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Asymptotic behaviour of words partition function

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Let \( n \) be a non-negative integer and \( r \) be a positive integer. We denote by \( w(n|r) \) the number of partitions into some words using any \( n \) letters in the alphabet that consists of \( r \) letters.

**EXAMPLE.** Let \( n=3 \) and \( r=2 \) (alphabet = \{a, b\}). Thus we have
\[
w(3|2)=20\text{ partitions:}\]
aaa, aab, aba, abb, baa, bab, bba, bbb,
a a a, a b a, b a a, b a b, a a b, a b b, b b a, b b b,
a a a a, a a b, a b b, b b b.

We have
\[
(1) \quad w(n|r) = \sum_{s_1,s_2,\ldots \geq 0} \prod_{t=1}^{n} \binom{r^{s_t}+s_t-1}{s_t},
\]
\[
= \prod_{n=s_1+2s_2+\ldots}^{\infty} \binom{r^{s_t}+s_t-1}{s_t}.
\]

By a combinatorial lemma (see Proposition in [1]), we have
\[
(2) \quad \sum_{n=0}^{\infty} w(n|r) x^n = \prod_{m=1}^{\infty} (1 - x^m)^{-\frac{m}{r}}, \quad |x| < 1/r.
\]

Therefore we have, by taking the logarithmic derivative of (2),
\[
(3) \quad n \cdot w(n|r) = \sum_{m=0}^{n-1} w(m|r) \sigma(n-m|r),
\]
where \( \sigma(n|r) = \sum_{d|n} d \cdot r^d \). Thus we may write
\[
w(n|r) = \frac{1}{n!} \sum_{m=0}^{n} W_{n,m} \cdot r^m,
\]
with non-negative integers \( W_{n,m} \). Particularly we have \( W_{n,0} = 1 \) (if \( n = 0 \)), \( = 0 \) (if \( n > 0 \)); \( W_{n,1} = (n-1)! \) (for any \( n > 0 \) and
\[ W_{n,n} = \sum_{s_1, s_2, \ldots} \frac{n!}{s_1! \ldots s_n!} s_1 s_2 \ldots \geq 0 \quad \text{if } n = 1s_1 + 2s_2 + \ldots \]

By Faà di Bruno's formula, we have

(4) \[ \exp \frac{x}{1-x} = \sum_{n=0}^{\infty} \frac{W_{n,n} x^n}{n!} x^n. \]

\[ p(n) = w(n\mid 1) = \sum_{m=0}^{n} \frac{W_{n,m}}{n!} \text{ is well known as the partition function. From now on, we consider } w(n\mid r) \text{ for any real } r > 1. \]

We may define \( w(n\mid r) \) by (1) for such \( r \). Then (2), (3) are valid also in this case. Let \( f(x\mid r) = \sum_{n=0}^{\infty} w(n\mid r) x^n \). We have

(5) \[ \log f(e^{-r}\mid r) = \sum_{m=1}^{\infty} \frac{r^m}{m!} \sum_{k=1}^{\infty} \frac{e^{-mk\tau}}{k} = - \sum_{k=1}^{\infty} \frac{1}{k} \frac{e^{-k\tau}}{e^{-r\tau} - a_{k,\ell}} \]

(Re \( \tau > \log r > 0 \)),

where \( a_{k,\ell} = r^{\ell-1/k} (\tau_k = e^{2\pi i/k}) \). From this, we have the following

**LEMMA.** If \( r > 1 \), the function \( \log f(e^{-r}\mid r) \) is regular for \( \Re \tau > 0 \) except \( \tau = (\log r - 2\pi \ell i)/k \) (\( k = 1, 2, \ldots; \ell \in \mathbb{Z} \)), where there are simple poles of the function with respective residues \( 1/k^2 \).

Our purpose is to get asymptotic expressions for \( w(n\mid r) \) with fixed \( r > 1 \). We are able to get following theorems:

**THEOREM 1.** For any \( r > 1 \),

\[ w(n\mid r) = \frac{e^{2\sqrt{n} r}}{2\sqrt{\pi} n^{3/4}} \left\{ \exp \sum_{h=2}^{\infty} \frac{1}{h(h-1)} \right\}^{N-1} \sum_{v=1}^{N-1} \frac{u_v(r)}{n^2} + O_r N(n^{-2}). \]

where \( \{u_v(r)\} \) is a sequence of functions of \( r \) only.
THEOREM 2. For any $r < 1$,

$$w(n|r) = \sum_{k=1}^{N-1} R_k + O_r (r^{n/N} e^{2\sqrt{n}/N} n^{-3/4}),$$

where $R_k = \sum_{k=1}^{k-1} R_k, k,$

$$R_k, k = r^{n/k} e^{V_0(r;k,l)} \sum_{v=0}^{\infty} U_v(r;k,l) (k^{1/2}/n)^{-v-1} I_{v+1}(2\sqrt{n}/k),$$

$$V_0(r;k,l) = -\frac{1}{2k} + \sum_{h=1,h\neq k}^{1} \frac{1}{h(rh/k-1)} \frac{1}{h(rh/k-1)},$$

$U_v(r;k,l)$ are the coefficients in the Taylor expansion

$$\exp\left(\frac{1}{k^2} - V_0(r;k,l)\right) \left(\frac{\alpha}{\tau}\right) = \sum_{v=0}^{\infty} U_v(r;k,l) \tau^v,$$

and $I_v(x)$ are modified Bessel functions.

Concerning this theorem, we have an equality as the following theorem.

THEOREM 3. If $r > e^{4/3}$, then the series $\sum_{k=1}^{\infty} R_k$ converges to $w(n|r)$.

I wish to publish the proof of these theorems on another day.

On the leading coefficient $W_{n,n}/n$ of the polynomial $w(n|r)$ in $r$,

We have the following theorem.

THEOREM 4.

$$W_{n,n}/n! = e^{-1/2} \sum_{v=0}^{\infty} b_v n^{-\frac{v+1}{2}} I_{v+1}(2\sqrt{n})$$

$$= \frac{e^{2\sqrt{n}}}{2\sqrt{\pi e n^{3/4}}} \left\{ 1 + \sum_{v=1}^{N-1} \left( u_v n^{-\frac{v}{2}} + O_n \left( \frac{N}{n^{3/2}} \right) \right) \right\},$$

where the numbers $b_v$ are the coefficients in the Taylor expansion

$$\exp\left(\frac{1}{\tau} + \frac{1}{e^\tau - 1}\right) = \sum_{v=0}^{\infty} b_v \tau^v,$$

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and \( u_\eta \) are given by \( u_\eta = \sum_{v+\mu=\eta; v, \mu \geq 0} (-1/4)^\mu (v+1, \mu) b_v \) with \[
(v, \mu) = \frac{\Gamma(v+\mu+\frac{1}{2})}{\mu! \Gamma(v-\mu+\frac{1}{2})} = \frac{(4v^2-1^2)(4v^2-3^2) \cdots (4v^2-(2\mu-1)^2)}{\mu! 4^\mu}.
\]

The numbers \( W_{\eta,n} \) have been treated by Motzkin[2] with his notation \( n^* \).

**Remark.** i) The functions \( U_v = U_v(r; k, \ell) \) \( (v = 0, 1, \ldots) \) in Theorem 2 are explicitly given by
\[
U_v = \sum_{v=1v_1+2v_2+\ldots} \frac{V_{v_1}^{v_1} V_{v_2}^{v_2} \cdots}{v_1! v_2! \cdots},
\]
where
\[
(7) \quad V_v = \frac{B_v+1}{(v+1)!} k^{v-1} + V^*_v (r; k, \ell),
\]
\[
V^*_v = \frac{1}{v!} \sum_{m=0}^{v} (-1)^m m! S(v, m) \sum_{h \geq 1, h \neq k} h^{v-1} \frac{r^m (h/k - 1) \frac{\zeta_k}{\zeta_k - h \ell - 1} \cdot m!}{(r)^{h/k}}
\]
with Bernoulli numbers \( B_v = \lim_{t \to 0} \frac{t^v}{(e^{t}-1)^v} \) and Stirling numbers \( S(v, m) = \left( (e^t - 1^m / m! ) \right)_v \) \( |t=0 \) of the second kind. ii) The functions \( u_\eta (r) \) \( (\eta = 0, 1, \ldots) \) in Theorem 1 are given by
\[
(8) \quad u_\eta (r) = \sum_{v+\mu=\eta; v, \mu \geq 0} (-1/4)^\mu (v+1, \mu) U_v(r; 1, 0).
\]
iii) The numbers \( b_v \) in Theorem 4 are given by
\[
(9) \quad b_v = \sum_{v=1v_1+2v_2+\ldots} \frac{(B_v/2!)^{v_1} (B_v/3!)^{v_2} \cdots}{v_1! v_2! \cdots}.
\]

**References**
