

Zero-viscosity Limit of the incompressible
Navier-Stokes Equation 2

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1. Problem and Result

It has been known that the solution $u(\nu, t, x)$ of the initial value problem for the incompressible Navier-Stokes equation tends to the (unique) solution $u(0, t, x)$ of the initial value problem for the incompressible Euler equation, as the viscosity coefficient $\nu > 0$ tends to zero. Moreover it is a smooth function of $\nu \in [0, 1]$ in some function spaces. For example, see [4], [3] and [1].

We consider the same problem for the initial boundary value problem (I.B.V.P.) in the half space $R_+^n = \{x = (x_1, \dots, x_{n-1}, x_n) = (x', x_n) ; x_n > 0\}$, $n \geq 2$. There has been no result for this problem, though the boundary layer originated by Prandtl in 1905 provides a good approximation method.

Let $u = {}^t(u_1, \dots, u_n) = {}^t(u', u_n) = u(\nu, t, x)$ be the velocity of the fluid at the time $t \geq 0$ and the point $x \in R_+^n$. The I.B.V.P. for the incompressible Navier-Stokes equation is written as follows :

$$(1.1) \quad (1) \quad \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = 0, \quad t > 0, \quad x_n \in R_+^n,$$

$$(2) \quad \nabla \cdot u = 0,$$

$$(3) \quad u|_{t=0} = u_0,$$

$$(4) \quad \gamma u \equiv u|_{x_n=0} = 0.$$

Here $\nu \in (0, 1]$ is the viscosity coefficient, $u \cdot \nabla u = u_1 \partial_1 u_1 + \dots + u_n \partial_n u_n$,
 $\nabla \cdot u = \partial_1 u_1 + \dots + \partial_n u_n$, $\Delta u = \partial_1^2 u + \dots + \partial_n^2 u$ and $\nabla p = (\partial_1 p, \dots, \partial_n p)$.

Similarly the I.B.V.P. for the incompressible Euler equation is written as

$$(1.2) \quad \begin{aligned} (1) \quad & \partial_t u + u \cdot \nabla u + \nabla p = 0, \quad t > 0, \quad x \in R_+^n, \\ (2) \quad & \nabla \cdot u = 0, \\ (3) \quad & u|_{t=0} = u_0, \\ (4) \quad & \gamma_n u \equiv u_n|_{x_n=0} = 0. \end{aligned}$$

As for the initial data u_0 , we assume the "compatibility":

$$(1.3) \quad \begin{aligned} (1) \quad & \nabla \cdot u_0 = 0, \\ (2) \quad & \gamma u = 0. \end{aligned}$$

We intend to get the solution of (1.1) in the following form:

$$(1.4) \quad \begin{aligned} u(\nu, t, x) &= u^0(\nu, t, x) + \varepsilon u^1(\varepsilon, t, x) + \varepsilon^2 u^2(\varepsilon, t, x) \\ &+ \tilde{u}^0(\varepsilon, t, x, x_n/\varepsilon) + \varepsilon \tilde{u}^1(\varepsilon, t, x, x_n/\varepsilon) + \varepsilon^2 \tilde{u}^2(\varepsilon, t, x, x_n/\varepsilon), \\ \tilde{u}^i(\varepsilon, t, x, x_n/\varepsilon) &= \begin{pmatrix} \tilde{u}^i(\varepsilon, t, x, x_n/\varepsilon) \\ \varepsilon \tilde{u}_n^i(\varepsilon, t, x, x_n/\varepsilon) \end{pmatrix}, \quad i = 0, 1, \\ \tilde{u}^2(\varepsilon, t, x, x_n/\varepsilon) &= {}^t(\tilde{u}^2(\dots, x_n/\varepsilon), \tilde{u}_n^2(\dots, x_n/\varepsilon)), \\ p(\nu, t, x) &= p^0(\nu, t, x) + \varepsilon p^1(\varepsilon, t, x) + \varepsilon^2 p^2(\varepsilon, t, x) \\ &+ \varepsilon \tilde{p}^1(\varepsilon, t, x, x_n/\varepsilon) + \varepsilon^2 \tilde{p}^2(\varepsilon, t, x, x_n/\varepsilon), \\ \varepsilon &= \sqrt{\nu} \in (0, 1]. \end{aligned}$$

Each term in the above expansion is determined to satisfy the following "Navier-Stokes equations" (1.5)-(1.9), respectively:

$$(1.5) \quad \begin{aligned} \partial_t u^0 + u^0 \cdot \nabla u^0 - \nu \Delta u^0 + \nabla p^0 &= 0, \quad t > 0, \quad x \in R_+^n, \\ \nabla \cdot u^0 &= 0, \\ u^0|_{t=0} &= u_0, \\ \gamma_n u^0 &= 0, \end{aligned}$$

$$(1.6) \quad \partial_t \tilde{u}^0 + (u^0 + \tilde{u}^0) \cdot \nabla \tilde{u}^0 + (u_n^0 + \varepsilon \tilde{u}_n^0 - \varepsilon \gamma \tilde{u}_n^0) \partial_n \tilde{u}^0 - \nu \Delta \tilde{u}^0 + \tilde{u}^0 \cdot \nabla u^0 = 0, \\ \tilde{u}_n^0(\varepsilon, t, x) = -\partial_n^{-1} \nabla \cdot \tilde{u}^0(\varepsilon, t, x) \equiv \int_{x_n}^{\infty} \nabla \cdot \tilde{u}^0(\varepsilon, t, x', \eta_n) d\eta_n,$$

$$\nabla \equiv {}^t(\partial_1, \dots, \partial_{n-1}) \equiv \partial',$$

$$\tilde{u}^0|_{t=0} = 0,$$

$$\gamma \tilde{u}^0 \equiv \tilde{u}^0|_{x_n=0} = -\gamma u^0,$$

$$(1.7) \quad \partial_t u^1 + u^0 \cdot \nabla u^1 - \nu \Delta u^1 + u^1 \cdot \nabla u^0 + \nabla p^1 = 0,$$

$$\nabla \cdot u^1 = 0,$$

$$u^1|_{t=0} = 0,$$

$$\gamma_n u^1 = -\gamma_n \tilde{u}^0,$$

$$(1.8) \quad (\partial_t + (u^0 + \varepsilon u^1 + \varepsilon^2 u^2) \cdot \nabla - \nu \Delta) u^2 + u^2 \cdot \nabla (u^0 + \varepsilon u^1) + \nabla p^2 = -u^1 \nabla \cdot u^1,$$

$$u^2|_{t=0} = 0,$$

$$(1.9) \quad (\partial_t + (u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \tilde{u}^0 + \varepsilon \tilde{u}^1 + \varepsilon^2 \tilde{u}^2) \cdot \nabla - \nu \Delta) (\tilde{u}^1 + \varepsilon \tilde{u}^2) + \nabla (\tilde{p}^1 + \varepsilon \tilde{p}^2)$$

$$+ (\tilde{u}^1 + \varepsilon \tilde{u}^2) \cdot \nabla (u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \tilde{u}^0) = -\tilde{u}^0 \cdot \nabla u^1 - \tilde{h}^0,$$

$$\tilde{u}^i|_{t=0} = 0, \quad i = 1, 2,$$

$$\tilde{h}^0 = \left(\begin{array}{l} (u_n^1 - \gamma u_n^1) \partial_n \tilde{u}^0 \\ \tilde{u}^0 \cdot \nabla u_n^0 + (u^0 + \varepsilon u^1 + \tilde{u}^0) \cdot \nabla \tilde{u}_n^0 - \partial_n^{-1} \nabla \cdot \tilde{u}^0 \end{array} \right), \quad (\text{See (6.2)}),$$

$$\nabla \cdot (\varepsilon u^2 + \tilde{u}^1 + \varepsilon \tilde{u}^2) = 0,$$

$$\gamma (\varepsilon u^2 + \tilde{u}^1 + \varepsilon \tilde{u}^2) = -{}^t(\gamma u^1, 0).$$

By using the notations described in the next section, our result is stated as follows:

Theorem. Let $u_0 \in H_a^{\ell, \rho, \theta}$ with $\ell > (n-1)/2+3$, $\rho > 0$, $0 < \theta < \pi/4$ and $a > 0$, and assume the "compatibility condition" (1.3). Then, there exists a time interval $[0, T]$, $T > 0$, independent of $\nu \in (0, 1]$, such that (1.1) has a (unique) solution $u(\nu, t, x)$ of the form (1.4) and each term satisfies (1.5)-(1.9), respectively, and the following:

$$\begin{aligned}
(1.10) \quad u^0(v, t, x) &\in X_{a, \beta_0, T}^{\ell, \rho, \theta}, \\
u^i(\varepsilon, t, x) &\in X_{a, \beta_i, T}^{\ell-i, \rho, \theta}, \quad i = 1, 2, \quad 0 < \beta_0 < \beta_1 < \beta_2, \\
\tilde{u}^0(\varepsilon, t, x) &= \left(\begin{array}{l} \tilde{u}^i(\varepsilon, t, x', x_n) \\ \tilde{u}_n^i(\varepsilon, t, x', x_n) \end{array} \right) \in X_{a/\varepsilon, \beta_1, T}^{\ell-1, \rho, \theta, (\mu)}, \quad \mu > 0, \\
\tilde{u}^i(\varepsilon, t, x', x_n) &\in X_{a/\varepsilon, \beta_2, T}^{\ell-i-1, \rho, \theta}, \quad i = 1, 2.
\end{aligned}$$

In particular, $\partial_n^j u(v, t, x', x_n)$, $0 \leq j \leq \ell$, is a continuous function of $(v, x_n) \in [0, 1] \times \Sigma(\theta', a) \setminus \{0\} \times \{0\}$ in the strong topology of $K_{\beta'}^{\ell-j, \rho, \theta'}$ with some $\beta' > 0$ and $\theta' > 0$, and $u^0(0, t, x)$ is the unique solution of (1.2).

2. Notations and Function spaces

First we introduce several notations :

$$\begin{aligned}
(2.1) \quad I(\rho) &= (-\rho, \rho)^{n-1} \quad (\text{open cube}), \\
D(\rho) &= \mathbb{R}^{n-1} + \sqrt{-1}I(\rho) = \{z' = x' + \sqrt{-1}y'; x' \in \mathbb{R}^{n-1}, y' \in I(\rho)\}, \\
\Sigma(\theta, a) &= \Sigma_1(\theta, a) \cup \Sigma_2(\theta, a), \quad 0 < \theta < \pi/4, a > 0, \\
\Sigma_1(\theta, a) &= \{z_n = x_n + \sqrt{-1}y_n; |y_n| \leq x_n \tan \theta, 0 \leq x_n \leq a\}, \\
\Sigma_2(\theta, a) &= \{z_n = x_n + \sqrt{-1}y_n; |y_n| \leq a \tan \theta, x_n \geq a\}, \\
\Omega(\rho, \theta, a) &= D(\rho) \times \Sigma(\theta, a), \\
L(y') &= \mathbb{R}^{n-1} + \sqrt{-1}y' \subset D(\rho), \\
L(\theta', a) &= L_1(\theta', a) \cup L_2(\theta', a) \subset \Sigma(\theta, a), \quad |\theta'| \leq \theta, \\
L_1(\theta', a) &= \{z_n = x_n + \sqrt{-1}y_n; y_n = x_n \tan \theta', 0 \leq x_n \leq a\}, \\
L_2(\theta', a) &= \{z_n = x_n + \sqrt{-1}y_n; y_n = a \tan \theta', x_n \geq a\}.
\end{aligned}$$

Next we introduce function spaces :

(2.2) For a Banach space X with the norm $\|\cdot\|_X$, $B^k([0, T]; X)$ is the set of all C^k -functions from $[0, T]$ to X with the norm

$$\|f\|_{X, k, T} = \sum_{j=0}^k \sup_{0 \leq t \leq T} |\partial_t^j f(t)|_X < \infty.$$

With $[0, T]$ replaced by $\Delta_T = [0, 1] \times [0, T]$ (resp. $\Sigma(\theta, a)$), we define $B^k(\Delta_T; X)$ (resp. $B^k(\Sigma(\theta, a); X)$) in a similar way.

(2.3) For a Banach scale $\tilde{X}_\rho = \{X_\rho; 0 \leq \rho \leq \rho_0\}$ (with the norm $\|\cdot\|_\rho$ of X_ρ) we define $B_\beta^k([0, T]; X_\rho)$ (resp. $B_\beta^k(\Delta_T; X_\rho)$) with the norm $\|f\|_{\rho_0, k, \beta, T} = \sum_{j=0}^k \sup_{0 \leq t \leq T} |\partial_t^j f(t)|_{\rho_0 - \beta t}$, $\beta \geq 0$, $\rho_0 - \beta T \geq 0$ (resp. $\|f\|_{\rho_0, k, \beta, T} = \sum_{i+j \leq k} \sup_{\Delta_T} |\partial_\varepsilon^i \partial_t^j f(\varepsilon, t)|_{\rho_0 - \beta t}$).

For further details, see 4.

(2.4) $H^{-\ell, \rho} \ni f \iff$ (1) $f(x' + \sqrt{-1}y')$ is analytic in $D(\rho)$,
 (2) $\partial^{-\alpha} f(x' + \sqrt{-1}y') \in L^2(L(y'))$ for $y' \in I(\rho)$, $|\alpha| \leq \ell$,
 (3) $\|f\|_{\ell, \rho} = \sum_{|\alpha| \leq \ell} \sup_{y' \in I(\rho)} |\partial^{-\alpha} f(\cdot + \sqrt{-1}y')|_{L^2(L(y'))} < \infty$.

(2.5) $H_a^{\ell, \rho, \theta} \ni f \iff$ (1) $f(z', z_n)$ is analytic inside $\Omega(\rho, \theta, a)$,
 (2) $\partial^\alpha f(z', z_n) \in B^0(\Sigma(\theta, a); H^{-0, \rho})$ for $|\alpha| \leq \ell$,
 (3) $\|f\|_{\ell, \rho, \theta} = \sum_{|\alpha| \leq \ell} \sup_{z_n \in \Sigma(\theta, a)} |\partial^\alpha f(\cdot, z_n)|_{0, \rho} < \infty$.

(2.6) $H_a^{\ell, \rho, \theta, (\mu)} \ni f$ ($\mu \geq 0$) \iff (1) $f \in H_a^{\ell, \rho, \theta}$,
 (2) $\|f\|_{\ell, \rho, \theta, (\mu)} = \sum_{|\alpha| \leq \ell} \sup_{z_n \in \Sigma(\theta, a)} e^{\mu x_n} |\partial^\alpha f(\cdot, z_n)|_{0, \rho} < \infty$.

(2.7) $K_{\beta, T}^{-\ell, \rho} = \bigcap_{j \leq \ell/2} B_\beta^j([0, T]; H^{-\ell-2j, \rho})$,
 $\|f\|_{\ell, \rho, \beta, T} = \sum_{j \leq \ell/2} \sup_{0 \leq t \leq T} |\partial_t^j f(t, \cdot)|_{\ell-2j, \rho - \beta t}$,
 $X_{\beta, T}^{-\ell, \rho} = \bigcap_{k \leq \ell} B^k([0, 1]; K_{\beta, T}^{-\ell-k, \rho})$.

$$(2.8) \quad K_{a,\beta,T}^{\ell,\rho,\theta} = \bigcap_{j \leq \ell/2} B_{\beta}^j([0,T]; H_a^{\ell-2j,\rho,\theta}),$$

$$|f|_{\ell,\rho,\theta,\beta,T} = \sum_{j \leq \ell/2} \sup_{0 \leq t \leq T} |\partial_t^j f(t, \cdot)|_{\ell-2j,\rho-\beta t,\theta-\beta t},$$

$$\tilde{K}_{a,\beta,T}^{\ell,\rho,\theta} = \bigcap_{k \leq \ell} B^k([0,1]; K_{a,\beta,T}^{\ell-k,\rho,\theta}), \quad ([0,1] \ni \varepsilon).$$

$$(2.9) \quad K_{a,\beta,T}^{\ell,\rho,\theta,(\mu)} = \bigcap_{j \leq \ell/2} B_{\beta}^j([0,T]; H_a^{\ell-2j,\rho,\theta,(\mu)}),$$

$$|f|_{\ell,\rho,\theta,(\mu),\beta,T} = \sum_{j \leq \ell/2} \sup_{0 \leq t \leq T} |\partial_t^j f(t, \cdot)|_{\ell-2j,\rho-\beta t,\theta-\beta t,(\mu-\beta t)},$$

$$\tilde{K}_{a,\beta,T}^{\ell,\rho,\theta,(\mu)} = \bigcap_{k \leq \ell} B_{\beta}^k([0,1]; K_{a,\beta,T}^{\ell-k,\rho,\theta,(\mu)}), \quad ([0,1] \ni \varepsilon).$$

$$(2.10) \quad \tilde{K}_{a,\beta,T}^{\ell,\rho,\theta} = \bigcap_{k \leq \ell/2} B^k([0,1]; K_{a,\beta,T}^{\ell-2k,\rho,\theta}), \quad ([0,1] \ni \nu).$$

$$(2.5) \quad H_{a,q}^{\ell,\rho,\theta} \ni f \Leftrightarrow \begin{aligned} (1) & \quad f(z', z_n) \text{ is analytic inside } \Omega(\rho, \theta, a), \\ (2) & \quad \partial^{\alpha} f(z', z_n) \in L^q(L(\theta', a); H^{-0,\rho}), \quad |\theta'| < \theta, \quad |\alpha| \leq \ell, \\ (3) & \quad |f|_{\ell,\rho,\theta,q} = \sum_{|\alpha| \leq \ell} \sup_{|\theta'| < \theta} \|\partial^{\alpha} f(\cdot, z_n)\|_{0,\rho} |L^q(\theta', a)|^{<\infty}. \end{aligned}$$

$$(2.6) \quad \tilde{H}_a^{\ell,\rho,\theta} = H_a^{\ell,\rho,\theta} \cap H_{a,1}^{\ell,\rho,\theta},$$

$$|f|_{\tilde{\ell},\rho,\theta} = |f|_{\ell,\rho,\theta} + |f|_{\ell,\rho,\theta,1}.$$

$$(2.7) \quad \tilde{K}_{a,\beta,T}^{\ell,\rho,\theta} = \bigcap_{j \leq \ell/2} B_{\beta}^j([0,T]; \tilde{H}_a^{\ell,\rho,\theta}),$$

$$|f|_{\tilde{\ell},\rho,\theta,\beta,T} = \sum_{j \leq \ell/2} \sup_{0 \leq t \leq T} |\partial_t^j f(t, \cdot)|_{\tilde{\ell}-2j,\rho-\beta t,\theta-\beta t},$$

$$\tilde{\tilde{K}}_{a,\beta,T}^{\ell,\rho,\theta} = \bigcap_{k \leq \ell} B^k([0,1]; \tilde{K}_{a,\beta,T}^{\ell,\rho,\theta}).$$

3. Operators and the Stokes equations

This section consists of three parts. First we introduce Poisson operator N or D (resp. $P_1(\nu)$ or $P_2(\nu)$) of the Neumann or Dirichlet Problem of the Laplace operator Δ (resp. the heat operator $\partial_t - \nu \Delta$) and other related operators.

Second we construct "evolution operators" solving the equation (1.5)-(1.9). In particular, we construct the "Poisson operator" $\mathcal{P}(v)$ of the Stokes equation, combining the operators defined above.

Finally we give the estimates for these operators in the function spaces introduced in 2, and the estimates of Cauchy-Kowalewski type for the first-order differential operators .

7. Various operators

(3.1) $rf = f|_{\mathbb{R}_+^n}$ (the restriction of the function f of \mathbb{R}^n onto \mathbb{R}_+^n).

$ef =$ a "nice" extension of the function f of \mathbb{R}_+^n to \mathbb{R}^n . Here

"nice" means the regularity preserving property (cf. [5]).

$\bar{e}f(z, z_n) = f(z, z_n)$ for $x_n = \operatorname{Re} z_n \geq 0$, $= 0$ for $x_n < 0$.

$\gamma f = f|_{D(\rho)} = f|_{z_n=0}$ (cf. (1.1) (4)).

(3.2) $U_0(v, t, x) = (4\pi vt)^{-n/2} e^{-|x|^2/(4\pi vt)}$ (heat kernel in \mathbb{R}^n),

$U_0(v, t)f(x) = \int_{\mathbb{R}^n} U_0(v, t, x-\eta)f(\eta)d\eta$,

$U_0(v)f(t, \cdot) = \int_0^t U_0(v, t-s)f(s, \cdot)ds$,

$U_0^-(v, t, x') = (4\pi vt)^{-(n-1)/2} e^{-|x'|^2/(4\pi vt)}$,

$\bar{U}_0(v, t, x) = U_0^-(v, t, x')(4\pi t)^{-1/2} e^{-x_n^2/(4\pi t)}$.

$\bar{U}_0(v, t)$ and $\bar{U}_0(v)$ are defined similarly as $U_0(v, t)$ and $U_0(v)$.

Let $u(x')$ (resp. $u(t, x')$) be a function on \mathbb{R}^{n-1} (resp. $[0, \infty) \times \mathbb{R}^{n-1}$).

We define the Fourier transform (resp. Fourier-Laplace transform)

$\hat{u}(\xi')$ (resp. $\tilde{u}(\lambda, \xi')$) of $u(x')$ (resp. $u(t, x')$) by

$$(3.3) \quad \hat{u}(\xi') = (2\pi)^{-(n-1)/2} \int e^{-ix' \cdot \xi'} u(x') dx' , \quad i = \sqrt{-1} ,$$

$$\text{(resp. } \tilde{u}(\lambda, \xi') = \int_0^\infty e^{-\lambda t} \hat{u}(t, \cdot) dt \text{)} .$$

We call the multiplier $\sigma(T)(\xi')$ in ξ' (resp. $\sigma(T)(\lambda, \xi')$ in (ξ', λ)) the symbol of the operator T , if there holds

$$(3.4) \quad (Tu)\hat{\ }(\xi') = \sigma(T)(\xi')\hat{\ }u(\xi')$$

(resp. $(Tu)\tilde{\ }(\lambda, \xi') = \sigma(T)(\lambda, \xi')\tilde{\ }u(\lambda, \xi')$).

Poisson operators N , D , $P_1(\nu)$ and $P_2(\nu)$ are defined by the following equations, respectively :

$$(3.5) \quad \Delta Nu = 0, \quad \gamma \partial_n Nu = g \text{ (given boundary data),}$$

$$\Delta Du = 0, \quad \gamma u = g,$$

$$(3.6) \quad (\partial_t - \nu \Delta) P_1(\nu) u = 0, \quad P_1(\nu) u|_{t=0} = 0, \quad \gamma \partial_n P_1(\nu) u = g,$$

$$(\partial_t - \nu \Delta) P_2(\nu) u = 0, \quad P_2(\nu) u|_{t=0} = 0, \quad \gamma P_2(\nu) u = g.$$

Clearly we have

$$(3.2) \quad \sigma(U_0(\nu, t))(\xi') = e^{-\nu t |\xi'|^2},$$

$$(3.5) \quad \sigma(N)(\xi', x_n) = -1/|\xi'| e^{-|\xi'| x_n},$$

$$\sigma(D)(\xi', x_n) = e^{-|\xi'| x_n},$$

$$D = \partial_n N, \quad \partial_n D = \partial_n^2 N = -\Delta' N = (\Lambda^{-2} N),$$

$$(3.6) \quad \sigma(P_1(\nu))(\lambda, \xi') = -(\lambda/\nu + |\xi'|^2)^{-1/2} e^{-(\lambda/\nu + |\xi'|^2)^{1/2} x_n},$$

$$\sigma(P_2(\nu))(\lambda, \xi') = e^{-(\lambda/\nu + |\xi'|^2)^{1/2} x_n}, \quad P_2(\nu) = \partial_n P_1(\nu),$$

For later use we define the Poisson operators $\bar{P}_j(\nu)$, $j=1,2$, by

$$(3.7) \quad \sigma(\bar{P}_j(\nu))(\lambda, \xi') = (-\lambda + \nu |\xi'|^2)^{1/2, j-2} e^{-(\lambda + \nu |\xi'|^2)^{1/2} x_n},$$

which is associated with the heat operator $(\partial_t - \nu \Delta' - \partial_n^2)$.

We introduce two kinds of singular integral operators. The first group, Q^∞ , P^∞ , $N' = {}^t(N_1, \dots, N_{n-1})$ and Λ' , are of Calderon-Zygmund type and act in the function spaces on R^n and R^{n-1} , respectively :

$$(3.8) \quad \sigma(Q^\infty)(\xi) = \begin{pmatrix} \xi^{-t} \xi' / |\xi|^2 & \xi' \xi_n / |\xi|^2 \\ \xi_n^t \xi' / |\xi|^2 & \xi_n^2 / |\xi|^2 \end{pmatrix}, \quad P^\infty = 1 - Q^\infty = \begin{pmatrix} P^\infty \\ P_n^\infty \end{pmatrix}.$$

(Only here in (3.8), we adopt the Fourier transform in R^n .)

$$(3.9) \quad \sigma(N')(\xi') = (i\xi'/|\xi'|), \quad i = \sqrt{-1},$$

$$(3.10) \quad \sigma(\Lambda')(\xi') = |\xi'|, \quad \Lambda'_1 = \Lambda' + 1.$$

$$(3.11) \quad \sigma(\omega(\nu))(\lambda, \xi') = |\xi'|/(\lambda/\nu + |\xi'|^2)^{1/2},$$

$$\Omega(\nu) = N'\omega(\nu) {}^t N' = \omega(\nu) N' N', \quad \omega_1(\nu) = \Lambda'^{-1} \omega(\nu).$$

$$(3.12) \quad \sigma(\tau(\nu))(\lambda, \xi') = |\xi'|/((\lambda/\nu + |\xi'|^2)^{1/2} + |\xi'|),$$

$$\tau_1(\nu) = \omega(\nu) - \tau(\nu) = \omega(\nu)\tau(\nu).$$

By symbol calculus, we have equalities :

$$(3.13) \quad Q^\infty = \nabla \Delta^{-1} \nabla \cdot, \quad \nabla \cdot Q^\infty = \nabla \cdot, \quad \nabla \cdot P^\infty = 0,$$

$$(3.14) \quad \sigma(\omega(\nu, t))(\xi') = \pi^{-1/2} \nu^{1/2} t^{-1/2} |\xi'| e^{-\nu t |\xi'|^2},$$

$$\sigma(\omega^2(\nu, t))(\xi') = \nu |\xi'|^2 e^{-\nu t |\xi'|^2}, \quad \omega^2 = \omega(\nu)\omega(\nu),$$

$$\sigma(\tau_1(\nu))(\xi') = \pi^{-1} \nu |\xi'|^2 \int_0^\infty e^{-\nu t |\xi'|^2} (1+s)^{-1/2} (1+s)^{-1} ds.$$

$$(3.15) \quad P_1(\nu) = -\omega_1(\nu) P_2(\nu), \quad \bar{P}_1(\nu) = -\nu^{1/2} \omega_1(\nu) \bar{P}_2(\nu),$$

$$\gamma U_0(\nu, t) = -1/(2\nu) \{ P_1(\nu, t)^* + P_1(\nu, t)^{* \nu} \},$$

where T^* means the adjoint of T , and $\check{f}(x', x_n) = f(x', -x_n)$ for $x_n > 0$.

For later use we define the modified operators \bar{Q}^∞ and \bar{P}^∞ by

$$(3.8) \quad \sigma(\bar{Q}^\infty)(\xi) = \begin{pmatrix} \varepsilon^2 \xi'^t \xi' & \varepsilon \xi' \xi_n \\ \varepsilon \xi_n^t \xi' & \xi_n^2 \end{pmatrix} (\varepsilon^2 |\xi'|^2 + \xi_n^2)^{-1}, \quad \bar{P}^\infty = 1 - \bar{Q}^\infty.$$

We note that the identity $1 = Q^\infty + P^\infty$ gives the Helmholtz decomposition in R^n . Similarly the following operators Q and P give the same decomposition in R_+^n (associated with the Euler equation):

$$(3.16) \quad P = r P^\infty e - \nabla N \gamma_n P^\infty e, \quad Q = 1 - P.$$

2. The "Stokes equations"

Define the "evolution operator" $V(\nu, t)$ (and $V(\nu)$) by

$$(3.17) \quad V(\nu, t) = r P^\infty U_0(\nu, t) e - \nabla N \gamma_n P^\infty U_0(\nu, t) e \quad (\text{or} = Pr U_0(\nu, t) e) \\ (V(\nu) f(t) = \int_0^t V(\nu, t-s) f(s, \cdot) ds).$$

Then $V(\nu, t)$ satisfies

$$(3.18) \quad \begin{aligned} (\partial_t - \nu \Delta) V(\nu, t) + \nabla N \gamma_n P^\infty \partial_t U_0(\nu, t) e &= 0, \quad t > 0, \quad x \in \mathbb{R}_+^n, \\ \nabla \cdot V(\nu, t) &= 0, \\ V(\nu, 0) &= P, \\ \gamma_n V(\nu, t) &= 0. \end{aligned}$$

The evolution operator $U_1(\nu)$ (resp. $U_2(\nu)$) of the Neumann (resp. Dirichlet) problem for the heat operator $\partial_t - \nu \Delta$ is given by

$$(3.19) \quad \begin{aligned} U_1(\nu) &= rU_0(\nu) \bar{e} - P_1(\nu) \gamma \partial_n U_0(\nu) \bar{e} \\ (\text{resp. } U_2(\nu) &= rU_0(\nu) \bar{e} - P_2(\nu) \gamma U_0(\nu) \bar{e} = rU_0(\nu) \bar{e} + P_1(\nu) \gamma \partial_n U_0(\nu) \bar{e}). \end{aligned}$$

These operators satisfy

$$(3.20) \quad \begin{aligned} (\partial_t - \nu \Delta) U_i(\nu) f &= f(t, x), \quad t > 0, \quad x \in \mathbb{R}_+^n, \\ U_i(\nu) f|_{t=0} &= 0, \\ \gamma \partial_n^{2-i} U(\nu) f &= 0, \quad i = 1, 2. \end{aligned}$$

By replacing $U_0(\nu)$ and $P_i(\nu)$ with $\bar{U}_0(\nu)$ and $\bar{P}_i(\nu)$, we also define $\bar{U}_i(\nu)$, which is associated with the heat operator $\partial_t - \nu \Delta - \partial_n^2$. Note

$$(3.21) \quad \partial_n^{-1} \bar{U}_2(\nu) = \bar{U}_1(\nu) \partial_n^{-1} = (r\bar{U}_1(\nu) \bar{e} - \bar{P}_1(\nu) \gamma \bar{U}_0(\nu) \bar{e}) \partial_n^{-1}.$$

The Poisson operator $\mathcal{P}(\nu)$ of the original Stokes equation is defined by solving

$$(3.22) \quad \begin{aligned} (\partial_t - \nu \Delta) w + \nabla p &= 0, \quad t > 0, \quad x \in \mathbb{R}_+^n, \\ \nabla \cdot w &= 0, \\ w|_{t=0} &= 0, \\ \gamma w &= g(t, x') = {}^t(g', g_n). \end{aligned}$$

Let $w = w^1 + w^2 + w^0$, and put

$$(3.23) \quad \begin{aligned} w^1 &= \nabla N f_0, \quad f_0|_{t=0} = 0, \\ w^2 &= P(\nu) f = {}^t(P_2(\nu) f', P_1(\nu) f_n), \quad \nabla \cdot f' + f_n = 0, \\ w^0 &= U(\nu) \nabla q = {}^t(U_2(\nu) \nabla' q, U_1(\nu) \partial_n q), \quad \Delta q = 0. \end{aligned}$$

Clearly each w^i satisfies the first three conditions of (3.22) and the following "boundary conditions"

$$\begin{aligned}
(3.24) \quad \gamma w^1 &= {}^t(-N' f_0, f_0), \\
\gamma w^2 &= {}^t(f', \gamma P_1(\nu) f_n), \\
\gamma w^0 &= {}^t(0', 1/\nu \gamma P_1(\nu) \gamma q + 2\gamma \partial_n U_0(\nu) \bar{e} q).
\end{aligned}$$

We determine q by the following condition

$$(3.25) \quad \Delta q = 0, \quad \gamma q = -\nu f_n, \quad \text{i.e. } q = -\nu Df_n = \nu D\nabla' f'.$$

Then, w satisfies (3.22), if the following equation is satisfied ;

$$\begin{aligned}
(3.26) \quad f' - N' f_0 &= g', \\
f_0 + P_2(\nu) {}^*DA' N' f' &= g_n.
\end{aligned}$$

Here we have used the equalities (cf. (3.15) or (3.31)) :

$$2\nu \gamma \partial_n U_0(\nu) \bar{e} = P_2(\nu) {}^* \quad \text{and} \quad \nabla' = \Lambda' N'.$$

Hence we obtain

$$(3.26)' \quad f' + N' P_2(\nu) {}^*DA' N' f' = g' + N_n' g \equiv Mg.$$

By symbol calculus (and the identity : $N' N' = -1$), we have

$$(3.27) \quad P_2(\nu) {}^*DA' = \tau(\nu), \quad \{1 - \tau(\nu)\}^{-1} = \omega(\nu),$$

$$(3.28) \quad \{1 + N' P_2(\nu) {}^*DA' N'\}^{-1} = \{1 + N' \tau(\nu) {}^tN'\}^{-1} = 1 + \Omega(\nu).$$

Thus (3.26) is solved by

$$\begin{aligned}
(3.29) \quad f' &= \{1 + \Omega(\nu)\} Mg, \\
f_0 &= g_n - \tau(\nu) N' \{1 + \Omega(\nu)\} Mg = g_n - \{\tau(\nu) - \tau_1(\nu)\} N' Mg.
\end{aligned}$$

Substituting (3.29) into (3.22) and rearranging the expression of w , we obtain

$$\begin{aligned}
(3.30) \quad w &= \mathcal{P}(\nu) g = \mathcal{P}_1(\nu) g + \mathcal{P}_2(\nu) g + \mathcal{P}_0(\nu) g \\
&= \nabla N g_n - \nabla N \tau(\nu) N' \{1 + \Omega(\nu)\} Mg \\
&\quad + P_2(\nu) \left(\frac{E' - \tau_1(\nu) N' {}^t/2}{\omega(\nu)/2} - \frac{\tau_1(\nu) N' {}^t/2}{\tau_1(\nu) N' /2} \right) \{1 + \Omega(\nu)\} Mg \\
&\quad + \nu \nabla r \nabla' U_0(\nu) \bar{e} D \{1 + \Omega(\nu)\} Mg.
\end{aligned}$$

We note that the boundary layer arises only from $\mathcal{P}_2(\nu) g$.

3. Estimates

Fix $\rho_0 > 0$ and $0 < \theta_0 < \pi/4$. Then we have the following estimates which hold uniformly in ρ and θ with $0 \leq \rho \leq \rho_0$ and $0 \leq \theta \leq \theta_0$.

Lemma 3.1. Let $v \in (0, 1)$, $\varepsilon = \sqrt{v}$, $l \geq 0$ and $a > 0$. Then : (7)

$$(3.31) \quad rU_0(v, t)e = O(1), \quad r\bar{U}_0(v, t)e = O(1), \\ (\varepsilon\Lambda')^\kappa rU_0(v, t)e = O(t^{-\kappa/2}), \quad (\varepsilon\Lambda')^\kappa r\bar{U}_0(v, t)e = O(t^{-\kappa/2}), \\ (\varepsilon\Lambda')^\kappa \partial_n r\bar{U}_0(v, t)e = O(t^{-1/2 - \kappa/2}), \quad \kappa \geq 0,$$

uniformly in ε , l and a in the function spaces $H_a^{\ell, \rho, \theta}$, $H_{a/\varepsilon}^{\ell, \rho, \theta, (\mu)}$ and $\tilde{H}_{a/\varepsilon}^{\ell, \rho, \theta}$. The same holds if the extension e is replaced by \bar{e} .

(2) There hold the following relations :

$$(3.32) \quad \bar{P}_1(v) = (\varepsilon\Lambda')^{-1} \bar{P}_2(v)\omega(v) = \bar{P}_1(v, t) *_t (\text{convolution in } t), \\ 2\gamma\bar{U}_0(v, t)\bar{e} = \bar{P}_1(v, t)^*, \quad 2\gamma\partial_n \bar{U}_0(v, t)\bar{e} = \bar{P}_2(v, t)^*.$$

As operators from $H^{-l, \rho}$ to $H_{a/\varepsilon}^{\ell, \rho, \theta, (\mu)}$ (resp. $H_{a/\varepsilon, 1}^{\ell, \rho, \theta}$),

$$(3.33) \quad (\varepsilon\Lambda')^\kappa \bar{P}_1(v, t) = O(t^{-1/2 - \kappa/2}) \quad (\text{resp. } O(t^{-\kappa/2})), \quad \kappa \geq 0, \\ \bar{P}_2(v, t) = O(t^{-1}) \quad (\text{resp. } O(t^{-1/2})),$$

uniformly in ε , l and a . From $H_a^{\ell, \rho, \theta}$ (resp. $H_{a, 1}^{\ell, \rho, \theta}$) to $H^{-l, \rho}$,

$$(3.34) \quad P_i(v, t)^* = O(\varepsilon^{3-i} t^{-(i-1)/2}) \quad (\text{resp. } O(\varepsilon^{2-i} t^{-i/2})), \\ \gamma\partial_n^{i-1} \bar{U}_0(v, t), \quad \bar{P}_i(v, t)^* = O(t^{-(i-1)/2}) \quad (\text{resp. } O(t^{-i/2})).$$

(3) As operators acting in $H^{-l, \rho}$,

$$(3.35) \quad (\varepsilon\Lambda')^{-\kappa} \omega(v, t), \quad (\varepsilon\Lambda')^{-\kappa} \tau(v, t), \quad (\varepsilon\Lambda')^{-\kappa} \tau_1(v, t) = O(t^{-1+\kappa/2}), \\ 0 \leq \kappa \leq 1, \\ N' = O(1)$$

uniformly in ε .

(4) As operators acting in $H_a^{\ell, \rho, \theta}$ and $H_{a/\varepsilon, 1}^{\ell, \rho, \theta}$,

$$Q^\infty, \quad P^\infty, \quad \bar{Q}^\infty, \quad \bar{P}^\infty = O(1).$$

(5) As the operators acting from $H^{-l, \rho}$ to $H_a^{\ell, \rho, \theta}$,

$$D = O(1), \quad \nabla N = D {}^t(-N', 1) = O(1).$$

Lemma 3.2. Under the corresponding conditions in Lemma 3.1, we have :

$$\begin{aligned}
 (3.36) \quad & \bar{u}_2(v) = r\bar{u}_0(v, t) \bar{e} *_{t^+} + \bar{u}_2(v, t) *_{t^+}, \\
 & (\varepsilon \Lambda^-)^{\kappa} \bar{u}_2(v, t) = o(t^{-\kappa/2}), \quad \partial_n \bar{u}_2(v, t) = o(t^{-1/2}), \\
 (3.37) \quad & \mathcal{P}_1(v) \gamma = \nabla N \gamma_n + \nabla N \bar{h}_1(v, t) *_{t^+} (\varepsilon \Lambda^-), \quad (\varepsilon \Lambda^-)^{\kappa} \bar{h}_1(v, t) = o(t^{-(1+\kappa)/2}), \\
 & \mathcal{P}_0(v) \gamma = \varepsilon \nabla \bar{h}_0(v, t) *_{t^+} (\varepsilon \Lambda^-), \quad (\varepsilon \Lambda^-)^{\kappa} \bar{h}_0(v, t) = o(t^{-\kappa/2}), \\
 & \varepsilon \partial_n \mathcal{P}_0(v) \gamma = -\mathcal{P}_0(v) \gamma \varepsilon \Lambda^- - \nabla P_1(v) \varepsilon \nabla \cdot (1 + \Omega(v)) M \gamma, \\
 & \bar{\mathcal{P}}_2(v) \gamma = \bar{P}_2(v) \begin{pmatrix} E^- & 0 \\ 0 & 0 \end{pmatrix} M \gamma + \bar{h}_2(v, t) *_{t^+} (\varepsilon \Lambda^-), \\
 & (\varepsilon \Lambda^-)^{\kappa} \bar{h}_2(v, t) = o(t^{-(1+\kappa)/2}).
 \end{aligned}$$

Here $*_{t^+}$ means convolution in t (on $[0, t]$), and $\bar{\mathcal{P}}_2(v)$ is defined by replacing $P_2(v)$ with $\bar{P}_2(v)$ in the definition (3.29) of $\mathcal{P}_2(v)$.

Lemma 3.3. Let $f(z') \in H^{-\ell, \rho}$. Then,

$$(3.38) \quad |\partial_j f|_{\ell, \rho'} \leq |f|_{\ell, \rho} / (\rho - \rho'), \quad \rho > \rho' \geq 0, \quad 1 \leq j \leq n-1.$$

Lemma 3.4. Let $f(z, z_n) \in H_a^{\ell, \rho, \theta}$, and put $\chi(z_n) = \min(1, |z_n|)$. Then there exists $C(\theta_0) > 0$, independent of a , such that

$$\begin{aligned}
 (3.39) \quad & |\chi(z_n) \partial_n f|_{\ell, \rho, \theta'} \leq C(\theta_0) |f|_{\ell, \rho, \theta} / (\theta - \theta'), \quad \theta > \theta' \geq 0, \\
 & |\chi(z_n) \partial_n f|_{\ell, \rho, \theta', (\mu)} \leq C(\theta_0) |f|_{\ell, \rho, \theta, (\mu)} / (\theta - \theta') + \mu |f|_{\ell, \rho, \theta', (\mu)}, \\
 & \frac{1}{\varepsilon} \chi(\varepsilon z_n) \partial_n f|_{\ell, \rho, \theta', (\mu')} \leq C(\theta_0) |f|_{\ell, \rho, \theta, (\mu)} \{1/(\theta - \theta') + 1/(\mu - \mu')\}.
 \end{aligned}$$

Remark. In what follows, we put $U_0(0, t) = 1$ and $U_0(0) = \delta(t) *_{t^+}$. Then, $rU_0(v, t)e$ (resp. $r\bar{U}_0(v)e$) is strongly continuous in $(v, t) \in [0, 1] \times [0, \infty)$ in $H_a^{\ell, \rho, \theta}$ (resp. in v in $K_a^{\ell, \rho, \theta, (\mu)}$ and $\tilde{K}_a^{\ell, \rho, \theta}$).

4. Abstract Cauchy-Kowalewski theorem

We give a survey on the abstract Cauchy-Kowalewski theorem ([2]).

Let $\tilde{X}_{\rho} = \{X_{\rho}; 0 \leq \rho \leq \rho_0\}$ be a Banach scale with the norm $|\cdot|_{\rho}$.

of X_ρ , i.e. $X_\rho \supset X_\rho$ and $\| \cdot \|_\rho \leq \| \cdot \|_\rho$ for $0 \leq \rho' \leq \rho \leq \rho_0$. Define $X_{\rho_0, \beta, T}$ and $Y_{\rho_0, \beta, T}$ by

$$(4.1) \quad X_{\rho_0, \beta, T} = B_\beta^0([0, T]; X_\rho) \ni f(t) \Leftrightarrow$$

$$(1) \quad f(t) \in B^0([0, T']; X_\rho) \text{ for } \rho \leq \rho_0 - \beta T', \quad T' \leq T,$$

$$(2) \quad \|f\|_{\rho_0, \beta} = \sup_{0 \leq t \leq T} |f(t)|_{\rho_0 - \beta t} < \infty,$$

$$(4.2) \quad Y_{\rho_0, \beta, T} \ni f(t) \Leftrightarrow (1) \quad f(t) \in B_\beta^0([0, T']; X_\rho) \text{ for } T' < T,$$

$$(2) \quad \|f\|_\beta = \sup_{0 \leq \rho \leq \rho_0 - \beta t} |f(t)|_\rho \varphi(\beta t / (\rho_0 - \rho)) < \infty, \quad \varphi(t) = (1-t)e^{-t}.$$

We also define

$$(4.3) \quad X_{\rho, \beta, T}(R) = \{f(t) \in X_{\rho, \beta, T} ; \|f\|_{\rho, \beta} \leq R\},$$

$$\tilde{X}_{\rho, \beta, T} = B^0([0, 1]; X_{\rho, \beta, T}), \quad ([0, 1] \ni \varepsilon),$$

$$\tilde{X}_{\rho, \beta, T}(R) : \text{similarly as } X_{\rho, \beta, T}(R).$$

Let $F(\varepsilon, t, u(\cdot))$ be a mapping from $[0, 1] \times [0, \tau] \times X_{\rho, \beta_0, \tau}(R)$ into $\tilde{X}_{\rho, \beta_0, \tau}$ for $0 \leq \rho' < \rho \leq \rho_0 - \beta_0 \tau$ and $0 < \tau \leq T_0$, and satisfy

$$(F.1) \quad \|F(\varepsilon, t, u(\cdot)) - F(\varepsilon, t, v(\cdot))\|_{\rho'} \leq \int_0^t C \|u(s) - v(s)\|_{\rho(s)} / (\rho(s) - \rho') ds$$

for each $u, v \in X_{\rho, \beta, \tau}$, $0 \leq \rho' < \rho(s) \leq \rho - \beta s$, $\beta \geq \beta_0$, $0 \leq t \leq T_0$,

$$(F.2) \quad \|F(\varepsilon, t, 0)\|_{\rho_0 - \beta t} \leq R_0 < R, \quad \varepsilon \in [0, 1], \quad 0 \leq t \leq \tau \leq T_0.$$

Here C, R and R_0 are constants independent of ε .

Consider the following equation

$$(4.4) \quad u(t) = F(\varepsilon, t, u(\cdot)), \quad 0 \leq t \leq T (\leq T_0).$$

Then, we have

Theorem ACK. (Abstract Cauchy-Kowalewski theorem). Assume (F.1) and (F.2). Then, there exist $\beta > \beta_0$ and $T \leq T_0$, $0 < T \leq \rho_0 / \beta$, such that the equation (4.4) has a unique solution $u(\varepsilon, t) \in \tilde{X}_{\rho_0, \beta, T}(R)$.

We can choose β such as

$$(4.5) \quad \beta = \max \{ 4\beta_0/3, 8Ce, 16Ce^2 R_0 / (R - R_0) \}.$$

Sketch of Proof. First we note

$$(4.6) \quad \|u\|_{\beta} \leq |u|_{\rho_0, \beta} \leq (1 - \beta/\beta_0)e \|u\|_{\beta_0}, \quad \beta > \beta_0 \geq 0.$$

By virtue of (F.1), we have

$$(4.7) \quad \|F(\varepsilon, t, u) - F(\varepsilon, t, v)\|_{\beta} \leq (2Ce/\beta) \|u-v\|_{\beta}, \quad \beta \geq \beta_0,$$

for each $u, v \in X_{\rho_0, \beta, T}(\mathbb{R})$.

Choose β satisfying (4.5), and put

$$(4.8) \quad \begin{aligned} u_0(t) &= F(\varepsilon, t, 0), \\ u_{n+1}(t) &= F(\varepsilon, t, u_n(\cdot)), \quad n \geq 0, \\ \beta_n &= \beta(1 - 2^{-n-1}), \quad \text{i.e. } \beta - \beta_n = \beta 2^{-n-1}. \end{aligned}$$

Then, $u_{n+1} \in \tilde{X}_{\rho_0, \beta_{n+1}, T}$, if $u_n \in \tilde{X}_{\rho_0, \beta_n, T}(\mathbb{R})$ and $T \leq \rho_0/\beta$. (4.7)

and (4.8) imply

$$(4.9) \quad \|u_{n+1} - u_n\|_{\beta_n} \leq (2Ce)/\beta_n \|u_n - u_{n-1}\|_{\beta_n},$$

$$(4.10) \quad \begin{aligned} |u_{n+1} - u_n|_{\rho_0, \beta} &\leq (1 - \beta_n/\beta)e \|u_{n+1} - u_n\|_{\beta_n} = 2^{n+1}e \|u_{n+1} - u_n\|_{\beta_n} \\ &= 2^{n+1}e(2Ce/\beta_n)^n \|u_1 - u_0\|_{\beta_n} \\ &\leq 8/3 e(4Ce/\beta)^n \|u_1 - u_0\|_{\beta_1}. \end{aligned}$$

On the other hand (F.2) implies

$$\|u_0\|_{\beta_1} \leq |u_0|_{\rho_0, \beta_0} \leq R_0, \quad \|u_1 - u_0\|_{\beta_1} \leq (2Ce/\beta_1) \|u_0\|_{\beta_1}.$$

Hence, because of the choice of β , we have

$$(4.11) \quad |u_{n+1} - u_n|_{\rho_0, \beta} \leq 16/9 e(4Ce/\beta)^{n+1} R_0,$$

$$|u_{n+1}|_{\rho_0, \beta} \leq \{1 + 4e(4Ce/\beta)\} R_0 \leq R.$$

This shows that $\{u_n\}$ converges in $\tilde{X}_{\rho_0, \beta, T}(\mathbb{R})$ and the limit $u(\varepsilon, t)$

satisfies (4.4), if $T \leq \min\{T_0, \rho_0/\beta\}$. Uniqueness is easily proved.

5. The first approximation $u^0(v, t, x)$

We solve the equation (1.5) by using the "evolution operator" $V(v, t)$ defined by (3.17). We consider the equation

$$(5.1) \quad u^0 = V(v, t)u_0 - V(v)u^0 \cdot \nabla u^0 \equiv F(v, t, u^0).$$

The solution u^0 of (5.1) is clearly a solution of (1.5).

We note that $V(v, t)$ (resp. $V(v)$) is strongly continuous in (v, t) in $H_a^{\ell, \rho, \theta}$ (resp. in v in $K_a^{\ell, \rho, \theta}$), by virtue of Remark in 3.

Fix ρ_0 and θ_0 so that $\rho_0 > 0$ and $0 < \theta_0 < \pi/4$, and put

$$(5.2) \quad G(u) = u \cdot \nabla u, \quad u \in \dot{H}_a^{\ell, \rho, \theta} = \{ u \in H_a^{\ell, \rho, \theta}; \gamma_n u = 0 \},$$

$$\ell \geq (n-1)/2 + 1, \quad 0 < \rho \leq \rho_0, \quad 0 < \theta \leq \theta_0, \quad a > 0.$$

By virtue of Sobolev embedding theorem, Lemma 3.3 and 3.4, we obtain the uniform estimates (in ρ, θ and $v \in (0, 1]$):

$$(5.3) \quad |G(u)|_{\ell, \rho, \theta} \leq C |u|_{\ell, \rho, \theta} \{ |u|_{\ell, \rho, \theta} / (\rho - \rho') + |u|_{\ell, \rho, \theta} / (\theta - \theta') \},$$

$$|G(u) - G(v)|_{\ell, \rho, \theta} \leq C (|u|_{\ell, \rho, \theta} + |v|_{\ell, \rho, \theta}) \times$$

$$\times \{ |u - v|_{\ell, \rho, \theta} / (\rho - \rho') + |u - v|_{\ell, \rho, \theta} / (\theta - \theta') \},$$

$$u, v \in \dot{H}_a^{\ell, \rho, \theta}, \quad 0 \leq \rho' < \rho \leq \rho_0, \quad 0 \leq \theta' < \theta \leq \theta_0, \quad 0 \leq v \leq 1.$$

The constant C is independent of $\rho', \rho, \theta', \theta, a > 0$ and v . Thus the mapping $F(v, t, u(\cdot)) = V(v, t)u_0 - V(v)G(u)$ appearing in (5.1) satisfies the conditions (F.1) and (F.2) in $\dot{H}_a^{\ell, \rho, \theta}$ with arbitrary $T_0 > 0$. Hence, applying Theorem ACK, we have

Theorem 5.1. Let $\ell \geq (n-1)/2 + 3$, $0 < \rho \leq \rho_0$, $0 < \theta \leq \theta_0 < \pi/4$ and $a > 0$. Assume $u_0 \in H_a^{\ell, \rho, \theta}$ and the condition (1.3). Then, there exist $T > 0$ and $\beta_0 > 0$ such that (5.1) has a unique solution $u^0(v, t) \in \dot{X}_{a, \beta_0, T}^{\ell, \rho, \theta}$, which is defined from $\dot{H}_a^{\ell, \rho, \theta}$ in the same way as in 2.

6. The first boundary layer $\tilde{u}^0(\varepsilon, t, x, x_n/\varepsilon)$

Let $\varepsilon = \sqrt{\nu} \in [0, 1]$. In order to solve (1.6), we change variables as follows :

$$(6.1) \quad \begin{aligned} x_n &\rightarrow \varepsilon x_n, \quad \partial_n \rightarrow \partial_n/\varepsilon, \\ \tilde{u}^0(\varepsilon, t, x, x_n/\varepsilon) &\rightarrow \tilde{u}^0(\varepsilon, t, x, x_n), \quad \tilde{u}_n^0(\dots, x_n/\varepsilon) \rightarrow \tilde{u}_n^0(\dots, x_n)/\varepsilon, \\ u^0(\nu, t, x) &\rightarrow \begin{pmatrix} u^0(\nu, t, x, \varepsilon x_n) \\ u_n^0(\nu, t, x, \varepsilon x_n)/\varepsilon \end{pmatrix} \equiv \bar{u}^0(\varepsilon, t, x, x_n). \end{aligned}$$

Then, the equation (1.6) is rewritten as

$$(6.2) \quad \begin{aligned} \partial_t \tilde{u}^0 + (\bar{u}^0 + \tilde{u}^0) \cdot \nabla \tilde{u}^0 + (\bar{u}_n^0 + \tilde{u}_n^0 - \gamma \tilde{u}_n^0) \partial_n \tilde{u}^0 - \nu \Delta \tilde{u}^0 - \partial_n^2 \tilde{u}^0 + \tilde{u}^0 \cdot \nabla \bar{u}^0 &= 0, \\ \tilde{u}_n^0(\varepsilon, t, x) &= -\partial_n^{-1} \nabla \cdot \tilde{u}^0(\varepsilon, t, x) \equiv \int_{x_n}^{\infty} \nabla \cdot \tilde{u}^0(\varepsilon, t, x', \eta_n) d\eta_n, \\ \tilde{u}^0|_{t=0} &= 0, \\ \gamma \tilde{u}^0 &\equiv \tilde{u}^0|_{x_n=0} = -\gamma \bar{u}^0 = -\gamma u^0. \end{aligned}$$

We put

$$(6.3) \quad \begin{aligned} \tilde{u}^0 &= \bar{U}_2(\nu) \tilde{v}^0 - \bar{P}_2(\nu) \gamma u^0, \\ \tilde{u}_n^0 &= -\partial_n^{-1} \nabla \cdot \tilde{u}^0 \equiv -\partial_n^{-1} \bar{U}_2(\nu) \nabla \cdot \tilde{v}^0 + \bar{P}_1(\nu) \gamma \nabla \cdot u^0 \\ &= -\{r \bar{U}_0(\nu) \bar{e} - \bar{P}_1(\nu) \gamma \bar{U}_0(\nu) \bar{e}\} \partial_n^{-1} \nabla \cdot \tilde{v}^0 + \bar{P}_1(\nu) \gamma u^0 \quad (\text{See (3.21)}). \end{aligned}$$

Then, the last three conditions of (6.2) are automatically satisfied.

Substituting (6.3) into (6.2), we have an equation for \tilde{v}^0 :

$$(6.4) \quad \begin{aligned} \tilde{v}^0 + (\bar{u}^0 + \tilde{u}^0) \cdot \nabla \bar{U}_2(\nu) \tilde{v}^0 + (\bar{u}_n^0 + \tilde{u}_n^0 - \gamma \tilde{u}_n^0) \partial_n \bar{U}_2(\nu) \tilde{v}^0 \\ + \{ \tilde{u}^0 \cdot \nabla + (\tilde{u}_n^0 - \gamma \tilde{u}_n^0) \partial_n \} \bar{u}^0 \\ - \{ (\bar{u}^0 + \tilde{u}^0) \cdot \nabla + (\bar{u}_n^0 + \tilde{u}_n^0 - \gamma \tilde{u}_n^0) \partial_n \} \bar{P}_2(\nu) \gamma u^0 = 0. \end{aligned}$$

Note (See (3.15).)

$$(6.5) \quad \tilde{u}_n^0 - \gamma \tilde{u}_n^0 = -\int_0^{x_n} \nabla \cdot \tilde{u}^0(\varepsilon, t, x, \xi_n) d\xi_n,$$

$$(6.6) \quad \begin{aligned} \bar{P}_2(\nu) \gamma u^0 &= \bar{P}_2(\nu) \gamma \{V(\nu, t) u_0 - V(\nu) u^0 \cdot \nabla u^0\}, \\ \bar{P}_2(\nu) \gamma V(\nu, t) &= \bar{P}_2(\nu) \gamma U_0(\nu, t) \{P^{\infty} + N P_n^{\infty}\} e \\ &= -\bar{P}_1(\nu) \nu^{-1/2} \{P_2(\nu, t)^* + P_2(\nu, t)^{*V}\} \{P^{\infty} + N P_n^{\infty}\} e. \end{aligned}$$

From (3.33) and (3.34), it follows $\bar{P}_1(v)v^{-1/2}P_2(v,t) = O(1)$, and $\bar{P}_2(v)\gamma^{-1}u^0 \in X_{\infty}^{\ell,\rho,\theta}$. Hence, only the underlined terms contain the first derivatives of \tilde{v}^0 in linear order, and other terms are continuous in \tilde{v}^0 in $K_{a/\varepsilon}^{\ell-1,\rho,\theta}$. Thus, by virtue of Lemma 3.3 and 3.4, we can apply Theorem ACK in order to solve (6.4). Since (6.4) has a unique solution $\tilde{v}^0 \in X_{a/\varepsilon, \beta_1, T_1}^{\ell-1,\rho,\theta,(\mu)}$ with some $\beta_1 > \beta_0$ and $0 < T_1 \leq T$, we have

Theorem 6.1. Under the assumptions of Theorem 5.1, the "Navier-Stokes equation" (6.2) has a solution $\tilde{u}^0(\varepsilon, t, x, x_n) \in X_{a/\varepsilon, \beta_1, T_1}^{\ell-1,\rho,\theta,(\mu)}$, where $\beta_0 < \beta_1$ and $0 < T_1 \leq T$.

7. The second approximation $u^1(\varepsilon, t, x)$

We solve the equation (1.7) in three steps. (7) First we solve

$$(7.1) \quad \begin{aligned} \partial_t u + u^0 \cdot \nabla u - \nu \Delta u + \nabla p &= 0, \quad t > 0, \quad x \in \mathbb{R}_+^n, \\ \nabla \cdot u &= 0, \\ u|_{t=0} &= v_0, \\ \gamma_n u &= 0. \end{aligned}$$

We write the solution u of (7.1) as $u = V(v, t; u^0)v_0$, which is the definition of the "evolution operator" $V(v, t; u^0)$. Since (7.1) is linear, it is easy to solve it in a framework of Theorem ACK. However, we sketch briefly how to construct $V(v, t; u^0)$ in order to get better estimates (cf. [11]).

First we consider the transport equation

$$(7.2) \quad \partial_t v + w \cdot \nabla v = 0 \quad (w = \varepsilon u^0), \quad t > s, \quad x \in \mathbb{R}^n, \quad v|_{t=s} = v_0.$$

We assume $w(s, x)$ and $v_0(x) \in H_a^{k, \rho - \beta s, \theta - \beta s}(\mathbb{R}^n)$ ($k \geq \ell - 1$), which is defined in a similar way as in (2.5) with $\Omega(\rho, \theta, a)$ replaced by

$$\Omega(\rho, \theta, a) \cup \check{\Omega}(\rho, \theta, a/2), \quad \check{\Omega}(\rho, \theta, a) = \{(x', x_n); (x', -x_n) \in \Omega(\rho, \theta, a)\}.$$

If $w = w(x)$ does not depend on t , $A = -w \cdot \nabla$ generates a (time-local) semigroup e^{tA} in $K_{a,\beta,T}^{k,\rho,\theta}(\mathbb{R}^n)$, since $e^{tA} = 1 + tA + (tA)^2/2! + \dots$ converges on $[0,T]$ strongly in $K_{a,\beta,T}^{k,\rho,\theta}$. Because, with the terminologies of 4, the estimate: $\|A\|_{\rho'} \leq C\|f\|_{\rho}/(\rho-\rho')$, implies

$$\|(tA)^j f\|_{\rho-\beta t} \leq t^j C^j \|f\|_{\rho} (\beta t/j)^{-j} \leq (C/\beta)^j j^j \|f\|_{\rho}.$$

Hence, Stirling's formula: $j! = (2\pi)^{-1/2} e^{-j} j^{j+1/2} (1 + o(1))$, gives our conclusion with $\beta \geq 2Ce$. If $w = w(\varepsilon, t, x) \in \mathcal{X}_{a,\beta_0,T}^{k,\rho,\theta}$, Cauchy's connected segment method can be applied to prove that $-w \cdot \nabla$ generates evolution operator $T(t,s;w)$ such that $v = T(t,s;w)v_0$ is the unique solution of (7.2). Second we put

$$(7.3) \quad \begin{aligned} V_0(v,t,s;w) &= P^\infty U_0(v,t-s)T(t,s;w), \\ V_1(v,t,s;w) &= V_0(v,t,s;w) + \int_s^t V_0(v,t,r;w)R_1(v,r,s;w)dr. \end{aligned}$$

If we choose $R_1(t,s) \equiv R_1(v,t,s;w)$ satisfying

$$(7.4) \quad \begin{aligned} R_1(t,s) - \int_s^t R_0(t,r)R_1(r,s)dr &= R_0(t,s), \\ R_0(t,s) \equiv R_0(v,t,s;w) &\equiv [P^\infty U_0(v,t-s), w(\varepsilon, t, \cdot) \cdot \nabla], \end{aligned}$$

then, we obtain the evolution operator of the linear N-S equation:

$$(7.5) \quad \begin{aligned} \partial_t V_1 + w \cdot \nabla V_1 - v \Delta V_1 + Q^\infty R_1 &= 0, \quad t > s, \\ \nabla \cdot V_1 &= 0, \\ V_1|_{t=s} &= P^\infty. \end{aligned}$$

Since $R_0(v,t,s;w) = O(1)$ in $\mathcal{X}_{a,\beta,T}^{k,\rho,\theta}(\mathbb{R}^n)$, $\beta \geq \beta_0$, the Volterra equation (7.4) is solved by successive approximation.

Next we put

$$(7.6) \quad \begin{aligned} V(v,t,s;u^0) &= rV_1(v,t,s;eu^0)e - \nabla N \gamma_n V_1(v,t,s;eu^0) \\ &\quad + \int_s^t (rV_1(v,t,r;u^0)e - \nabla N \gamma_n V_1(v,t,r;eu^0))S(v,r,s;u^0)dr, \end{aligned}$$

$$(7.7) \quad \begin{aligned} S(t,s) - \int_s^t S_1(t,r)S(r,s)ds &= S_1(t,s), \\ S_1(t,s) \equiv S_1(v,t,s;u^0) &\equiv \{[u^0(t) \cdot \nabla, \nabla]N + u_n^0(t) \partial_n \nabla\} \gamma_n V_1(t,s). \end{aligned}$$

In the same way as above, (7.7) has a unique solution $S(v, t, s; u^0)$, which is $O(1)$ in $\mathcal{X}_{a, \beta, T}^{k, \rho, \theta}$. Thus, we have the evolution operator $V(v, t, s; u^0)$ of (7.1).

(2) The Poisson operator $\mathcal{Q} = \mathcal{Q}(v; u^0)$ of the equation (7.1) is given by solving

$$(7.8) \quad \begin{aligned} \partial_t v + u^0 \cdot \nabla v - v \Delta v + \nabla p &= 0, \quad t > 0, \quad x \in \mathbb{R}_+^n, \\ \nabla \cdot v &= 0, \\ v|_{t=0} &= 0, \\ \gamma_n v &= g \quad (g|_{t=0} = 0). \end{aligned}$$

We put

$$(7.9) \quad v(t) = \nabla N g + \int_0^t V(v, t, s; u^0) f(s, \cdot) ds.$$

then, $v(t)$ satisfies the last three conditions of (7.9). The first equation will be satisfied, if f is determined by

$$(7.10) \quad f(t) + [u^0(t) \cdot \nabla, \nabla] N g = 0, \quad [u^0 \cdot \nabla, \nabla] N = O(1).$$

$\mathcal{Q}(v; u^0)$ is defined by $v(t) = \mathcal{Q}(v; u^0) g$, which is $O(1)$ from $\mathcal{X}_{\beta, T}^{-k, \rho}$ to $\mathcal{X}_{a, \beta, T}^{k, \rho, \theta}$, if $k \leq \ell$ and $\beta \geq \beta_0$.

(3) The solution u^1 of the equation (1.7) is described as

$$(7.11) \quad u^1(t) = -\int_0^t V(v, t, s; u^0) u^1(s) \cdot \nabla u^0(s) ds - \mathcal{Q}(v; u^0) \gamma_n \tilde{u}^0.$$

Since this is a linear Volterra equation in $\mathcal{X}_{a, \beta_1, T_1}^{\ell-1, \rho, \theta}$, we have a unique solution $u^1(\varepsilon, t, x)$ in the same space. thus, we have

Theorem 7.1. Under the assumptions of Theorem 5.1 the "N-S equation" (1.7) has a solution $u^1(\varepsilon, t, x) \in \mathcal{X}_{a, \beta_1, T_1}^{\ell-1, \rho, \theta} \cap \mathcal{X}_{a, 2, \beta_1, T_1}^{\ell-1, \rho, \theta}$.

8. The complementary terms

We are at the final stage, though we can continue our procedure which was used to get \tilde{u}^0 and u^1 . It provides an asymptotic solution

of (1.1). In order to solve (1.8) and (1.9), we put

$$(8.1) \quad (7) \quad u^2 = rP^\infty U_0(v) e v^2 - (\mathcal{P}_1(v) + \mathcal{P}_0(v)) \gamma P^\infty U_0(v) e (v^2 + \tilde{v}^1/\varepsilon) \\ + (\mathcal{P}_1(v) + \mathcal{P}_0(v)) g/\varepsilon,$$

$$(2) \quad \tilde{u}^1 + \varepsilon \tilde{u}^2 = rP^\infty U_0(v) e (\tilde{v}^1 + \varepsilon \Lambda_1 \tilde{v}^2) - \mathcal{P}_2(v) \gamma P^\infty U_0(v) e (\varepsilon v^2 + \tilde{v}^1) \\ - \mathcal{P}(v) \gamma P^\infty U_0(v) e \varepsilon \Lambda_1 \tilde{v}^2 + \mathcal{P}_2(v) g,$$

$$(3) \quad g = {}^t(\gamma u^1, 0) \in \mathcal{X}_{\beta_1, T_1}^{-\ell-1, \rho}, \quad (\text{and } \varepsilon g \in \mathcal{X}_{\beta_1, T_1}^{-\ell, \rho, \theta}).$$

Then, all conditions of (1.8)-(1.9) are satisfied except for the first two equations. Later we will show that we may assume

$$(8.2) \quad u^2(\varepsilon, t, x) = {}^t(u^2(\varepsilon, t, x), u_n^2(\varepsilon, t, x)), \\ \tilde{u}^1(\varepsilon, t, x, x_n/\varepsilon) = {}^t(\tilde{u}^1(\varepsilon, t, x, x_n/\varepsilon), \varepsilon \tilde{u}_n^1(\varepsilon, t, x, x_n/\varepsilon)), \\ \tilde{u}^2(\varepsilon, t, x, x_n/\varepsilon) = {}^t(\tilde{u}^2(\varepsilon, t, x, x_n/\varepsilon), \tilde{u}_n^2(\varepsilon, t, x, x_n/\varepsilon)).$$

We make the change of the variables :

$$(8.3) \quad (x, x_n) \rightarrow (x, \varepsilon x_n), \quad \nabla = {}^t(\nabla, \partial_n) \rightarrow \bar{\nabla} = {}^t(\nabla, \partial_n/\varepsilon), \\ Q^\infty \rightarrow \bar{Q}^\infty = \bar{\nabla}(\nabla^2 + \partial_n^2/\varepsilon^2)^{-1} \bar{\nabla}, \quad P^\infty \rightarrow \bar{P}^\infty = 1 - \bar{Q}^\infty, \\ u^i(x, x_n) \rightarrow u^i(x, \varepsilon x_n) \equiv \bar{u}^i(x, x_n), \\ \tilde{u}^i(x, x_n/\varepsilon) \rightarrow \tilde{u}^i(x, x_n) \equiv \tilde{u}^i.$$

Then, we have (See (3.8).)

$$(8.4) \quad \bar{Q}^\infty = \begin{pmatrix} \bar{Q}^{\infty''} & \bar{R}^\infty \\ {}^t\bar{R}^\infty & \bar{S}^\infty \end{pmatrix}, \quad \bar{P}^\infty = \begin{pmatrix} \bar{P}^{\infty''} & -\bar{R}^\infty \\ -{}^t\bar{R}^\infty & \bar{T}^\infty \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} + \bar{W}^\infty, \\ \sigma(\bar{R}^\infty)(\xi) = (\varepsilon \xi' \xi_n / (\varepsilon^2 |\xi'|^2 + \xi_n^2)) = \varepsilon \sigma(\bar{S}^\infty)(\xi) \xi' / \xi_n, \\ \sigma(\bar{S}^\infty)(\xi) = \xi_n^2 / (\varepsilon^2 |\xi'|^2 + \xi_n^2), \\ \sigma(\bar{T}^\infty)(\xi) = \varepsilon^2 |\xi'|^2 / (\varepsilon^2 |\xi'|^2 + \xi_n^2) = \varepsilon \sigma({}^t N \bar{R}^\infty)(\xi) |\xi'| / \xi_n^{-1}, \\ \sigma(\bar{Q}^{\infty''})(\xi) = \varepsilon \sigma(\bar{R}^{\infty t} N)(\xi) |\xi'| / \xi_n^{-1}, \\ \text{i.e. } \bar{R}^\infty = \varepsilon \Lambda \bar{S}^\infty N \partial_n^{-1}, \quad \bar{T}^\infty = \varepsilon \Lambda N \bar{R}^\infty \partial_n^{-1}, \quad \bar{Q}^{\infty''} = \varepsilon \Lambda \bar{R}^{\infty t} N \partial_n^{-1}, \\ \bar{R}^\infty, \bar{S}^\infty, \bar{T}^\infty, \bar{W}^\infty (= {}^t(\bar{W}^\infty, \bar{W}_n^\infty) = \varepsilon \Lambda \bar{w}^\infty \partial_n^{-1}) = O(1), \quad \bar{w}^\infty = O(1).$$

Substitute (8.1) into (1.8), and drop the boundary layers and potential parts. Then, we obtain an equation for v^2 :

$$\begin{aligned}
(8.5) \quad & v^2(t) + \{ \underline{u}^0 + \varepsilon u^1 + \varepsilon^2 u^2 \} \cdot \underline{\nabla} r P^\infty U_0(v) e v^2 \\
& - [(u^0 + \varepsilon u^1) \cdot \underline{\nabla}, \underline{\nabla}] \{ N \gamma_n + N \kappa_1(v) \varepsilon \Lambda' + \varepsilon \kappa_0(v) \varepsilon \Lambda' \} P^\infty U_0(v) e v^2 \\
& - [(u^0 + \varepsilon u^1) \cdot \underline{\nabla}, \underline{\nabla}] N \gamma_n \bar{U}_0(v) \bar{w}_n^\infty e \Lambda' \partial_n^{-1} \tilde{v}^1 \\
& - [(u^0 + \varepsilon u^1) \cdot \underline{\nabla}, \underline{\nabla}] \{ N \kappa_1(v) + \varepsilon \kappa_0(v) \} \bar{U}_0(v) \{ e \Lambda'^t(\tilde{v}^1, 0) + \bar{w}_n^\infty e \Lambda' \partial_n^{-1} \tilde{v}^1 \} \\
& - \varepsilon u^2 \cdot \underline{\nabla} \underline{\nabla} \{ N \gamma_n + N \kappa_1(v) \varepsilon \Lambda' + \varepsilon \kappa_0(v) \varepsilon \Lambda' \} \bar{P}^\infty \bar{U}_0(v) e \{ \varepsilon \tilde{v}^2 + \tilde{v}^1 \} \\
& - [(u^0 + \varepsilon u^1) \cdot \underline{\nabla}, \underline{\nabla}] \{ N \kappa_1(v) + \varepsilon \kappa_0(v) \} \Lambda' g - \varepsilon u^2 \cdot \underline{\nabla} \{ \underline{\nabla} N \kappa_1(v) + \varepsilon \underline{\nabla} \kappa_0(v) \} \varepsilon \Lambda' g \\
& + u^2 \cdot \underline{\nabla} (u^0 + \varepsilon u^1) = f^1 \equiv -u^1 \cdot \underline{\nabla} u^1,
\end{aligned}$$

where only the underlined terms contain the first derivatives of v^2 and \tilde{v}^1 in linear order. Note

$$\begin{aligned}
(8.6) \quad & g^- = -\gamma^- u^1 = -\gamma^- v(v, t, s; u^0) *_{\underline{s}} u^1 \cdot \underline{\nabla} u^0 + (-N^-) \gamma \tilde{u}_n^0, \\
& \varepsilon \Lambda' \gamma^- v(v, t, s; u^0) = O((t-s)^{-1/2}), \\
& \gamma \tilde{u}_n^0 = \gamma \{ \bar{U}_0(v) \bar{e} - \bar{P}_1(v) \gamma \bar{U}_0(v) \bar{e} \} \partial_n^{-1} \underline{\nabla} \tilde{v}^0 - \bar{P}_1(v) \gamma^- u^0 \quad ((6.3)).
\end{aligned}$$

This and (3.36) of Lemma 3.2 imply that the last term containing g on the left hand side of (8.5) is in $\mathcal{X}_{a, \beta_1, T_1}^{l-2, \rho, \theta}$.

Taking (8.1)-(2) and (8.4) into account, we set for \tilde{u}^1 and \tilde{u}^2

$$\begin{aligned}
(8.1) \quad (2) \quad & \tilde{u}^1 = r \bar{U}_0(v) e \tilde{v}^1 - \bar{\mathcal{F}}_2(v) \gamma \bar{P}^\infty \bar{U}_0(v) e \{ \varepsilon \tilde{v}^2 + \tilde{v}^1 \} + \bar{\mathcal{F}}_2(v) g, \\
& \varepsilon \tilde{u}_n^1 = \varepsilon r \bar{U}_0(v) \bar{w}_n^\infty e \Lambda' \partial_n^{-1} \tilde{v}^1 \quad (= r \bar{U}_0(v) \bar{w}_n^\infty e \tilde{v}^1) \\
& \quad - \varepsilon \bar{\kappa}_2(v) \Lambda' \gamma \bar{P}^\infty \bar{U}_0(v) e \{ \varepsilon \tilde{v}^2 + \tilde{v}^1 \} - \varepsilon \bar{\kappa}_2(v) \Lambda' g, \\
& \tilde{u}^2 = r \bar{U}_0(v) \left\{ \left(\bar{w}_n^\infty e \Lambda' \partial_n^{-1} \tilde{v}^1 \right) + \bar{P}^\infty e \Lambda' \tilde{v}^2 \right\} - \bar{\mathcal{F}}(v) \gamma \bar{P}^\infty \bar{U}_0(v) e \Lambda' \tilde{v}^2.
\end{aligned}$$

Substitute (8.1) into (1.9) and change the variables by (8.3) (and by (8.1) (2)). Then, we obtain the equations for \tilde{v}^1 and \tilde{v}^2 :

$$\begin{aligned}
(8.7) (7) \quad & \tilde{v}^1 + \{ \underline{\bar{u}}^0 \cdot \underline{\nabla} + \tilde{u}^0 \cdot \underline{\nabla} + \tilde{u}_n^0 \partial_n + \varepsilon (\underline{\bar{u}}^1 + \varepsilon \underline{\bar{u}}^2 + \tilde{u}^1 + \varepsilon \tilde{u}^2) \cdot \underline{\nabla} \} r \bar{U}_0(v) e \tilde{v}^1 \\
& - \{ (\underline{\bar{u}}^0 + \tilde{u}^0) \cdot \underline{\nabla} + \varepsilon (\underline{\bar{u}}^1 + \varepsilon \underline{\bar{u}}^2 + \tilde{u}^1 + \varepsilon \tilde{u}^2) \cdot \underline{\nabla} \} \bar{\mathcal{F}}_2(v) \gamma \bar{P}^\infty \bar{U}_0(v) e \{ \varepsilon \tilde{v}^2 + \tilde{v}^1 \} \\
& - \{ (\underline{\bar{u}}_n^0 / \varepsilon + \underline{\bar{u}}_n^1 + \tilde{u}_n^0) \partial_n + (\varepsilon \underline{\bar{u}}_n^2 + \varepsilon \tilde{u}_n^1 + \varepsilon \tilde{u}_n^2) \partial_n \} \bar{\mathcal{F}}_2(v) \gamma \bar{P}^\infty \bar{U}_0(v) e \{ \varepsilon \tilde{v}^2 + \tilde{v}^1 \} \\
& + (\underline{\bar{u}}^0 + \varepsilon \underline{\bar{u}}^1 + \varepsilon \underline{\bar{u}}^2 + \tilde{u}^0 + \varepsilon \tilde{u}^1 + \varepsilon \tilde{u}^2) \cdot \underline{\nabla} \bar{\mathcal{F}}_2(v) g \\
& + \tilde{u}^1 \cdot \underline{\nabla} (\underline{\bar{u}}^0 + \varepsilon \underline{\bar{u}}^1 + \varepsilon \underline{\bar{u}}^2 + \tilde{u}^0) + \varepsilon \tilde{u}^2 \cdot \underline{\nabla} \tilde{u}^0 + (\underline{\bar{u}}_n^2 + \underline{\bar{u}}_n^1 + \tilde{u}_n^2) \partial_n \tilde{u}^0 \\
& = \tilde{h}^0 - (\tilde{u}^0 \cdot \underline{\nabla} + \tilde{u}_n^0 \partial_n) \tilde{u}^1,
\end{aligned}$$

$$\begin{aligned}
 (2) \quad \tilde{v}_n^1 &+ \{ \underline{u^0} \cdot \underline{\tilde{v}} + \underline{u^0} \cdot \underline{\tilde{v}} + \underline{u_n^0} \partial_n + \varepsilon (\underline{u^1} + \varepsilon \underline{u^2} + \underline{u^1} + \varepsilon \underline{u^2}) : \underline{\tilde{v}} \} r \underline{U}_0(v) \underline{\tilde{w}}_n^\infty e \underline{v}^1 \\
 &- (\underline{u^0} + \varepsilon \underline{u^1} + \varepsilon^2 \underline{u^2} + \underline{u^0} + \varepsilon \underline{u^1} + \varepsilon^2 \underline{u^2}) \cdot \underline{\tilde{v}} \cdot \underline{\tilde{h}}_2(v)_n \varepsilon \Lambda' \gamma \underline{P}^\infty \underline{U}_0(v) (\varepsilon v^2 + \underline{v}^1) \\
 &+ (\underline{u^0} + \varepsilon \underline{u^1} + \varepsilon^2 \underline{u^2} + \underline{u^0} + \varepsilon \underline{u^1} + \varepsilon^2 \underline{u^2}) \cdot \underline{\tilde{v}} \underline{\tilde{h}}_2(v)_n \varepsilon \Lambda' g \\
 &+ (\underline{u^1} + \varepsilon \underline{u^2}) \cdot \underline{\tilde{v}} (\underline{u_n^0} + \varepsilon \underline{u_n^1} + \varepsilon^2 \underline{u_n^2} + \varepsilon \underline{u_n^0}) = \underline{h}_n^0 - \underline{u^0} \cdot \underline{\tilde{v}} \underline{u_n^1} , \\
 (3) \quad \Lambda_1 \underline{\tilde{v}}^2 &+ \{ \underline{u^0} \cdot \underline{\tilde{v}} + \underline{u^0} \cdot \underline{\tilde{v}} + \underline{u_n^0} \partial_n + \varepsilon (\underline{u^1} + \varepsilon \underline{u^2} + \underline{u^1} + \varepsilon \underline{u^2}) : \underline{\tilde{v}} \} r \underline{P}^\infty \underline{U}_0(v) \varepsilon \Lambda_1 \underline{\tilde{v}}^2 \\
 &- (\underline{u^0} + \varepsilon \underline{u^1} + \varepsilon^2 \underline{u^2} + \underline{u^0} + \varepsilon \underline{u^1} + \varepsilon^2 \underline{u^2}) \cdot \underline{\tilde{v}} \underline{\tilde{h}}_2(v) \varepsilon \Lambda' \gamma \underline{P}^\infty \underline{U}_0(v) \varepsilon \Lambda_1 \underline{\tilde{v}}^2 \\
 &+ \{ (\underline{u_n^1} + \underline{u_n^2}) \partial_n / \varepsilon + \underline{u^2} \cdot \underline{\tilde{v}} \}^t (\underline{u^0} + \varepsilon \underline{u^1} + \varepsilon^2 \underline{u^2}, 0) .
 \end{aligned}$$

Here the underlined terms contain the first derivatives of \tilde{v}^1 and \tilde{v}^2 in linear order. We note that the (8.8) results from (8.9) :

$$\begin{aligned}
 (8.8) \quad (u^1, \tilde{u}^1, \tilde{u}_n^1, \tilde{u}^2) &\in K_{a, \beta, T}^{\ell-1, \rho, \theta} \times K_{a/\varepsilon, \beta, T}^{\ell-2, \rho, \theta, (\mu)} \times K_{a/\varepsilon, \beta, T}^{\ell-2, \rho, \theta} \times K_{a/\varepsilon, \beta, T}^{\ell-3, \rho, \theta} , \\
 (\varepsilon u^1, \varepsilon \tilde{u}^1, \varepsilon \tilde{u}_n^1, \varepsilon \tilde{u}^2) &\in K_{a, \beta, T}^{\ell, \rho, \theta} \times K_{a/\varepsilon, \beta, T}^{\ell-1, \rho, \theta, (\mu)} \times K_{a/\varepsilon, \beta, T}^{\ell-1, \rho, \theta} \times K_{a/\varepsilon, \beta, T}^{\ell-2, \rho, \theta} , \\
 (8.8) \quad (v^1, \tilde{v}^1, \tilde{v}_n^1, \tilde{v}^2) &\in K_{a, \beta, T}^{\ell-1, \rho, \theta} \times K_{a/\varepsilon, \beta, T}^{\ell-2, \rho, \theta, (\mu)} \times \tilde{K}_{a/\varepsilon, \beta, T}^{\ell-2, \rho, \theta} \times K_{a/\varepsilon, \beta, T}^{\ell-2, \rho, \theta} \\
 &= L_{a, \beta, T}^{\ell-1, \rho, \theta} ,
 \end{aligned}$$

The same holds if the symbol letter K (or L) is replaced by \mathcal{X} (\mathcal{L}).

(\mathcal{L} is defined from L as in 2). We put

$$(8.9) \quad w = {}^t (v^1, \tilde{v}^1, \tilde{v}_n^1, \tilde{v}^2) .$$

Then, the equations (8.5) and (8.7) (1), (2), (3) are written as

$$(8.10) \quad w = G(\varepsilon, t, w(\cdot)) ,$$

in the function space $L_{a, \beta, T}^{\ell-1, \rho, \theta}$ (or in $\mathcal{L}_{a, \beta, T}^{\ell-1, \rho, \theta}$), with $\beta > \beta_1$ and $0 < T < T_1$. Thus, we can apply Theorem ACK in order to solve (8.9), and we obtain

Theorem 8.1. Under the assumptions of Theorem 5.1 the "Navier-Stokes equation" (1.8)-(1.9) has a (unique) solution $(u^1, \tilde{u}^1 + \varepsilon \tilde{u}^2)$ with $\tilde{u}^1 = {}^t (\tilde{u}^1, \varepsilon \tilde{u}_n^1)$ such that $(u^2, \tilde{u}^1, \tilde{u}_n^1, \tilde{u}^2) \in \mathcal{X}_{a, \beta_2, T_2}^{\ell-2, \rho, \theta} \times \mathcal{X}_{a/\varepsilon, \beta_2, T_2}^{\ell-2, \rho, \theta, (\mu)} \times \tilde{\mathcal{X}}_{a/\varepsilon, \beta_2, T_2}^{\ell-2, \rho, \theta} \times \mathcal{X}_{a/\varepsilon, \beta_2, T_2}^{\ell-3, \rho, \theta}$ with $\beta_1 < \beta_2$ and $0 < T_2 < T_1$.

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References

- [1] Asano, k., Zero-viscosity limit of the incompressible Navier-Stokes equation 1, Preprint (1985).
- [2] Asano, K., A note on the abstract Cauchy Kowalewski theorem, to appear in Proc. Japan Acad. (1988).
- [3] Beale, J.T. and Majda, A., Rates of convergence for viscous splitting of the Navier-Stokes equations, Math. Comp. 37 (1981), 243-259.
- [4] Kato, T., Quasilinear equations of evolutions with applications to partial differential equations, Lec. Notes in Math. 448 (1975), 25-70. Springer.
- [5] Solonikov, V.A., Estimates of the solutions of a nonstationary linearized system of Navier-Stokes equations, Amer. Math. Soc. Transl. (2) 75 (1968), 1-116.
- [6] Ukai, S., A solution formula for the Stokes equation in R_+^n , Comm. Pure Appl. Math. 40 (1987) 611-621.