Expressions and Subexpressions

Let $V$ be a finite set of symbols. Let $V^*$ denote the set of all the strings on $V$ and let $V^+ = V^* - \{\varepsilon\}$, where $\varepsilon$ is the empty string. The length of a string $u$ is denoted by $|u|$. For strings $u$ and $w$, if $w = v_1uv_2$ for some $v_1$ and $v_2$ then $u$ is called a substring of $w$ at occurrence $|v_1|$, or simply, a substring of $w$. For strings $w$, $u'$ and a substring $u$ of $w$, let $w[u \leftarrow u', i]$ denote the string obtained from $w$ by replacing $u$ with $u'$, i.e.,

\[ w[u \leftarrow u', i] = v_1 u' v_2 \]

where $i = |v_1|$. If there is no ambiguity, $w[u \leftarrow u', i]$ may be abbreviated as $w[u \leftarrow u']$ by omitting the occurrence $i$.  

4
Let $E$ be a subset of $V^*$. A string in $E$ is called an expression (in $E$), and $E$ is called a set of expressions (on $V$). Let $Z$ denote the set of all the non-negative integers. Let $\sigma$ be a 3-ary relation on $E \times E \times Z$ (If $e_1, e_2, i$ satisfies $\sigma$, then we write $e_1 \sigma e_2$ at $i$) which satisfies (1) $e_1$ is a substring of $e_2$ at $i$ whenever $e_2 \sigma e_1$ at $i$, and (2) $e \sigma e$ at 0 holds for any expression $e$. $e_1$ is called a subexpression of $e_2$ at occurrence $i$, or simply, a subexpression of $e_2$ if and only if $e_2 \sigma e_1$ at $i$. $e_2 \sigma e_1$ at $i$ is abbreviated as $e_2 \sigma e_1$ if there is no ambiguity. We also say $e_2$ contains $e_1$ as a subexpression if $e_2 \sigma e_1$. The relation $\sigma$ is called the syntax relation on $E$. We assume that, when a set $E$ of expressions is introduced, a syntax relation on $E$ is also introduced (either explicitly or implicitly). Section 3 describes how a set of expressions and a syntax relation on it are specified.

### 2.2 Definition of a Congruence Relation

Let $E$ be a set of expressions. A subset $R$ of $E \times E$ is called a binary relation on $E$ and we write $e R e'$ when $(e, e') \in R$. A binary relation $C$ on $E$ is said to be a congruence relation on $E$ if the following conditions (1) and (2) are satisfied.

1. $C$ is an equality relation. i.e., $C$ is reflexive, symmetric and transitive relation.
2. [Congruency] For expressions $e_1, e_1'$ and $e_2$ in $E$ such that $e_1 C e_1'$ and $e_2 \sigma e_1$,

\[
\text{if } e_2[e_1 \leftarrow e_1'], \text{ denoted } e_2', \text{ is in } E \text{ and } e_2' \sigma e_1' \text{ holds, then } e_2 C e_2' \text{ also holds.}
\]

Suppose that $e_2 \sigma e_1$, i.e., $e_1$ is a subexpression of $e_2$, and let $e_2' = e_2[e_1 \leftarrow e_1']$ (the string obtained from $e_2$ by replacing $e_1$ with $e_1'$). It does not necessarily follow that $e_2'$ is an expression in $E$. Furthermore $e_1'$ is not always a subexpression of $e_2'$ even if $e_2' \in E$. Congruency means that for expressions $e_1$ and $e_1'$ which are already known to be 'congruent with each other', and an expression $e_2$ which contains $e_1$ as a subexpression, $e_2$ and $e_2' = e_2[e_1 \leftarrow e_1']$ must also be congruent if $e_2'$ is an expression in $E$ and $e_1'$ is a subexpression of $e_2'$.

Let $R$ be a binary relation on $E$. $R$ is sometimes called a set of equations. The congruence relation on $E$ generated by $R$ (denoted by $\bar{R}$) is defined to be the least congruence relation on $E$ containing $R$. For any set $R$ of equations, $\bar{R}$ always exists and is uniquely determined [6].

### 2.3 Operational Semantics of a Congruence Relation
In this subsection, for any set $R$ of equations, the congruence relation $\hat{R}$ generated by $R$ is characterized operationally. For a relation $R$, let $R^\#$ be the reflexive-transitive closure of $R \cup R^{-1}$.

[Definition 2.1] For a set $E$ of expressions and a relation $R$ on $E$, define the relations $R_0, R_1, ..., R_\infty$ as follows.

1. $R_0 = R$
2. For each $i (i \geq 1)$, $e_2 R_i e_2'$ if and only if there exist expressions $e_1$ and $e_1'$ in $E$ such that $e_1 R_i e_1'$, $e_2 \triangleright e_1$, $e_2' = e_2[e_1 \leftarrow e_1']$ is in $E$ and $e_2' \triangleright e_1'$.
3. $e R_\infty e'$ if and only if $e R_i e'$ for some $i (i \geq 1)$.

Intuitively, $e R_i e'$ means that, when pairs of expressions in the reflexive-transitive closure of $R \cup R^{-1}$ are regarded as rewrite rules, $e'$ is obtained from $e$ by one step rewriting. Similarly, $e R_i e'$ means that, when pairs of expressions in $R_i^{-1}$ are regarded as rewrite rules, $e'$ is obtained from $e$ by one step rewriting, and $e R_\infty e'$ means that $e'$ is obtained from $e$ by one step rewriting under such rewrite rules.

[Lemma 2.1] For a set $E$ of expressions and a relation $R$ on $E$, $\hat{R} = R_\infty^\# = R_\infty$.

Let us define the relation $\overrightarrow{R}$ as follows:

$$e_2 \overrightarrow{R} e_2'$$

if and only if there exist expressions $e_1$ and $e_1'$ in $E$ such that $e_1 R e_1', e_2 \triangleright e_1, e_2' = e_2[e_1 \leftarrow e_1']$ is in $E$ and $e_2' \triangleright e_1'$.

The difference between $\overrightarrow{R}$ and $R_i$ is that only $R$ is used as a set of rewrite rules in the definition of $\overrightarrow{R}$ while $R_i^{-1}$ (i.e., reflexive-transitive closure of $R_i^{-1}$) is used in the definition of $R_i$. It is well-known that, if the syntax of expressions is restricted to the prefix form such as $f(e_1, e_2, ..., e_n)$, then $\hat{R} = (\overrightarrow{R})^\#$ holds [4], which means that the operational semantics of the congruence relation $\hat{R}$ can be defined simply. However $R_\infty^\#$ does not coincide with $(\overrightarrow{R})^\#$ in general. The reasons are as follows.

1. Even if $e_1$ is a subexpression of $e_2$ and $e_2$ is a subexpression of $e_3$, $e_1$ is not always a subexpression of $e_3$. For example, assume that $e_2 \triangleright e_1$ and $e_3 \triangleright e_2$ hold but $e_3 \triangleright e_1$ does not hold. Let $e_2' = e_2[e_1 \leftarrow e_1']$ and $e_3' = e_2[e_2 \leftarrow e_2']$ and assume that $e_2' \triangleright e_1'$ and $e_3' \triangleright e_2'$. Let $R = \{(e_1, e_1')\}$. Then, since $e_2 \triangleright e_1$ and $e_2' \triangleright e_1'$ hold by the assumption, $e_2 R_1 e_2'$ holds by definition 2.1. Similarly, $e_3 R_2 e_3'$ and hence $e_3 R_\infty^\# e_3'$ hold since $e_3 \triangleright e_2$ and $e_3' \triangleright e_2'$. On the other hand, since $e_1$ is
not a subexpression of $e_3$, $(e_1, e_1')$ cannot be applied to $e_3$ as a rewrite rule. Similarly, $(e_1', e_1)$ cannot be applied to $e_3'$. It follows that $e_3$ ($\frac{R}{R}$)$^#$ $e_3'$ does not hold and $R_{\infty}^#$ does not coincide ($\frac{R}{R}$)$^#$. 

(2) For expressions $e_1$ and $e_2$ satisfying $e_2 \bowtie e_1$, $e_2' = e_2[e_1 \leftarrow e_1'] \bowtie e_1'$ does not always hold even if $e_2'$ is an expression. For example, let $R = \{(e_1, e_1'), (e_1', e_1'')\}$. Suppose $e_2 \bowtie e_1$ and let $e_2' = e_2[e_1 \leftarrow e_1']$ and $e_2'' = e_2[e_1 \leftarrow e_1'']$. Suppose that $e_2'' \bowtie e_1''$ but $e_1'$ is not a subexpression of $e_2'$. $e_2 R_1^# e_2''$ and $e_2 R_{\infty}^# e_2''$ hold since $e_2 \bowtie e_1$, $e_2'' \bowtie e_1''$ and $e_1 R^# e_1''$ (by the definition of $R$). On the other hand, $e_1'$ is not a subexpression of $e_2'$ and so $e_2''$ (or $e_2''$) cannot be obtained from $e_2$ (or $e_2''$) by rewriting under rewrite rules in $R \cup R^{-1}$. Hence $e_2$ ($\frac{R}{R}$)$^# e_2''$ does not hold and $R_{\infty}^#$ does not coincide with ($\frac{R}{R}$)$^#$. ■

Suppose that a set $E$ of expressions satisfies the following condition.

[Syntactic Transitivity] For any expressions $e_1$, $e_2$ and $e_3$, if $e_2 \bowtie e_1$ and $e_3 \bowtie e_2$, then $e_3 \bowtie e_1$. ■

Then, in the example in (1) above, $e_3 \bowtie e_1$ and $e_3' \bowtie e_1'$ hold. Hence $(e_1, e_1') \in R$ can be applied to $e_3$ as a rewrite rule and $(e_2, e_2') \in R_1$ is not needed as a rewrite rule. For (2) above, let us assume that a set $R$ of equations on $E$ satisfies the following condition.

[Local Syntax Compatibility] For any pair $(e_1, e_1')$ in $R \cup R^{-1}$ and any expression $e_2$ containing $e_1$ as a subexpression, let $e_2' = e_2[e_1 \leftarrow e_1']$. Then $e_2'$ is also an expression and $e_2' \bowtie e_1'$. ■

Local syntax compatibility ensures that, if $e_2 \bowtie e_1$ and $(e_1, e_1') \in R \cup R^{-1}$ then $e_2' = e_2[e_1 \leftarrow e_1']$ is always an expression and $e_2' \bowtie e_1'$. Hence in the definition of $R_1$, $R$ is sufficient as a set of rewrite rules ($R^#$ is not needed). Notice that, even if a set $R$ of equations satisfies local syntax compatibility, it does not necessarily follow that $R_1$, $R_2$, ... also satisfy it. However, the latter is not required since $(\cup R_i) - R$ is not needed as rewrite rules if syntactic transitivity is satisfied. To summarize the above discussion, a sufficient condition for $\hat{R} = (\frac{R}{R})^#$ can be derived as follows.

[Definition 2.2] For a set $E$ of expressions, define the relations $\leq$ and $\approx$ as follows:

(1) $e_1 \leq e_1'$ if and only if

for any expression $e_2$ in $E$ such that $e_2 \bowtie e_1$, $e_2' = e_2[e_1 \leftarrow e_1']$ is also an expression in $E$ and $e_2' \bowtie e_1'$.

(2) $e = e' \approx e'$ if and only if $e \leq e'$ and $e' \leq e$. ■
Lemma 2.2] For a set $E$ of expressions and a relation $R$ on $E$, if the following conditions (1) and (2) are satisfied, then operational completeness $\bar{R} = (\bar{R})^\#$ holds.

(1) $E$ satisfies syntactic transitivity.
(2) $R$ satisfies local syntax compatibility "$eR e' \implies e = e'$". □

3. A Set of Expressions Generated by a Grammar

3.1 A Set of Expressions Generated by a Grammar

This subsection describes how a set of expressions together with a syntax relation (see 2.1) is defined by a context-free grammar. Let $G = (V_N, V_T, P)$ be a context-free grammar (cfg) without a start symbol, where $V_N, V_T (V_N \cap V_T = \phi, \phi$ denotes the empty set) and $P$ are a finite set of non-terminal symbols, a finite set of terminal symbols, and a finite set of productions, respectively. A production in $P$ has a form $A \rightarrow \alpha$, where $A \in V_N$ and $\alpha \in (V_N \cup V_T)^+$. Let $V = V_N \cup V_T$. We write $\alpha \xrightarrow{G} \beta$ if $\alpha = uAv, \beta = u\gamma v$ and $A \rightarrow \gamma \in P$ for some $u, v \in V^*$. Let $\xrightarrow{i}$ and $\xrightarrow{G}^*$ denote $\xrightarrow{G}^i$ and $\xrightarrow{G}^*$, respectively, and $\xrightarrow{G}, \xrightarrow{i}$ and $\xrightarrow{G}^*$ are abbreviated as $\Rightarrow, \Rightarrow^i$ and $\Rightarrow^*$, respectively if there is no ambiguity. For each $A \in V_N$, let

$$L_G(A) = \{ e \mid A \xrightarrow{G}^* e, e \in V_T^+ \}$$

and let

$$E_G = \bigcup_{A \in V_N} L_G(A).$$

For $e \in E_G$, let

$$N_G(e) = \{ A \mid A \in V_N, A \xrightarrow{G} e \}.$$

A cfg $G$ specifies $E_G$ as a set of expressions.

A derivation tree is defined to be a tree $tr$ satisfying the following conditions (1) to (3). A node with at least one child is called an internal node.

(1) Each node in $tr$ has a symbol in $V$ as the label.
(2) The label of an internal node is a non-terminal symbol in $V_N$, and, for a tree which consists of a single node $r$, the label of $r$ is a non-terminal symbol.
(3) If a node $n$ has children $n_1, n_2, \ldots, n_k$ from left to right, and the labels of $n, n_1, n_2, \ldots, n_k$ are $X, X_1, X_2, \ldots, X_k$, respectively, then $X \rightarrow X_1X_2 \ldots X_k \in P$. □

For a derivation tree $tr$, $tr$ is called a derivation tree for $\alpha$, where $\alpha$ is the string obtained by concatenating the labels of the leaves in $tr$ from left to right.
If \( tr \) is a derivation tree for \( \alpha \) and the label of the root of \( tr \) is \( A \), then we write \( tr : A \xrightarrow{G} \alpha \) by using the same notation as for derivations. Similarly, if the number of the internal nodes of a derivation tree \( tr \) is \( i \), then \( tr \) may be written as \( tr : A \xrightarrow{G} \alpha \). For a string \( \alpha \in E_G \), a substring \( \beta \) of \( \alpha \) and a derivation tree \( tr : A \xrightarrow{G} \alpha \) for \( \alpha \), if \( tr \) contains a derivation tree for \( \beta \) as a subtree, then \( \beta \) is said to be derived from a non-terminal symbol in \( tr \).

A cfg \( G \) specifies the following relation \( \beta \) on \( E_G \) as a syntax relation:

\[
e_2 \beta e_1 \text{ if and only if}
\]

there exists a derivation tree \( tr \) for \( e_2 \) such that \( e_1 \) is derived from a non-terminal symbol in \( tr \).

For a given cfg \( G \), a set of expressions together with a syntax relation can be specified in a natural way as above by using the phrase structures defined by \( G \). Unfortunately, however, it is undecidable whether \( e = e' \) (see 2.3) holds in general:

[Lemma 3.1] It is undecidable for a given cfg \( G \) and expressions \( e \) and \( e' \) in \( E_G \) whether \( e = e' \) holds.

(Proof) For a given cfg \( G = (V_N, V_T, P) \) and non-terminal symbols \( A_1 \) and \( A_2 \) in \( V_N \), we can construct another cfg \( G' \) such that \( e_1 = e_2 \) holds for two distinguished expressions \( e_1 \) and \( e_2 \) in \( E_G \) if and only if \( L_G(A_1) = L_G(A_2) \), which is undecidable. Let \( G' = (V'_N, V'_T, P') \), where

\[
V'_N = V_N \cup \{ S, H_1, H_2 \} \quad (V_N \cap \{ S, H_1, H_2 \} = \phi),
V'_T = V_T \cup \{ e_1, e_2 \} \quad (V_T \cap \{ e_1, e_2 \} = \phi),
P' = P \cup \{ S \rightarrow H_1 A_1, S \rightarrow H_2 A_2, H_1 \rightarrow e_1, H_2 \rightarrow e_2 \}.
\]

Then, it is easy to see that

\[
e_1 = e_2 \text{ if and only if } L_G(A_1) = L_G(A_2) .
\]

3.2 A cfg with Unambiguous Structure

In general, for a cfg \( G \) and an expression \( e = ue_1v \), there may be derivation trees \( tr1 \) and \( tr2 \) for \( e \) such that \( e_1 \) is derived from a non-terminal symbol in \( tr1 \) while it is not the case in \( tr2 \).

[Definition 3.1] Let \( G \) be a cfg \( (V_N, V_T, P) \), and let \( tr1 \) and \( tr2 \) be derivation trees for \( \alpha \) and \( \beta \), respectively, where \( \alpha, \beta \in (V_N \cup V_T)^* \) and \( |\alpha| = |\beta| \). \( tr1 \) and \( tr2 \) are said to have the same structure if the following condition holds:

Let \( \gamma \) be a substring of \( \alpha \) and let \( \alpha = \zeta \gamma \eta \) and \( \beta = \zeta' \delta \eta' \) where \( |\zeta'| = |\zeta| \) and \( |\eta'| = |\eta| \). \( \gamma \) is derived from a non-terminal symbol in \( tr1 \) if and only if \( \delta \) is derived from a non-terminal symbol in \( tr2 \).
For a cfg $G$, if, for each expression $e$ in $E_G$, all the derivation trees for $e$ have the same structure, then we say that $G$ has unambiguous structure, or $G$ is with unambiguous structure. If $G$ has unambiguous structure and an expression $e_1$ is a subexpression of an expression $e_2$, then $e_1$ is always derived from a non-terminal symbol in every derivation tree for $e_2$.

[Lemma 3.2] It is undecidable for a given cfg $G$ whether $G$ has unambiguous structure.

(Proof) For a given instance $I$ of Post's correspondence problem, a cfg $G$ can be effectively constructed such that $G$ has unambiguous structure if and only if $I$ does not have any solution. ■

[Lemma 3.3] If a cfg $G$ has unambiguous structure then $E_G$ satisfies syntactic transitivity (see 2.3).

(Proof) Obvious by the definition of 'unambiguous structure'. ■

3.3 Global Syntax Compatibility

A Sufficient Condition for Local Syntax Compatibility

We have already given a sufficient condition for a set $R$ of equations to satisfy $\hat{R} = (R)^\#$ as lemma 2.2. The condition requires that $R$ satisfies local syntax compatibility "$R \subseteq =$". As will be described in 4.1, a text in ASL/* specifies a set $R$ of equations by a set of axioms. However, it is undecidable for a given text $t$ in ASL/* whether the set of equations specified by $t$ satisfies local syntax compatibility (see lemma 4.1).

By the way, the sufficient condition in lemma 2.2 also requires that the set of expressions under consideration satisfies syntactic transitivity. In 4.2, we assume that a cfg $G$ which specifies a set $E_G$ of expressions has unambiguous structure so that $E_G$ satisfies syntactic transitivity (we believe the assumption is a reasonable one as will be explained in 4.2). However, to the best of the authors' knowledge, even if a cfg is assumed to have unambiguous structure, it is open whether it is decidable or not for a text $t$ whether $R \subseteq =$ holds, where $R$ is the set of equations specified by $t$. If the problem is decidable, it becomes also decidable for given cfg's $G_1, G_2$ with unambiguous structures whether $E_{G_1} = E_{G_2}$. Since the class of cfl's generated by cfg's with unambiguous structures contains the class of deterministic cfl's, and the equality problem for deterministic cfl's has been one of the most famous open problems, it seems to be difficult to know whether the problem "$R \subseteq =$" is decidable or not.
Considering above discussions, we will define a relation, denoted by $\emptyset$, which satisfies $\emptyset \subseteq \prec$. Hence $R \subseteq \prec$ if $R \subseteq \emptyset$. Furthermore, it will be shown in 4.2 that it is decidable for a text $t = (G, AX)$ whether $R \subseteq \emptyset$ under the assumption that $G$ has unambiguous structure. Intuitively, $e_1 \emptyset e_1'$ holds if and only if the following two conditions hold:
(1) $e_1 = e_1'$.
(2) For any expression $e_2$ containing $e_1$ as a subexpression, let $e_2' = e_2[e_1 \leftarrow e_1']$ ($e_2'$ is always an expression by (1)). Then, for any derivation tree $tr$ for $e_2$, there exists a derivation tree $tr'$ for $e_2'$ such that $tr$ and $tr'$ have the same structure "outside of " the subtrees for $e_1$ and $e_1'$. That is, $tr$ and $tr'$ have the same structure when pruning all the nodes other than the roots in the subtrees for $e_1$ and $e_1'$, and all the edges in them. Similar condition holds also for any $e_3$ containing $e_1'$ as a subexpression.

To define the relation $\emptyset$ on $E_G$ more precisely, we first define the relation, denoted by $\prec$ (or $\prec$, if no ambiguity), on the powerset of the set of non-terminal symbols of $G$ as follows.

[Definition 3.2] For a cfg $G = (V_N, V_T, P)$ and subsets $N_1$ and $N_2$ of $V_N$,
$N_1 \prec N_2$ if and only if
for any derivation tree
$tr1 : A \stackrel{\ast}{\Rightarrow}_G uBv \ (A \in V_N, B \in N_1, u, v \in V_T^*)$,
there exists a derivation tree
$tr2 : C \stackrel{\ast}{\Rightarrow}_G uDv \ (C \in V_N, D \in N_2)$
such that $tr1$ and $tr2$ have the same structure.

[Definition 3.3] For a cfg $G$, define the relations $\leq$ and $\emptyset$ on $E_G$ as follows:
(1) $e \leq e'$ if and only if $N_G(e) \prec N_G(e')$.
(2) $e \emptyset e'$ if and only if $e \leq e'$ and $e' \leq e$.

The property "$R \subseteq \emptyset$" is called global syntax compatibility. Obviously, global syntax compatibility implies local syntax compatibility.

[Lemma 3.4] It is decidable for a given cfg $G = (V_N, V_T, P)$ and subsets $N_1, N_2$ of $V_N$ whether $N_1 \prec N_2$ holds.

(Proof) We can decide whether $N_1 \prec N_2$ by a technique similar to the one used in the proof of Theorem 1 and 3 in [7]. For simplicity, suppose $L_G(A) \neq \emptyset$ for each non-terminal symbol $A$ in $V_N$. The decision algorithm stated below can be extended easily so as to be applied to a cfg which may have a non-terminal $A$ such that $L_G(A) = \emptyset$. For a given cfg $G = (V_N, V_T, P)$ and subsets $N_1, N_2$ of $V_N$, execute the next procedure, PROC1. $N_1 \prec N_2$ holds if the output is "YES" and $N_1 \prec N_2$ does not hold if "NO".

[PROC1]
(Input) A cfg $G = (V_N, V_T, P)$ and subsets $N_1, N_2$ of $V_N$

(Output) "YES" or "NO"

(Step 1) Construct a cfg $G_1 = (V_N, V_T, P_1)$ from $G$, where

$P_1 = \{ A \rightarrow \alpha | A, B \in V_N, A \xrightarrow{\delta} B, B \rightarrow \alpha \in P \text{ and } B \rightarrow \alpha \text{ is not a unit production (i.e., } \alpha \text{ is not a string consisting of a single non-terminal symbol)} \}.$

By the definition of $G_1$, $E_{G_1} = E_G$, $G_1 = G_2$ and $G_1$ has no unit production.

(Step 2) Construct a cfg $G_2 = (V_{N_2}, V_T, P_2)$, where

1. $V_{N_2}$ consists of all the (newly introduced) non-terminal symbols each of which corresponds (one to one) to a non-empty subset of $V_N$. For each $B \in V_{N_2}$, let $\sigma(B)$ denote the subset of $V_N$ corresponding to $B$.
2. $B \rightarrow u_0 B_1 u_1 ... B_n u_n \in P_2$ ($B, B_1, ..., B_n \in V_{N_2}, u_0, u_1, ..., u_n \in V_T^*$), if and only if $\sigma(B)$ consists of exactly all the non-terminal symbols $A \in V_N$ such that there exists $A_i \in \sigma(B_i)$ ($1 \leq i \leq n$) and $A \rightarrow u_0 A_1 u_1 ... A_n u_n \in P_1$.

(Step 3) Construct a cfg $G_3 = (V_{N_3}, V_T, P_3)$ from $G_2$ by removing from $G_2$ (1) useless non-terminal symbols (i.e., which do not generate any expression), and (2) productions which have at least one useless non-terminal symbol in left-hand side or right-hand side. Details are described in [3].

(Step 4) If the following condition holds, output "YES", and otherwise, output "NO".

(Condition) Let $m = 2 | V_{N_3} | - 1$. Let $C_1$ be an arbitrary non-terminal symbol of $G_3$ such that $A_1 \in \sigma(C_1)$ for some $A_1 \in N_1$. For any derivation tree $tr_1 : B_1 \xrightarrow{G_3} \alpha C_1 \beta$ ($B_1 \in V_{N_3}, \alpha, \beta \in (V_{N_3} \cup V_T)^*, i \leq m$), there exists a derivation tree $tr_2 : B_2 \xrightarrow{G_3} \alpha C_2 \beta$ ($B_2, C_2 \in V_{N_3}$) which satisfies the following conditions (1) and (2):

1. $A_2 \in \sigma(C_2)$ for some $A_2 \in N_2$,
2. $tr_1$ and $tr_2$ have the same structure.

[Corollary 3.5] It is decidable for a given cfg $G$ and expressions $e, e'$ in $E_G$ whether $e \preceq e'$, and hence so is whether $e \preceq e'$. (Proof) Obvious from lemma 3.4 and the definitions of $\preceq$ and $\preceq$. ■

4. **A Congruence Relation Generated by Axioms**

4.1 **The Algebraic Language ASL**
ASL/* is an algebraic description language used for writing programs and/or their specifications. A pair \((G, AX)\) is called a text in ASL/*, where \(G\) is a cfg without a start symbol and \(AX\) is a finite set of axioms defined below. A text \(t = (G, AX)\) specifies (a) a set of expressions together with a syntactic relation by \(G\), and (b) a set of equations (and, in turn, a congruence relation) by \(AX\) as follows:

(a) A text \(t = (G, AX)\) in ASL/* specifies \(E_G\) as a set of expressions and \(\overline{Q}_t\) as a syntax relation on \(E_G\) (see 3.1).

(b) Let \(V_{var}\) be a set of variables such that \(V_{var} \cap (V_N \cup V_T) = \emptyset\). For each variable \(x\) in \(V_{var}\), a non-terminal symbol in \(V_N\), denoted by \(M_x\), is associated with \(x\). \(M_x\) is called the type of \(x\). Let 

\[ G(V_{var}) = (V_N, V_T \cup V_{var}, P \cup \{ M_x \to x \mid x \in V_{var} \} ). \]

An expression in \(E_G(V_{var})\) is called an expression with variables (with respect to \(V_{var}\)).

Let \((x_1/u_1, x_2/u_2, \ldots, x_n/u_n)\) denote the substitution which substitutes strings \(u_1, u_2, \ldots, u_n\) for variables \(x_1, x_2, \ldots, x_n\), respectively. For a string \(u\) and a substitution \(\theta\), let \(u\theta\) denote the string obtained from \(u\) by the substitution \(\theta\).

An axiom in ASL/* consists of

(1) variables \(x_1, x_2, \ldots, x_n\) and their types \(M_{x_1}, M_{x_2}, \ldots, M_{x_n}\),

and

(2) a pair \((l, r)\) of expressions with variables (w.r.t. \((x_1, x_2, \ldots, x_n)\)).

An axiom is sometimes written as

\[ x_1 : M_{x_1}, x_2 : M_{x_2}, \ldots, x_n : M_{x_n}, \  l = r. \]

Let \(t = (G, AX)\) be a text in ASL/*. Let \(ax\) be an axiom in \(AX\) which is

\[ x_1 : M_{x_1}, x_2 : M_{x_2}, \ldots, x_n : M_{x_n}, \  l = r. \]

Define \(Q_t(ax)\) as

\[ Q_t(ax) = \{ (l\theta, r\theta) \mid \theta = (x_1/e_1, x_2/e_2, \ldots, x_n/e_n) \text{ is a substitution satisfying } e_i \in L_G(M_{x_i}) \ (1 \leq i \leq n) \}. \]

and let

\[ Q_t = \bigcup_{ax \in AX} Q_t(ax). \]

\(Q_t\) is called the set of equations specified by the text \(t\). The congruence relation \(\overline{Q}_t\) generated by \(Q_t\) (see 2.2) is called the congruence relation specified by \(t\) and denoted by \(\equiv_t\). If \(e \equiv_t e'\) holds for expressions \(e\) and \(e'\) in \(E_G\), then we say that \(e\) is congruent with \(e'\) in \(t\).

4.2 A Sufficient Condition for a Text to Satisfy Operational Completeness
Let $t = (G, AX)$ be a text in ASL/*. Since $t$ specifies $Q_t$ as a set of equations and $\equiv_t (= \bar{Q}_t)$ as a congruence relation, operational completeness means $\equiv_t = (\bar{Q}_t)^\#$. By lemma 2.2, $\equiv_t = (\bar{Q}_t)^\#$ holds if $E_G$ satisfies syntactic transitivity, and $E_G$ and $Q_t$ satisfy local syntax compatibility "$Q_t \subseteq \approx". However, the latter is undecidable in general.

[Lemma 4.1] It is undecidable for a given text $t = (G, AX)$ in ASL/* whether $E_G$ and $Q_t$ satisfy local syntax compatibility "$Q_t \subseteq \approx"

(Proof) For a given cfg $G$ and expressions $e, e'$ in $E_G$, construct a text $t = (G, AX)$, where $AX$ consists of only one axiom $e = e'$, which contains no variables. Since $Q_t = \{ (e, e') \}$ by the definition of $t$, $Q_t \subseteq \approx$ if and only if $e = e'$, which is undecidable by lemma 3.1.

In the following, a sufficient condition is given for a text $t$ in ASL/* to satisfy operational completeness $\equiv_t = (\bar{Q}_t)^\#$ with the help of lemma 2.2.

Hereafter, for a text $t = (G, AX)$, $G$ is assumed to have unambiguous structure, which we believe is a reasonable assumption for the following reasons:

(1) If $G$ has unambiguous structure, then $E_G$ satisfies syntactic transitivity (lemma 3.3).

(2) As mentioned in 3.3, global syntax compatibility "$R \subseteq 0$" implies local syntax compatibility "$R \subseteq \approx". As shown in lemma 4.2, it is decidable for a text $t = (G, AX)$ to satisfy global syntax compatibility "$Q_t \subseteq 0$" under the assumption that $G$ has unambiguous structure.

(3) Even if the class of cfg's used for specifying the syntax of expressions is restricted to the class of cfg's with unambiguous structure, we can write specifications (texts) in a natural way which deal with error handling (see example 4.1 at the end of this section, where the 'popping' operation on the empty stack are considered), or sort inclusion (such as integer and real). For example, the syntax of reals (including integers) can be specified by a cfg with unambiguous structure as follows:

```
real → real + real
real → 0.1
real → int
int → int + int
int → 0
```

On the other hand, if the class of cfg's is restricted to a smaller class than the class of cfg's with unambiguous structure, e.g., to the class of unambiguous cfg's, then the grammar for specifying the syntax of expressions in such a text as the example above becomes complicated.
[Lemma 4.2] Let $t = (G, AX)$ be a text in ASL/*, where $G$ has unambiguous structure. It is decidable whether $t$ satisfies global syntax compatibility "$Q_t \subseteq \emptyset".

(Proof) Let $G = (V_N, V_T, P)$. By the definition of $Q_t$, global syntax compatibility is logically equivalent to

(a) for each $ax$ in $AX$, $e Q_t(ax) e'$ implies $e \emptyset e'$.

Let $AX^{-1}$ be the set of the axioms in $AX$, left-hand sides and right-hand sides interchanged. Then, (a) in turn is equivalent to

(b) for each $ax$ in $AX \cup AX^{-1}$, $e Q_t(ax) e'$ implies $e \subseteq e'$.

Since $AX$ is finite, it is enough to show that it is decidable for a given axiom $ax$ whether $e Q_t(ax) e'$ implies $e \subseteq e'$. Let $ax$ be an axiom in $AX$ which is

$x_1 : M_{x_1}, x_2 : M_{x_2}, \ldots, x_n : M_{x_n}, \ l == r$.

Then the condition (b1) holds if and only if

(c) for any substitution $\theta = \{x_1 / e_1, x_2 / e_2, \ldots, x_n / e_n\}$ satisfying $e_i \in L_G(M_{x_i})$ $(1 \leq i \leq n)$, $N_G(l \theta) \overset{G}{=} N_G(r \theta)$.

Let

$\Sigma = \{(N_G(l \theta), N_G(r \theta)) \mid \theta = \{x_1/e_1, x_2/e_2, \ldots, x_n/e_n\} \text{ is an arbitrary substitution satisfying } e_i \in L_G(M_{x_i}) \ (1 \leq i \leq n)\}$.

Then, the condition (c) can be paraphrased as

(c') for each pair $(S_1, S_2)$ in $\Sigma$, $S_1 \overset{G}{=} S_2$.

Although the number of substitutions $\theta$'s in (c) is infinite in general, $\Sigma$ is finite since it is a set of subsets of $V_N \times V_N$. $\Sigma$ is effectively obtained as follows.

For each variable $x_i \ (1 \leq i \leq n)$, let

$\hat{N}_x = \{N_G(e_i) \mid e_i \in L_G(M_{x_i})\}$

and for the left-hand side $l$ of $ax$ and sets of non-terminal symbols $N_1, N_2, \ldots, N_n$ satisfying $N_i \in \hat{N}_x \ (1 \leq i \leq n)$, let

$N[l, N_1, N_2, \ldots, N_n] = \{A \mid \text{there exists } \hat{l} \text{ such that } A \overset{G}{*} \hat{l}, \text{ where } \hat{l} \text{ is obtained from } l \text{ by replacing each variable } x_i \text{ with } N_i \ (\text{distinct occurrences of the same variable } x_i \text{ may be replaced with distinct non-terminal symbols in } N_i)\}$

and define $N[r, N_1, N_2, \ldots, N_n]$ for the right-hand side $r$ in the same way.

Since $G$ has unambiguous structure, $\Sigma$ coincides with

$\{(N[l, N_1, N_2, \ldots, N_n], N[r, N_1, N_2, \ldots, N_n]) \mid N_i \in \hat{N}_x (1 \leq i \leq n)\}$.

Let $G_2$ be a CFG obtained by executing Steps 1 and 2 of PROC1 in lemma 3.4 with $G$ as input. Let

$\hat{N}_x' = \{\sigma(B) \mid M_{x_i} \in \sigma(B) \text{ and } L_{G_2}(B) \neq \emptyset\}$.
for each $i$ $(1 \leq i \leq n)$, then $\tilde{N}_{x_i} = \tilde{N}_{x_i}'$ holds for each $i$ $(1 \leq i \leq n)$ since $G$ has unambiguous structure. Hence $\Sigma$ can be effectively obtained. To summarize above arguments, we can decide whether $Q_t \subseteq \emptyset$ as follows: Execute the following procedure PROC2 with a given text $t = (G, AX)$ as input. If the answer is "YES" then $Q_t \subseteq \emptyset$, and otherwise, $Q_t \subseteq \emptyset$ does not hold.

[PROC2]
(Input) A text $t = (G, AX)$ in $\text{ASL/*}$
(Output) "YES" or "NO"

(Step 1) Execute Steps 1 and 2 of PROC1 in the proof of lemma 3.4 with $G$ as input. Let $G_2$ be a cfg constructed in Step 2 of PROC1.

(Step 2) For each axiom $ax$ in $AX \cup AX^{-1}$, execute Steps 3 and 4 below. If the answer is always "YES", then output "YES", and otherwise, output "NO".

(Step 3) Let a given axiom be $x_1:M_{x_1}, x_2:M_{x_2}, \ldots, x_n:M_{x_n}, l == r$. Compute $\tilde{N}_{x_i}$ for each $i$ $(1 \leq i \leq n)$, where

$\tilde{N}_{x_i} = \{ \sigma(B) \mid M_{x_i} \in \sigma(B) \text{ and } L_{G_2}(B) \neq \emptyset \}$.

(Step 4) For each $N_i \in \tilde{N}_{x_i} (1 \leq i \leq n)$, determine whether

$N[l, N_1, N_2, \ldots, N_n] \in G[N[r, N_1, N_2, \ldots, N_n]}$ (*)

which is decidable by lemma 3.4. If (*) holds for every pair $N_1, N_2, \ldots, N_n$, then output "YES", and otherwise, output "NO". ■

Now, the main theorem can be derived as a corollary of lemma 2.2, lemma 3.3 and lemma 4.2.

[Theorem 4.3] Let $t = (G, AX)$ be a text in $\text{ASL/*}$, where $G$ has unambiguous structure. It is decidable whether $t$ satisfies global syntax compatibility $Q_t \subseteq \emptyset$ which implies operational completeness $\mathcal{C} = (Q_t)^\#$. ■

[Example 4.1] Figure 1 is a text $t = (G, AX)$ in $\text{ASL/*}$ which defines a data type, stack of non-negative integers, such as:

Popping the empty stack results in the error state (denoted by $\text{ERRSTACK}$) and no operation is defined on $\text{ERRSTACK}$. If one tries to know the top of the empty stack, the distinguished value, $\text{ERRINT}$, is returned.

Lines (1) to (14) define the productions of $G$. The symbols appearing in the left-hand side of some production are non-terminal symbols, and the other symbols appearing in the right-hand sides of the productions are terminal symbols. Lines (17) to (22) define the axioms in $AX$. Line (15) and (16) defines that the type of variables $s$ and $i$ are $\text{stack}$ and $\text{int}$, respectively.
(1) stack \rightarrow \text{NEW\_STACK}
(2) stack \rightarrow \text{PUSH}(stack, \text{int})
(3) stack\&err \rightarrow \text{stack}
(4) stack\&err \rightarrow \text{ERRSTACK}
(5) stack\&err \rightarrow \text{PUSH}(stack\&err, \text{int})
(6) stack\&err \rightarrow \text{POP}(stack\&err)
(7) int \rightarrow 0
(8) int \rightarrow \text{SUCC}(\text{int})
(9) int\&err \rightarrow \text{int}
(10) int\&err \rightarrow \text{ERRINT}
(11) int\&err \rightarrow \text{TOP}(stack\&err)
(12) bool \rightarrow \text{TRUE}
(13) bool \rightarrow \text{FALSE}
(14) bool \rightarrow \text{ISNEW}(stack\&err)
(15) stack s
(16) int i
(17) \text{ISNEW(NEW\_STACK)} == \text{TRUE}
(18) \text{ISNEW(PUSH(s, i))} == \text{FALSE}
(19) \text{TOP(NEW\_STACK)} == \text{ERRINT}
(20) \text{TOP(PUSH(s, i))} == i
(21) \text{POP(NEW\_STACK)} == \text{ERRSTACK}
(22) \text{POP(PUSH(s, i))} == s

Figure 1 A Specification of Stack with Error State
Non-terminal symbol int generates all the non-negative integers; 
$L_G(int) = \{ \text{SUCC}^n(0) \mid n \geq 0 \}$. Let $n \geq 0$ denote $\text{SUCC}^n(0)$ for notational simplicity. int&err generates all the non-negative integers together with the distinguished value ERRINT, i.e., $L_G(int&err) = L_G(int) \cup \{ \text{ERRINT} \}$. Similarly, stack generates all the expressions each of which denotes a legal state of the stack and $L_G(stack&err) = L_G(stack) \cup \{ \text{ERRSTACK} \}$.

Operations ISNEW, POP, and TOP are defined only on the expressions in $L_G(stack)$. For example,

$TOP(PUSH(NEW_STACK, 3))$

is congruent with 3 by the axiom (20) while

$TOP(PUSH(POP(NEW_STACK), 3)) \quad (*)$

is not congruent with any $n \geq 0$. That is, the value of (*) is not defined, which means that no operation is defined once the stack goes into error state.

Let us execute PROC2 with the text shown in Figure 1 as input. By executing Steps 1 to 3, we know

$\hat{N}_s = \{ \text{stack, stack} & \text{err} \}$

$\hat{N}_i = \{ \text{int, int} & \text{err} \}$

If we execute Step 4 for the axiom (18), only the pair $S_1 = \{ \text{bool} \}$ and $S_2 = \{ \text{bool} \}$ is obtained and both $S_1 \propto S_2$ and $S_2 \propto S_1$ hold obviously. Similar arguments hold for (17), (19) and (21). For (20), the pair $S_1 = \{ \text{int} & \text{err} \}$ and $S_2 = \{ \text{int, int} & \text{err} \}$ is obtained and $S_1 \propto S_2$ holds while $S_2 \propto S_1$ does not hold. The reason why the latter does not hold is that PUSH can take as its second argument only expressions in $L_G(int)$ (and so is the first argument of SUCC). For example, if in the expression $PUSH(NEW.Stack, 3)$, the subexpression 3 is replaced with the congruent expression $TOP(PUSH(NEW.Stack, 3))$, the resulting string

$PUSH(NEW.Stack, TOP(PUSH(NEW.Stack, 3)))$

is not an expression. The reason is that 3 is in $L_G(int)$ while $TOP(PUSH(NEW_STACK, 3))$ is not. For (22), we obtain the pair $S_1 = \{ \text{stack} & \text{err} \}$ and $S_2 = \{ \text{stack, stack} & \text{err} \}$ and in this case both $S_1 \propto S_2$ and $S_2 \propto S_1$ hold. The reason why the latter holds is that every operation which can take an expression in $L_G(stack)$ as its argument can also take an expression in $L_G(stack&err)$ as its argument.

The text in Figure 1 does not satisfy global syntax compatibility since $\{ \text{int, int} & \text{err} \} \not\propto \{ \text{int} & \text{err} \}$ does not hold. For a text $t = (G, AX)$ which does not satisfy global syntax compatibility, by adding certain productions to the set of productions of $G$, $t$ may be transformed into a text $t' = (G', AX)$ such that

(#1) $t'$ satisfies global syntax compatibility, and

(#2) for any expressions $e$ and $e'$ in $E_G$, $e \equiv_t e'$ if and only if $e \equiv_t e'$. 

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Let \( t' = (G', AX) \), where \( AX \) is the same as in Figure 1 and \( G' \) is obtained from \( G \) by adding the following productions to the set of productions of \( G \):

\[
\begin{align*}
\text{stack} & \quad \rightarrow \text{PUSH}(\text{stack}, \text{int}&\text{err}) \\
\text{stack}&\text{err} & \quad \rightarrow \text{PUSH}(\text{stack}&\text{err}, \text{int}&\text{err}) \\
\text{int}&\text{err} & \quad \rightarrow \text{SUCC}(\text{int}&\text{err})
\end{align*}
\]

Then, \( \{ \text{int}, \text{int}&\text{err} \} \not\subseteq \{ \text{int}&\text{err} \} \) holds, and the conditions (#1) and (#2) are satisfied although \( E_G \) is properly contained in \( E_{G'} \), e.g., \( \text{PUSH(NEW\_STACK, ERRINT)} \) is in \( E_{G'} \) while \( E_G \) does not contain it.

In general, if a text \( t = (G, AX) \) is transformed into a text \( t' = (G', AX) \) by adding certain productions, \( E_G \subseteq E_{G'} \) and \( \mathfrak{X}_G \subseteq \mathfrak{X}_{G'} \) always hold but \( E_{G'} \subseteq E_G \) and \( \mathfrak{X}_{G'} \subseteq \mathfrak{X}_G \) are not always true. It is under study to investigate a transformation from a given text into a text which satisfies the conditions (#1) and (#2) above.

5. Conclusion

In this paper, a congruence relation is extended on a set of expressions generated by a cfg so that the semantics of specifications or programs which deal with error handling and/or sort inclusion can be defined in a simple algebraic framework. Next, a sufficient condition is shown under which the operational semantics of the congruence relation generated by a set of axioms can be defined simply in the sense that operational completeness holds. Although only unconditional axioms are considered in this paper, conditional axioms can also be used in ASL/*. A conditional axiom has a form such as

\[
\begin{align*}
x_1:M_{x_1}, x_2:M_{x_2}, \ldots, x_n:M_{x_n}, \quad l_1 = r_1, \quad l_2 = r_2, \quad \ldots, \quad l_m = r_m \Rightarrow l = r
\end{align*}
\]

which means "if \( l_1, l_2, \ldots, \) and \( l_m \) are congruent with \( r_1, r_2, \ldots, \) and \( r_m \), respectively, then \( l \) must be congruent with \( r \)." The congruence relation generated by a text which contains conditional axioms can be defined as simply as in the unconditional case [6]. Intuitively, the operational semantics of the above axiom is that, if \( r_1, r_2, \ldots, \) and \( r_m \) are obtained by rewriting from \( l_1, l_2, \ldots, \) and \( l_m \), respectively, then \( l \) can be rewritten as \( r \). It is an open problem to find a sufficient condition under which the operational semantics of the congruence relation generated by a text which contains conditional axioms can be defined simply.

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References


