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Asymptotic behaviour of words partition function

小樽商大　兼岩 龍二
(Ryuji Kaneiwa)

Let n be a non-negative integer and r be a positive integer. We denote by $w(n|r)$ the number of partitions into some words using any n letters in the alphabet that consists of r letters.

EXAMPLE. Let $n=3$ and $r=2$ (alphabet = $\{a,b\}$). Thus we have $w(3|2)=20$ partitions:

aaa, aab, aba, abb, baa, bab, bba, bbb,

aa a, ab a, ba a, bb a, aa b, ab b, ba b, bb b,

a a a, a a b, a b b, b b b.

We have

$$(1)\qquad w(n|r) = \sum_{\substack{s_1,s_2,\dots \geq 0 \\ n=1s_1+2s_2+\dots}} \prod_{t=1}^{n} \binom{r^t+s_t-1}{s_t}.$$

By a combinatorial lemma (see Proposition in [1]), we have

$$(2)\qquad \sum_{n=0}^{\infty} w(n|r)\, x^n = \prod_{m=1}^{\infty} (1-x^m)^{-r^m}, \quad |x| < 1/r.$$

Therefore we have, by taking the logarithmic derivative of (2),

$$(3)\qquad n\cdot w(n|r) = \sum_{m=0}^{n-1} w(m|r)\, \sigma(n-m|r),$$

where $\sigma(n|r) = \Sigma_{d|n}\, d\cdot r^d$. Thus we may write

$$w(n|r) = \frac{1}{n!} \sum_{m=0}^{n} W_{n,m}\, r^m,$$

with non-negative integers $W_{n,m}$. Particularly we have $W_{n,0} = 1$ (if $n = 0$), $= 0$ (if $n > 0$); $W_{n,1} = (n-1)!$ (for any $n > 0$) and

$$W_{n,n} = \sum_{\substack{s_1, s_2, \cdots \geqq 0 \\ n=1s_1+2s_2+\cdots}} n!/s_1!\cdots s_n!.$$

By Faà di Bruno's formula, we have

$$(4) \qquad \exp \frac{x}{1-x} = \sum_{n=0}^{\infty} \frac{W_{n,n}}{n!} x^n.$$

$p(n) = w(n|1) = \Sigma_{m=0}^{n} W_{n,m}/n!$ is well known as the partition function. From now on, we consider $w(n|r)$ for any real $r > 1$. We may define $w(n|r)$ by (1) for such r. Then (2),(3) are valid also in this case. Let $f(x|r) = \Sigma_{n=0}^{\infty} w(n|r)\, x^n$. We have

$$\log f(e^{-\tau}|r) = \sum_{m=1}^{\infty} r^m \sum_{k=1}^{\infty} \frac{e^{-mk\tau}}{k} = - \sum_{k=1}^{\infty} \frac{1}{k} \frac{e^{-k\tau}}{e^{-k\tau} - r^{-1}}$$

$$(5) \qquad = - \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{\ell=0}^{k-1} \frac{e^{-\tau}}{e^{-\tau} - a_{k,\ell}} \quad (\operatorname{Re} \tau > \log r > 0),$$

where $a_{k,\ell} = \zeta_k^{\ell}\, r^{-1/k}$ ($\zeta_k = e^{2\pi i/k}$). From this, we have the following

LEMMA. If $r > 1$, the function $\log f(e^{-\tau}|r)$ is regular for $\operatorname{Re} \tau > 0$ except $\tau = (\log r - 2\pi\ell i)/k$ ($k=1,2,\ldots;\ell \in \mathbf{Z}$), where there are simple poles of the function with respective residues $1/k^2$.

Our purpose is to get asymptotic expressions for $w(n|r)$ with fixed $r > 1$. We are able to get following theorems:

THEOREM 1. For any $r > 1$,

$$w(n|r) = \frac{e^{2\sqrt{n}}\, r^n}{2\sqrt{\pi e}\, n^{3/4}} \left\{\exp \sum_{h=2}^{\infty} \frac{1}{h(r^{h-1} - 1)}\right\} \left\{1 + \sum_{\nu=1}^{N-1} u_\nu(r)\, n^{-\frac{\nu}{2}} + O_{r,N}(n^{-\frac{N}{2}})\right\},$$

where $\{u_\nu(r)\}$ is a sequence of functions of r only.

THEOREM 2. For any $r < 1$,

$$w(n|r) = \sum_{k=1}^{N-1} R_k + O_{r,N}(r^{n/N} e^{2\sqrt{n}/N} n^{-3/4}),$$

where $R_k = \Sigma_{\ell=0}^{k-1} R_{k,\ell}$,

$$R_{k,\ell} = r^{n/k} \zeta_k^{-\ell n} e^{V_0(r;k,\ell)} \sum_{\nu=0}^{\infty} U_\nu(r;k,\ell) \; (k\sqrt{n})^{-\nu-1} I_{\nu+1}(2\sqrt{n}/k),$$

$$V_0(r;k,\ell) = -\frac{1}{2k} + \sum_{h\geqq 1, h\neq k} \frac{1}{h(r^{h/k-1}\zeta_k^{-h\ell} - 1)},$$

$U_\nu(r;k,\ell)$ are the coefficients in the Taylor expansion

$$\exp(-\frac{1}{k^2\tau} - V_0(r;k,\ell))f(a_{k,\ell}\, e^{-\tau}|r) = \sum_{\nu=0}^{\infty} U_\nu(r;k,\ell)\, \tau^\nu,$$

and $I_\nu(x)$ are modified Bessel functions.

Concerning this theorem, we have an equality as the following

THEOREM 3. If $r > e^{4/3}$, then the series $\Sigma_{k=1}^{\infty} R_k$ converges to $w(n|r)$.

I wish to publish the proof of these theorems on another day. On the leading coefficient $W_{n,n}/n$ of the polynomial $w(n|r)$ in r, We have the following

THEOREM 4.

$$W_{n,n}/n! = e^{-1/2} \sum_{\nu=0}^{\infty} b_\nu\, n^{-\frac{\nu+1}{2}} I_{\nu+1}(2\sqrt{n})$$

$$= \frac{e^{2\sqrt{n}}}{2\sqrt{\pi e}n^{3/4}} \left\{1 + \sum_{\nu=1}^{N-1} u_\nu\, n^{-\frac{\nu}{2}} + O_N(n^{-\frac{N}{2}})\right\},$$

where the numbers b_ν are the coefficients in the Taylor expansion

$$\exp(-\frac{1}{\tau} + \frac{1}{2} + \frac{1}{e^\tau - 1}) = \sum_{\nu=0}^{\infty} b_\nu\, \tau^\nu,$$

and u_η are given by $u_\eta = \Sigma_{\nu+\mu=\eta;\nu,\mu\geqq 0} (-1/4)^\mu (\nu+1,\mu) b_\nu$ with

$$(\nu,\mu) = \frac{\Gamma(\nu+\mu+\frac{1}{2})}{\mu!\ \Gamma(\nu-\mu+\frac{1}{2})} = \frac{(4\nu^2-1^2)(4\nu^2-3^2)\cdots(4\nu^2-(2\mu-1)^2)}{\mu!\ 4^\mu}$$

The numbers $W_{n,n}$ have been treated by Motzkin[2] with his notation $!^{n+}$.

REMARK. i) The functions $U_\nu = U_\nu(r;k,\ell)$ $(\nu = 0,1,\ldots)$ in Theorem 2 are explicitly given by

$$(6) \qquad U_\nu = \sum_{\nu=1\nu_1+2\nu_2+\cdots} \frac{V_1^{\nu_1} V_2^{\nu_2}\cdots\cdot}{\nu_1!\ \nu_2!\cdots\cdot},$$

where

$$(7) \qquad V_\nu = \frac{B_{\nu+1}}{(\nu+1)!} k^{\nu-1} + V_\nu^*(r;k,\ell),$$

$$V_\nu^* = \frac{1}{\nu!} \sum_{m=0}^{\nu} (-1)^m m!\ S(\nu,m) \sum_{h\geqq 1, h\neq k} h^{\nu-1} \frac{r^{m(h/k-1)} \zeta_k^{-mh\ell}}{(r^{h/k-1}\zeta_k^{-h\ell}-1)^{m+1}},$$

with Bernoulli numbers $B_\nu = \lim_{t\to 0}(t/(e^t-1))^{(\nu)}$ and Stirling numbers $S(\nu,m) = ((e^t-1)^m/m!)^{(\nu)}|_{t=0}$ of the second kind. ii) The functions $u_\eta(r)$ $(\eta = 0,1,\ldots)$ in Theorem 1 are given by

$$(8) \qquad u_\eta(r) = \sum_{\nu+\mu=\eta;\nu,\mu\geqq 0} (-1/4)^\mu (\nu+1,\mu)\ U_\nu(r;1,0).$$

iii) The numbers b_ν in Theorem 4 are given by

$$(9) \qquad b_\nu = \sum_{\nu=1\nu_1+2\nu_2+\cdots} \frac{(B_2/2!)^{\nu_1}(B_3/3!)^{\nu_2}\cdots\cdot}{\nu_1!\ \nu_2!\cdots\cdot}.$$

References

[1] Kaneiwa, R., An Asymptotic Formula for Cayley's Double Partition function p(2;n), Tokyo J. of Math., **2** (1979), 137-158.

[2] Motzkin, T.S., Sorting Numbers for Cylinders and other Classification Numbers, Proc. of Symposia in Pure Math., **19** (1971), 167-176.