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Some Results and Problems on the Diophantine Equations

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1. On Diophantine equations $x_1^{x_1} x_2^{x_2} \dots x_k^{x_k} = z^z$

Erdös asked for integer solutions of the equation

$$x^x y^y = z^z \tag{1}$$

with $x>1$, $y>1$. In 1940, Ko Chao[1] proved that when $(x,y)=1$, equation (1) has no solutions in positive integers $x>1$, $y>1$, $z>1$ and when $(x,y)>1$, equation (1) has infinitely many solutions

$$x = 2^{2^{n+1}(2^n-n-1)+2n}(2^n-1)^{2(2^n-1)}$$
$$y = 2^{2^{n+1}(2^n-n-1)}(2^n-1)^{2(2^n-1)+2}$$
$$z = 2^{2^{n+1}(2^n-n-1)+n+1}(2^n-1)^{2(2^n-1)+1},$$

$n>1$, with $4xy=z$. Other solutions have not been found yet. Recently, Erdös pointed out that it is possible that these should be all the solutions of the equation (1).

In 1984, Uchiyama (内山)[2] proved that there can only be a finite number of solutions for any fixed value of $Q=xy/z<\frac{1}{4}$.

Anderson conjectured that the equation $w^w x^x y^y = z^z$ has no solution with $1<w<x<y$.

In 1964, Ko Chao and Sun Qi[3] proved the equation

$$\prod_{i=1}^{k} x_i^{x_i} = z^z, \quad x_i>1,\ k\geq 2,\ i=1,\dots,k,$$

has infinitely many solutions

$$x_1 = k^{k^n(k^{n+1}-2n-k)+2n}\,(k^n-1)^{2(k^n-1)}$$

$$x_2 = k^{k^n(k^{n+1}-2n-k)}\,(k^n-1)^{2(k^n-1)+2}$$

$$x_3 = \cdots = x_k = k^{k^n(k^{n+1}-2n-k)+n}\,(k^n-1)^{2(k^n-1)+1}$$

$$z = k^{k^n(k^{n+1}-2n-k)+n+1}\,(k^n-1)^{2(k^n-1)+1},$$

of which the first one is $x_1=3^{14}2^4$, $x_2=3^{12}2^6$, $x_3=3^{13}2^5$, $z=3^{14}2^5$ for $k=3$. It gives a counter-example to Anderson's Conjecture.

For $2\nmid xy$ are there any solutions of equation (1)? This still remains unproved. For the equation $x_1^{x_1}x_2^{x_2}x_3^{x_3}=z^z$ we asked that are there any solution with $1<x_1<x_2<x_3$, $2\nmid x_1x_2x_3$?

2. Some exponential equations

Jesmanowicz conjectured that the diophantine equation

$$a^x+b^y=c^z$$

has no integer solution except $x=y=z=2$, where a, b, c satisfy $a^2+b^2=c^2$. Ko Chao[4]-[6] made a lot of investigations about it in 1958-1965. Lu Wenduan[7], Chen Jingrun[8] and Sun Qi[9] also have studied this conjecture. For example, Lu proved that if $a=4n^2-1$, $b=4n$, $c=4n^2+1$, then Jesmanowicz's conjecture is true.

For the Diophantine equation

$$a^x+b^y=c^z \tag{2}$$

where a, b, c are different primes. In 1958-1976, Nagell, Makowski, Hadano (石谷野), Uchiyama (内山) studied this equation. They gave all the solution (x,y,z) for $\max(a,b,c)\leqslant 17$. In 1984, Sun Qi and Zhou Xiaoming[10] gave all the nonnegative integral solutions of equation (2) for $\max(a,b,c)=19$. We proved also that the equation (2) has no solutions in non-negative integers $x>1$, y, z, if that $a=2$, $b=p$, $c=q$, where $p\equiv 5 \pmod 8$ or $p\equiv 1 \pmod 8$ and $p=u^2+16v^2$, $2\nmid v$, and $q\equiv 3 \pmod 4$, $q\equiv 2 \pmod p$. In 1985, for $\max(a,b,c)=23$, it has been solved by Yang Xiaozuo.[11] In 1987, for $29\leq\max(a,b,c)<100$, it has been solved by Cao Zhenfu[12].

Selfridge asks for what a and b

$$2^a-2^b \mid n^a-n^b, \tag{3}$$

is true for all n?

Sun Qi and Zhang Mingzhi[13] proved that for $0\leq b<a$ if and only if $(a,b)=(1,0)$, $(2,1)$, $(3,1)$, $(4,2)$, $(5,3)$, $(5,1)$, $(6,2)$, $(7,3)$, $(8,4)$, $(8,2)$, $(9,3)$, $(14,2)$, $(15,3)$, $(16,4)$, (3) is ture for all n.

3. Some diophantine equations which arise in the combinatorial theory and the theory of finite groups

Hall asked for the integer solution of the Diophantine equation

$$p^r+2=q^s \tag{4}$$

which arises in the combinatorial theory, where p, q are prime numbers. This includes that $5^2+2=3^3$, but we know no other case in which both $r>1$ and $s>1$. Sun Qi and Zhou Xiaoming[14] studied the case $p+2=q$ in (4). Cao Zhenfu[15] proved that when $p+2=q$, then the equation (4) has no solution for $r>1$, $s>1$.

Crescenzo[16] investigated the equation (5) below, which arise in the theory of finite groups.

$$p^m - 2q^n = \pm 1, \quad p,\ q \text{ primes},\ m>1,\ n>1. \qquad (5)$$

Crescenzo proved that with the exception of the relation $(239)^2 - 2(13)^4 = -1$, every solution of (5) has exponents $m=n=2$.

However, it should be noted that Crescenzo's theorem is wrong. Because the equation (5) has another solution $p=3$, $q=11$, $m=5$, $n=2$.

For diophantine equation

$$3^m - 2q^n = 1, \quad m>1,\ n>1,\ q \text{ is an odd prime},\ 2\nmid m, \qquad (6)$$

we conjectured that equation (6) has no solution except $q=11$, $m=5$, $n=2$.

For equation (6), the proof of following results are easy.

1) If $2|n$, then the conjecture is true.

2) If the equation (6) has solution, then $q \equiv 1 \pmod{12}$.

3) If the equation (6) has solution then $\left(\frac{q}{7}\right)=\left(\frac{q}{13}\right)=\left(\frac{q}{757}\right)=1$, where $\left(\frac{a}{q}\right)$ denotes legendre symbol.

Recently, Sun Qi studies further equations (7) and (8) below, which include (4) and (6) respectively.

For diophantine equation

$$a^m - kb^n = 2, \quad k>0,\ 2\ k, \qquad (7)$$

and diophantine equation

$$a^m - lb^n = 1, \quad l>0, \qquad (8)$$

we proved the following results.

1) If the equation (7) has positive integer solution a_1, b_1, m_1, n_1 with $2\nmid m_1 n_1$, $m_1>1$, $n_1>1$, then $\frac{a_1^{m_1} + kb_1^{n_1}}{2} + a_1^{\frac{m_1-1}{2}} b_1^{\frac{n_1-1}{2}} \sqrt{k a_1 b_1}$ is the fundamental solution of the Pell's equation $x^2 - k a_1 b_1 y^2 = 1$.

2) If the equation (8) has positive integer solution a_2, b_2, m_2, n_2, with $2\nmid m_2n_2$, $m_2>1$, $n_2>1$, then $a_2^{m_2}+l b_2^{n_2}+2a_2^{\frac{m_2-1}{2}}b_2^{\frac{n_2-1}{2}}\sqrt{l a_2 b_2}$ is the fundamental solution of the Pell's equation $x^2-la_2b_2y^2=1$.

From above results, we proved that the equation (7) has no positive integer solution a_1, b_1, m_1, n_1, $2\nmid m_1n_1$, $m_1>1$, $n_1>1$, if that $a_1=kb_1t^2+2$ or $b_1=ka_1t^2+2$, etc. For the equation (8), we proved similar results also. If $q=pt^2+2$, where p, q denote odd prime numbers, then equation $q^m=p^n+2$ has no integer solution m, n with $m>1$, $n>1$, $2\nmid mn$.

We proved also the following result.

If $q=6s^2+1$, then the equation (6) has no solution.

4. Some Cubic equations and Quartic equations

From the well known identity

$$(x+1)^3+(x-1)^3-2x^3=6x,$$

Mordell suggested that perhaps most of the numbers can be expressed as $x^3+y^3+2z^3$ with integers x, y, z. In 1936, Ko Chao[18] gave the decompositions into four cubes in this form for $n\leqslant 100$ except the numbers 76, 99.

For the diophantine equations $x^3+y^3+2z^3=76$ and $x^3+y^3+2z^3=99$, we asked that ~~ake~~ are there any integeral solution x, y, z? This still remains unproved.

A interesting equation is

$$x^3+y^3+z^3=n \tag{9}$$

When $n=3$, there are solutions given by $(x,y,z)=(1,1,1)$, $(4,4,-5)$, $(4,-5,4)$, $(-5,4,4)$. In 1984, Scarowsky and Boyarsky proved that the equation (9) has no new solutions were found for $|m|\leq 50000$, where $x+y+z=3m$, $m\in Z$. In 1985, Cassels proved that any integral solution of the equation (9) has $x\equiv y\equiv z \pmod 9$. Recently, Sun Qi[19] proved that if $n=9a^3$, where a is not divisible by primes of the form $6k+1$, then any integral solution of the equation $x^3+y^3+z^3=9a^3$ satisfies $9\left|\frac{xyz}{d^3}\right.$, where $(x,y,z)=d$. If $n=3a^3$, $3\nmid a$, then any integral solution of the equation $x^3+y^3+z^3=3a^3$ satisfies $\frac{x}{d}\equiv\frac{y}{d}\equiv\frac{z}{d} \pmod 9$.

Ljunggren proved that if $D>2$ be a square-free integer which is not divisible by primes of the form $6n+1$, then equations

$$x^3\pm 1=Dy^2 \tag{10}$$

have

has at most one solution in positive integers x, y. In 1981, Ko Chao and Sun Qi[20],[21] proved that the only solution in integers of the equations (10) is $x=1$, $y=0$. In 1975-1981, Ko Chao and Sun Qi[22]–[26] studied the equation

$$x^4-Dy^2=1,\quad D>1,\quad \mu(D)\neq 0. \tag{11}$$

We

They proved that 1) If $D\equiv 3 \pmod 8$, $\varepsilon=x_0+y_0\sqrt{D}$ is the fundamental solution of the equation $x^2-Dy^2=1$ and if $x_0\equiv 0 \pmod 2$, then the equation (11) has no solutions in positive integers x, y. 2) If $D=2p$, p is an odd prime number, then the equation (11) has no positive integral solutions except $p=3$, $x=7$, $y=20$. 3) If D is not divisible by primes of the form

$4n+1$, then the equation (11) has no positive integral solutions. In 1979 and 1981, Ko Chao and Sun Qi[27]-[29] also studied the equations $x^2-Dy^4=1$, $x^4+4=Dy^2$ and $x^3\pm 8=Dy^2$.

For equation $6y^2=x(x+1)(2x+1)$ Mordell asked if there was an elementary proof. In 1985, Ma Degang[30] have answered the Mordell's question.

In 1942, Ljunggren ~~has~~ showed that the only solutions of $x^2=2y^4-1$ in positive integers are (1,1) and (239,13) but his proof is difficult. Mordell asks if it is possible to find a simple or elementary proof. This still remains unproved.

In 1967, Bumby proved that the diophantine equation

$$2y^2=3x^4-1 \tag{12}$$

has only integer solutions $x=\pm1, \pm3$. The proof depends upon an application of the law quadratic reciprocity in the quadratic fields $Q(\sqrt{-2})$.

In 1979, Bremner proved that the diophantine equation

$$3x^4-4y^4-2x^2+12y^2-9=0 \tag{13}$$

only positive integer solutions $x=1$, $y=1$, and $x=3$, $y=3$. The proof depends upon an application of the Skolem's p-adic method.

We asks if it is possible to find an elementary proof for the equation (12) or the equation (13).

5. The equation $\frac{1}{x_1}\ \ldots\ \frac{1}{x_s}\pm\frac{1}{x_1\ldots x_s}=1$. Znām problem. S-problem.

For the equation

$$\sum_{i=1}^{s}\frac{1}{x_i}\quad\frac{1}{x_1\ldots x_s}=1, \quad 0<x_1<\ldots<x_s, \tag{14}$$

in 1964, Ko Chao and Sun Qi [31] gave all solutions for $s=5$ and $s=6$.

Let $\Omega(s)$ be the number of positive integral solutions of the equation (14). In 1978, Sun Qi [32] proved that when $s\geqslant 4$, then $\Omega(s)<\Omega(s+1)$.

In 1978, Janàk and Skula gave eighteen solutions of the system of congruences

$$x_1\ldots x_{i-1}\,x_{i+1}\ldots x_n+1\equiv 0 \pmod{x_i},\quad x_i>1,\quad i=1,\ldots,n,\quad n>1, \tag{15}$$

for $n=7$. Let $H(n)$ be the number of solutions of the system of congruences (15). In 1983, Sun Qi [33] proved that if $n\geqslant 4$, then $H(n)<H(n+1)$. As a corollary one obtains: If $n\geqslant 7$, then $H(n)\geqslant n+11$.

In 1972, Znām asked whether for every positive integer $n>1$ there exist integers $x_i>1$ $(i=1,\ldots,n)$ such that x_i is a proper divisor of the numbers $x_1\ldots x_{i-1}\,x_{i+1}\ldots x_n+1$ for every i. In 1983, Sun Qi [34] proved that let $Z(n)$ be the number of solutions of the Znām problem with $1<x_1<\ldots<x_n$, we have $Z(n)\geqslant\Omega(n)-\Omega(n-1)>0$, when $n\geqslant 5$. Hence the problem of Znām is completely solved. It is difficult to prove $Z(n+1)>Z(n)$, when $n\geqslant 5$.

In 1985, Sun Qi and Cao Zhenfu [35] studied the equation

$$\sum_{i=1}^{s}\frac{1}{x_i}-\frac{1}{x_1\ldots x_s}=1,\quad 0<x_1\ldots<x_s. \tag{16}$$

Let $A(s)$ be the number of solutions of the equation (16).

We They proved that if $t\geqslant 9$, then $A(t+1)\geqslant\Omega(t)+\Omega(t-1)+6$. For $s=6$, we gave that there are 17 solutions of equation (16) in all. I conjecture that if $n\geqslant 3$, then $A(n+1)>A(n)$.

In 1984, Sun Qi and Cao Zhenfu[36] proved that $\Omega(s+1) \geq \Omega(s)+3$, when $s \geq 10$. From this result, we also have $Z(s) \geq 3$ and $H(s) \geq 3s-9$, when $s \geq 10$.

In 1986, Sun Qi and Cao Zhenfu[37] proved the following theorems.

1) If $s \geq 2$, then $Z(s)=H(s)-H(s-1)$.

2) If $s \geq 10$, then $\Omega(s+1) \geq \Omega(s)+5$.

3) If $s \geq 11$, then $Z(s) \geq 5$.

For $Z(s)$, we conjecture that if $s \geq 4$, then

$$Z(s+1) > Z(s).$$

In 1984, Sun Qi posed such a problem that, for each integer $n>1$, if there are n integers $x_i>1$ $(i=1,\dots,n)$ such that each x_i is a proper divisor of integer $x_1\dots x_{i-1}x_{i+1}\dots x_n-1$. We call the problem as $\mathcal{S}$-problem for simplicity. For each integer $n>1$, we use the symbol $X(n)$ to express the number of solutions to $\mathcal{S}$-problem for the case of n integers. Recently, Li-Shuguang[38] proved that if $n \geq 4$, then $X(n)>0$ and if $n=2$, 3, then $X(n)=0$. Thus, the $\mathcal{S}$-problem is solved. Is this ture that $X(n+1)>X(n)$ for $n \geq 4$?

6. The equation $\sum_{i=1}^{n} \frac{y_i}{d_i} \equiv 0 \pmod 1$. Diophantine equation over Finite fields.

Let $d_1, \dots, d_n$ be fixed positive integers. It is well-know that the number $I(d_1,\dots,d_n)$ of solutions of the equation

$$\frac{y_1}{d_1}+\frac{y_2}{d_2}+\dots+\frac{y_n}{d_n} \equiv 0 \pmod 1,\ y_i \text{ integers,}$$

$$1 \leq y_i < d_i \quad (i=1,\dots,n) \qquad (17)$$

play an important role in the study of diagonal equations over finite fields.

In 1948 and in 1949, Hua and Vandiver,[39][40] Furtado, Weil at about the same time proved the following results. If N denote the number of solutions of the equation

$$\sum_{i=1}^{n} a_i x_i^{d_i} = 0, \quad \text{where } d_i \mid q-1 \quad (i=1,\dots,n) \tag{18}$$

over finite field F , then

$$\left| N - q^{n-1} \right| \leq I(d_1,\dots,d_n)(q-1)q^{\frac{n-2}{2}} \tag{19}$$

Hence the value of $I(d_1,\dots,d_n)$ heavily affects the estimate of the number N of solutions of the equation (18).

For the case $d_1 = d_2 = \dots = d_n$, it is proved that $I(d,\dots,d) = \frac{d-1}{d}((d-1)^{n-1} + (-1)^n)$. For the more general case $d_1 | d_2 \dots d_{n-1} | d_n$, in 1986, Sun Qi, Wan Daqing, Ma Degang[41] proved that

n-1

$$I(d_1,\dots,d_n) = \prod_{j=1}^{n-1}(d_j-1) - \prod_{j=1}^{n-2}(d_j-1) \quad \dots + (-1)^{n-1} \quad (d_2-1)(d_1-1) + (-1)^n(d_1-1).$$

A complicated formula for $I(d_1,\dots,d_n)$ was obtained independently by Lide and Niederreiter, Stanly, and us[42] with different methods. The formula can be stated as follows

$$I(d_1,\dots,d_n) = (-1)^n + \sum_{r=1}^{n}(-1)^{(n-r)} \sum_{1 \leq i_1 < \dots < i_r \leq n} \frac{d_{i_1} \cdots d_{i_r}}{\operatorname{lcm}[d_{i_1},\dots,d_{i_r}]}.$$

Form (19), it is interesting to determine when $I(d_1,\dots,d_n) = 0$, for if $I(d\ ,\dots,d\ ) = 0$, then (18) has exactly q^{n-1} solutions. Some partial results have been obtained by Joly. In 1985, Sun Qi and Wan Daqing[43] proved following theorem: Let $n > 2$, then (17) has no solutions if and only if one of the following conditions holds. 1) For some d_i, $(d_i, \frac{d_1 \cdots d_n}{d_i})=1$ or 2) If $d_{i_1}, \ldots, d_{i_k}$ $(1 \leq i_1 < \cdots < i_k \leq n)$ is the set of all even integers among $\{d_1, \ldots, d_n\}$, then $2 \nmid k$, $\frac{d_{i_1}}{2}, \ldots, \frac{d_{i_k}}{2}$ are pairwise prime, and d_{i_j} is prime to any odd number in $\{d_1, \ldots, d_n\}$ $(j=1,\ldots,k)$ if $k<n$.

Recently, Sun Qi[44] have proved the following theorems.

Theorem 1. Suppose GF(p) is a finite field, where p is an odd prime number, $\{u,v,\theta\} \subset GF(p)$, $uv\theta \neq 0$, if

$$p > \sqrt{p}\,(2^{\omega(p-1)}-1)^2 + \sqrt{p} - (\sqrt{p}-3)\frac{(p-1)}{\varphi(p-1)} - 1,$$

then there are two primitive roots α and β in GF(p) such that $u\alpha + v\beta = \theta$, where μ is Mobius function, φ is Euler's totient function, $\omega(p-1)$ denotes the number of distinct prime factors of p-1.

Theorem 2. If $p > 2^{60}$, then there are two primitive roots α and β such that $u\alpha + v\beta = \theta$.

Theorem 3. If $p>3$, and

$$p > \sqrt{p}\,(2^{\omega(p-1)}-1)^2 - (\sqrt{p}-1)\left(\frac{(p-1)}{\varphi(p-1)} - 1\right),$$

then there are two primitive roots α and β such that $\alpha - \beta = 1$.

Vegh asks whether, for all primes $p > 61$, every integer can be expressed as the difference of two primitive roots of p.

From theorem 2, we easily deduce the following corollary.

Corollary. If $p > 2^{60}$, then every integer can be expressed as the difference of two primitive roots of p.

We can extend theorems 1-3 to $GF(p^n)$ $(n>1)$ without difficulty. For example, we have the following theorem.

Theorem 4. If $p^n > 2^{60}$, then there are two primitive roots

α and β in $GF(p^n)$ such that $u\alpha+v\beta=\theta$, where $\{u,v,\theta\}\subset GF(p^n)$, $uv\theta\neq 0$.

In order to prove our theorems we need the following lemmas.

Lemma 1. Let x and λ be characters of GF(p), and set

$J_{u,v}(x,\lambda,\theta)=\sum_{\substack{ul+vm=\theta\\ l,m\in GF(p)}}\chi(l)\lambda(m)$, then

$$|J_{u,v}(x,\lambda,\theta)|=\begin{cases}\sqrt{p}, & \text{if } \chi\lambda\neq\chi_0.\\ 1, & \text{if } \chi\lambda=\chi_0.\end{cases}$$

where χ_0 denotes the trivial character of GF(p).

Lemma 2. If $\delta>1$, $\eta>1$, $\delta\mid p-1$, $\eta\mid p-1$, $(a,\delta)=(b,\eta)=1$, $1\leq a\leq\delta$, $1\leq b\leq\eta$, then $\chi_{a(p-1)/\delta}$, $\chi_{b(p-1)/\eta}$ are not equal to χ_0, and $\chi_{a(p-1)/\delta}\chi_{b(p-1)/\eta}=\chi_0$ if $\frac{a}{\delta}+\frac{b}{\eta}\equiv 0 \pmod 1$, $\chi_{a(p-1)/\delta}\cdot\chi_{b(p-1)/\eta}\neq\chi_0$, if $\frac{a}{\delta}+\frac{b}{\eta}\not\equiv 1 \pmod 1$.

Lemma 3. Let $n\in GF(p)$, $n\neq 0$, then

$$\sum_{k\mid p-1}\frac{\mu(k)}{\varphi(k)}\sum_{\substack{a=1\\(a,k)=1}}^{k}e^{\frac{2\pi i a\,\mathrm{ind}\,n}{k}}=\begin{cases}0, & \text{if } n \text{ is not a primitive root,}\\ \frac{p-1}{\varphi(p-1)}, & \text{if } n \text{ is a primitive root,}\end{cases}$$

Lemma 4.

$$\sum_{\delta\mid p-1}\frac{|\mu(\delta)|}{\varphi(\delta)}=\frac{p-1}{\varphi(p-1)}.$$

A natural problem is that, is the result above true to primitive roots modulo p^l ($l\geq 2$, the p is an odd prime)? The problem is solved in [45]. Sun Qi and Li Shuguang have proved the following theorem.

Theorem . Let p be an odd prime and integer $l\geq 2$. When $p>2^{60}$, there exist at least $(p-2)p^{l-2}$ pairs of primitive roots

α and β modulo p^l such that

$$a\alpha + b\beta \equiv c \pmod{p^l}.$$

Recently, Sun Qi, Han Wenbao,[4b] have proved the following theorem.

Theorem Let p be an odd prime and $h \in GF(p)$, $h \neq 0$. If $m \geq 3$, then there is a primitive root g in $GF(p^m)$ such that

$$g + g^p + \cdots + g^{p^{m-1}} = h$$

except $m=3$, $p=11$.

For $m=3$, $p=11$, the problem above still remains unproved.

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