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**On Gevrey Singularities of solutions of equations with non symplectic characteristics**

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In this note we shall construct parametrices for a specific class of differential operators with non symplectic characteristics and clarify the structure of Gevrey singularities of solutions of the corresponding equations using constructed parametrices.

## 0. Notation and preliminaries

If $X$ is an open set of $\mathbb{R}^N$ and $\nu \geq 1$, the Gevrey class of order $\nu$; which we denote by $G^{\nu}(X)$, is the set of all $u \in C^{\infty}(X)$ such that for every compact set $K \subset X$ there is a constant $C_K$ with

$$|\partial_x^{\alpha} u(x)| \leq C_K^{|\alpha|+1}(\alpha!)^{\nu}, \quad x \in K,$$

for all multi-indices $\alpha \in \mathbb{N}^N$.

We use the following definition of the Gevrey wave front set given by Hörmander [14].

**Definition 0.1.** If $X \subset \mathbb{R}^N$ and $u \in \mathcal{D}'(X)$ we denote by $WF_{\nu}(u)$ the complement in $T^*(X)\backslash 0$ of the set of $(\mathring{x}, \mathring{\xi})$ such that there exist a neighborhood $U \subset X$ of $\mathring{x}$, a conic neighborhood $V \subset \mathbb{R}^N\backslash 0$ of $\mathring{\xi}$ and a bounded sequence $u_k \in \mathcal{E}'(X)$ which is equal to $u$ in $U$ and satisfies

$$|\hat{u}_k(\xi)| \leq C^{k+1}(k^{\nu}/|\xi|)^k, \quad k = 1,2,\cdots$$

for some constant $C$ when $\xi \in V$, where $\hat{u}_k$ denotes the Fourier transform of $u_k$.

$WF_1(u)$ is also denoted by $WF_A(u)$ since this is one of the definition of the analytic wave front set known to be equivalent to the others; see e.g. Bony [3].

If $\pi$ denotes the canonical projection of $T^*(X)\backslash 0$ on $X$ then $u \in G^\nu(X\backslash\pi(WF_\nu(u)))$ and for a differential operator $P$ with analytic coefficients, we have

$$WF_\nu(Pu) \subset WF_\nu(u) \subset \text{Char } P \cup WF_\nu(Pu),$$

where Char $P$ denotes the characteristic set of $P$. We say that $P$ is $G^\nu$ microhypoelliptic at $(\mathring{x},\mathring{\xi})$ if there is a conic neighborhood $V \subset T^*(X)\backslash 0$ of $(\mathring{x},\mathring{\xi})$ such that

$$WF_\nu(Pu) \cap V = WF_\nu(u) \cap V.$$

## 1. Statement of the results

Let $\Sigma$ be the submanifold in $T^*(\mathbb{R}^N)\backslash 0$ of codimension $2d+d'$ given by

$$\Sigma = \{(x,\xi)\in T^*(\mathbb{R}^N)\backslash 0;\ x_1=\cdots=x_d=0,\ \xi_1=\cdots=\xi_{d+d'}=0\},$$

where $0 < d < d+d' < N$. With this $\Sigma$ we set

$$\mathbb{R}^N_x = \mathbb{R}^d_t\times\mathbb{R}^n_y = \mathbb{R}^d_t\times\mathbb{R}^{d'}_{y'}\times\mathbb{R}^{d''}_{y''} \qquad (d+n=N,\ d'+d''=n)$$

and denote by $\xi = (\tau,\eta) = (\tau,\eta',\eta'')$ the dual variables of $x = (t,y)$ $= (t,y',y'') \in \mathbb{R}^d_t\times\mathbb{R}^{d'}_{y'}\times\mathbb{R}^{d''}_{y''}$. (In this coordinate $\Sigma = \{(t,y,\tau,\eta',\eta'');$ $t=\tau=\eta'=0,\ \eta''\neq 0\}$.)

For a fixed integer $h \geq 1$ we shall consider a differential operator of order $m$ with polynomial coefficients of the form:

$$(1.1) \qquad P = p(t,D_t,D_y) = \sum_{\substack{|\alpha|+|\beta|\leq m \\ |\gamma|=|\alpha|+|\beta'|+(1+h)|\beta''|-m}} a_{\alpha\beta\gamma}t^\gamma D_y^\beta D_t^\alpha,$$

where $(\alpha,\beta,\gamma) = (\alpha,\beta',\beta'',\gamma) \in \mathbb{N}^d\times\mathbb{N}^{d'}\times\mathbb{N}^{d''}\times\mathbb{N}^d$ and $(D_t,D_y) =$ $(-i\partial_t,-i\partial_y)$. Note that the symbol $p(t,\tau,\eta)$ has the following quasi-homogeneity:

$$(1.2)\qquad p(t/\lambda^{\rho},\lambda^{\rho}\tau,\lambda^{\rho}\eta',\lambda\eta'') = \lambda^{\rho m}p(t,\tau,\eta',\eta''),\quad \lambda>0$$

with $\rho = 1/(1+h)$.

Let $p_0$ denote the principal symbol given by

$$(1.3)\qquad p_0(t,\tau,\eta) = \sum_{\substack{|\alpha|+|\beta|=m\\ |\gamma|=h|\beta''|}} a_{\alpha\beta\gamma}t^{\gamma}\eta^{\beta}\tau^{\alpha}.$$

For a point $(\mathring{x},\mathring{\xi}) = (0,\mathring{y};0,0,\mathring{\eta}'') \in \Sigma$ $(|\mathring{\eta}''| \neq 0)$ we suppose:

(H-1) There exists a constant $c > 0$ such that

$$|p_0(t,\tau,\eta',\mathring{\eta}'')| \geq c(|\tau|+|\eta'|+|t|^{h})^{m},\quad (t,\tau,\eta') \in \mathbb{R}^{d}\times\mathbb{R}^{d}\times\mathbb{R}^{d'}.$$

We also consider the following condition due to Grušin.

(H-2) For all $\eta' \in \mathbb{R}^{d'}$, $\operatorname{Ker} p(t,D_t,\eta',\mathring{\eta}'') \cap \mathcal{S}(\mathbb{R}^{d}_{t}) = \{0\}$.

Here $p(t,D_t,\eta',\mathring{\eta}'')$ is considered as an operator acting on $\mathcal{S}(\mathbb{R}^{d}_{t})$ with a parameter $\eta' \in \mathbb{R}^{d'}$.

*Remark.* If $h = 1$, (H-2) is known to be equivalent to $C^{\infty}$ micro-hypoellipticity with loss of $m/2$ derivatives; see e.g. Boutet de Monvel-Grigis-Helffer [4], see also Grušin [10],[11],[12] and the other authors [8],[15],[28].

**Theorem I.** *Let* $P$ *be an operator of the form* (1.1) *satisfying* (H-1), (H-2) *for* $(\mathring{x},\mathring{\xi}) \in \Sigma$. *If* $\nu \geq 1+h$ *then* $P$ *is* $G^{\nu}$ *micro-hypoelliptic at* $(\mathring{x},\mathring{\xi})$.

The condition $\nu \geq 1+h$ is the best in the sence that

**Theorem II.** *Let* $P$ *be an operator of the form:*

$$(1.4)\qquad P = p'(t,D_t,D_{y''}) + q(D_{y'})$$

$$= \sum_{\substack{|\alpha|+|\beta''|\leq m\\ |\gamma|=|\alpha|+(1+h)|\beta''|-m}} a_{\alpha\beta''\gamma}t^{\gamma}D_{y''}^{\beta''}D_t^{\alpha} + \sum_{|\beta'|=m} b_{\beta'}D_{y'}^{\beta'}.$$

*sastisfying* (H-1) *for* $(0,\mathring{\xi}) \in \Sigma$. *Then one can find a neighborhood* $U$ *of the origin in* $\mathbb{R}^N$ *and a solution* $u \in C^\infty(U)$ *of* $Pu = 0$ *in* $U$ *such that for every* $\nu < 1+h$

$$(1.5) \qquad (0,\mathring{\xi}) \in WF_\nu(u) \subset WF_A(u) \subset \{(x,\lambda\mathring{\xi});\ x\in U,\ \lambda>0\}.$$

If $1 \le \nu < 1+h$ we can get a result on propagation of singularities of solutions for these operators.

Let $\Lambda$ be the involutive submanifold of $T^*(\mathbb{R}^N)\backslash 0$ containing $\Sigma$ given by

$$\Lambda = \{(t,y;\tau,\eta',\eta'')\in T^*(\mathbb{R}^N)\backslash 0;\ \eta'=0\}.$$

Then in the canonical way $\Lambda$ defines a bicharacteristic foliation in $\Sigma$ as well as in $\Lambda$; that is, each leaf $\Gamma_0$ is an integral submanifold of dimention $d'$ of the vector fields generated by $\{\partial_{y_1},\cdots,\partial_{y_{d'}}\}$. (Note that $T_\rho(\Gamma_0) = T_\rho(\Sigma) \cap T_\rho(\Sigma)^\perp$ for all $\rho \in \Gamma_0$.)

**Theorem III**. *Let* $\Gamma_0$ *be the bicharacteristic leaf passing through* $(\mathring{x},\mathring{\xi}) \in \Sigma$ *defined as above and* $W$ *be an open set containing* $(\mathring{x},\mathring{\xi})$ *such that* $\Gamma_0 \cap W$ *is connected. Suppose that* $P$ *is an operator of the form* (1.1) *satisfying* (H-1) *for* $(\mathring{x},\mathring{\xi})$ *and that* $1 \le \nu < 1+h$. *If* $u \in \mathcal{D}'(\mathbb{R}^N)$ *and* $WF_\nu(Pu) \cap \Gamma_0 \cap W = \phi$ *then either* $\Gamma_0 \cap W \cap WF_\nu(u) = \phi$ *or* $\Gamma_0 \cap W \subset WF_\nu(u)$.

*Remark*. If $h = 1$ and $\nu = 1$ this is a spacial case of Theorem 2 in Grigis-Schapira-Sjöstrand [9]. See also Sjöstrand [29],[30] and Hasegawa [13] in this connexion.

*Example*. Let

$$(1.6) \qquad P = \sum_{j=1}^{N-1} \partial_{x_j}^2 + \sum_{j=1}^{d} x_j^{2h}\partial_{x_N}^2 + h\sum_{j=1}^{d} c_j x_j^{h-1}\partial_{x_N},$$

where $c = (c_1,\cdots,c_d) \in \mathbb{C}^d$. If

$$(1.7)\qquad \sup_{\substack{\langle\sigma,\mathrm{Re}c\rangle=0\\ |\sigma|_1=1}} |\langle\sigma,\mathrm{Im}c\rangle| < 1 \quad (\sigma\in\mathbb{R}^d,\ |\sigma|_1=|\sigma_1|+\cdots+|\sigma_d|),$$

then $P$ is $G^{1+h}$ microhypoelliptic at every point in $T^*(\mathbb{R}^N)\backslash 0$.

In fact, noticing that $hx_j^{h-1}\partial_{x_N} = [\partial_{x_j}, x_j^h\partial_{x_N}]$ we get by Theorem 1' of Rothschild-Stein [25]

$$(1.8)\qquad \sum_{j=1}^{N-1}\|\partial_{x_j}u\|^2 + \sum_{j=1}^{d}\|x_j^h\partial_{x_N}u\|^2 \le C|(Pu,u)|$$

if (1.7) is fulfilled. This implies (H-2) while (H-1) is evident.

## 2. A study of the Grušin operator

We shall construct a right parametrix $K$ for a self-adjoint operator $Q=(P^*P)^k$ with $2km\ge d+1$. (Note that the quasi-homogeneity (1.1), (1.2) and the conditions (H-1), (H-2) are preserved for $Q$ with the order $m$ replaced by $M=2km$.) Then clearly $K^*(P^*P)P^*$ is a left parametrix of $P$ and the micro-hypoellipticity of $P$ follows immediately from that of $Q$.

In the construction of the parametrix we follow closely Métivier [21] and Ōkaji [22]. In this section we shall derive the estimates for the inverse of $\hat{Q}=\mathcal{F}_yQ\mathcal{F}_y^{-1}$.

**2.1. Grušin operator.** Let $Q=q(t,D_t,D_y)$ be an operator of the form (1,1) satisfying (H-1), (H-2) for $(\mathring{x},\mathring{\xi})\in\Sigma$. We may assume $(\mathring{x},\mathring{\xi})=(0,e_N)=(0;0,\cdots,0,1)$ without loss of generality; henceforth we let $\mathring{\xi}=(0,\mathring{\eta})=(0,0,\mathring{\eta}'')=(0,\cdots,0,1)\in\mathbb{R}^N$.

By the fourier transform in y, we consider the equation:

$$(2.1)\qquad q(t,D_t,\eta)v(t,\eta)=u(t,\eta)$$

in a conic neighborhood $U_\varepsilon\times V_\varepsilon$ of $(0;\mathring{\eta})\in\mathbb{R}^d\times(\mathbb{R}^n\backslash 0)$ given by

$$(2.2)\qquad U_\varepsilon=\{t\in\mathbb{R}^d;\ |t|<1\},$$

$$V_\varepsilon = \{\eta=(\eta',\eta'')\in\mathbb{R}^n\setminus 0;\ |\eta'|<\varepsilon\eta_n,\ |\eta''-\mathring{\eta}''\eta_n|<\varepsilon\eta_n\}.$$

$q(t,D_t,\eta)$ is essentially the same operator that was studied by Grušin [12]; so we call it *Grušin operator*.

Now we shall start with the following lemma due to Grušin (: Lemma 3.4 in [12]).

**Lemma 2.1.** *Let* $Q = q(t,D_t,D_y)$ *be an operator of order* $M$ *of the form* (1.1) *satisfying* (H-1) *and* (H-2) *for* $\mathring{\xi} = (0,0,\mathring{\eta}'')$. *Then there exist a conic neighborhood* $V''$ *of* $\mathring{\eta}''$ *and a constant* $C$ *such that for all* $\eta = (\eta',\eta'') \in \mathbb{R}^{d'}\times V''$

$$(2.3)\qquad \sum_{|\beta|\le M}\int|(|\eta''|^\rho+|\eta'|+|t|^h|\eta''|)^{M-|\beta|}D_t^\beta v(t)|^2\,dt$$
$$\le C\int|q(t,D_t,\eta)v(t)|^2\,dt$$

*for* $v \in \mathcal{S}(\mathbb{R}_t^d)$, *where* $\rho = 1/(1+h)$.

Let us introduce new variables

$$\bar{t} = t\eta_n^\rho,\ \bar{\eta}' = \eta'/\eta_n^\rho,\ \bar{\eta}'' = \eta''/\eta_n,\ (\eta_n>0)$$

and set

$$\bar{v}(\bar{t},\bar{\eta},\eta_n) = v(\bar{t}/\eta_n,\bar{\eta}'\eta_n,\bar{\eta}''/\eta_n).$$

Then in view of (1.2) we have

$$(2.4)\qquad q(\bar{t},D_{\bar{t}},\bar{\eta})\bar{v}(\bar{t},\bar{\eta},\eta_n) = \eta_n^{-\rho m}q(t,D_t,\eta)v(t,\eta),$$

and the conic neighborhood $U_\varepsilon\times V_\varepsilon$ blows up into $\mathbb{R}_{\bar{t}}^d\times\bar{V}_\varepsilon$, where $\bar{V}_\varepsilon = \mathbb{R}_{\bar{\eta}'}^{d'}\times\{\bar{\eta}''\in\mathbb{R}^{d''};\ |\bar{\eta}''-\mathring{\eta}''|<\varepsilon\}$.

By multiplying $\eta_n^{-\rho(M-d)}$, (2.3) becomes

$$(2.5)\qquad \sum_{|\beta|\le M}\int|(1+|\bar{\eta}'|+|\bar{t}|^h|\bar{\eta}''|)^{M-|\beta|}D_{\bar{t}}^\beta\bar{v}(\bar{t},\eta_n)|^2\,d\bar{t}$$
$$\le C\int|q(\bar{t},D_{\bar{t}},\bar{\eta})\bar{v}(\bar{t},\eta_n)|^2\,d\bar{t}.$$

Moreover, we have

**Proposition 2.2.** *Let*

$$\bar{V}_\varepsilon^{\mathbb{C}} = \{\bar{\eta}=(\bar{\eta}',\bar{\eta}'')\in\mathbb{C}^{d'}\times\mathbb{C}^{d''};\ |\mathrm{Im}\ \bar{\eta}'|<\varepsilon(1+|\mathrm{Re}\ \bar{\eta}'|),\ |\bar{\eta}''-\hat{\eta}''|<\varepsilon\}.$$

*If* $\varepsilon$ *is chosen sufficiently small then for all* $\bar{\eta}\in\bar{V}_\varepsilon^{\mathbb{C}}$ *we have* (2.5) *with another constant* $C$ *and there exists a left inverse* $\bar{K}(\bar{\eta})$ *of* $q(\bar{t},D_{\bar{t}},\bar{\eta})$ *depending holomorphically on* $\bar{\eta}\in\bar{V}_\varepsilon^{\mathbb{C}}$.

**2.2 Commutator estimates.** We consider the operators

$$T_j = \partial_{\bar{t}_j} \quad \text{and} \quad T_{-j} = i\bar{t}_j \quad (j = 1,2,\cdots,d).$$

For a sequence $I = (j_1,\cdots,j_k) \in \{\pm1,\cdots,\pm d\}^k$ we denote by $T_I$ the operator

$$T_I = T_{j_1}T_{j_2}\cdots T_{j_k}$$

and $\langle I\rangle = |I_+| + (1/h)|I_-| = \#\{j_l>0\} + (1/h)\#\{j_l<0\}$.

We define the space

$$B^k(\bar{\eta}) = \{u\in L^2(\mathbb{R}^d);\ \forall I,\ \langle I\rangle \le k,\ T_I u \in L^2(\mathbb{R}^d)\}$$

for $k \in \mathbb{N}/h$ equipped with the norm:

$$|u|_{k,\bar{\eta}} = \underset{\langle I\rangle+j\le k}{\mathrm{Max}} (1+|\bar{\eta}'|)^j \|T_I u\|_{L^2(\mathbb{R}^d)}$$

depending on $\bar{\eta}\in\bar{V}_\varepsilon^{\mathbb{C}}$. Note that $|u|_{0,\bar{\eta}}$ is the usual $L^2$ norm independent of $\bar{\eta}$; hence denoted by $|u|_0$. We also define $B^{-k}(\bar{\eta})$ the dual space of $B^k(\bar{\eta})$.

If $L$ is an operator acting from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$ we set

$$(\mathrm{ad}\ T_j)(L) = [T_j,L] = T_jL - LT_j \quad (j = \pm1,\cdots,\pm d)$$

and because the ad $T_j$'s commute, we denote for a multi-index $\alpha = (\alpha_+,\alpha_-) = (\alpha_1,\cdots,\alpha_d;\alpha_{-1},\cdots,\alpha_{-d}) \in \mathbb{N}^d\times\mathbb{N}^d$

$$(\mathrm{ad}\ T)^\alpha = \prod_j (\mathrm{ad}\ T_j)^{\alpha_j}.$$

If the operator $L$ from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$ can be extended as a bounded operator in $L^2(\mathbb{R}^d)$ we denote by $\|L\|_0$ the norm of this extension, otherwise we agree with $\|L\|_0 = +\infty$.

At last, we introduce the norm:

$$\|L\|_{k,\bar{\eta}} = \underset{\langle I\rangle+\langle J\rangle+j\le k}{\mathrm{Max}} (1+|\bar{\eta}'|)^j \|T_I L T_J\|_0$$

for $k \in \mathbb{N}/h$, then $\|L\|_{k,\bar{\eta}} < +\infty$ only means that $L$ is bounded from $B^{-p}(\bar{\eta})$ to $B^{-p+k}(\bar{\eta})$ for all $p = 0, 1/h, 2/h, \cdots, k$.

Now let $\bar{Q} = \bar{Q}(\bar{\eta}) = q(\bar{t}, D_{\bar{t}}, \bar{\eta})$. Then we can write

$$(2.6) \qquad \bar{Q}(\bar{\eta}) = \sum_{\langle I\rangle+|\beta|\le M} b_{I,\beta}\bar{\eta}^{\beta} T_I$$

and (2.5) by:

$$(2.7) \qquad |u|_{M,\bar{\eta}} \le C_0 |\bar{Q}u|_0 \quad \text{for } \bar{\eta} \in \bar{V}_\varepsilon^{\mathbb{C}}$$

We obtain as in Ōkaji [22]

**Lemma 2.3.** *If* $\bar{Q}$ *is a self-adjoint operator satisfying* (2.7) *then there exists a constant* $C_1$ *such that*

$$(2.8) \qquad \|L\|_{M,\bar{\eta}} \le C_1(\|\bar{Q}L\|_0 + \|L\bar{Q}\|_0) \quad \text{for } \bar{\eta} \in \bar{V}_\varepsilon^{\mathbb{C}}.$$

Let $p$ be an integer. For real $R > 1$, $\mathcal{L}_R^p(\bar{V}_\varepsilon^{\mathbb{C}})$ denotes the space of operators $L$ for which there is a constant $C$ such that for all $\alpha = (\alpha_+, \alpha_-) \in \mathbb{N}^d \times \mathbb{N}^d$ and $\bar{\eta} \in \bar{V}_\varepsilon^{\mathbb{C}}$

$$\|(\mathrm{ad}\ T)^{\alpha}(L)\|_{>\alpha<+p,\bar{\eta}} \le C|\alpha|!R^{|\alpha|},$$

where $>\alpha< = (1/h)|\alpha_+| + |\alpha_-|$. Then $\mathcal{L}_R^p(\bar{V}_\varepsilon^{\mathbb{C}})$ becomes a Banach space in an obvious way.

**Lemma 2.4.** *Let* $\bar{Q}$ *be as in Lemma* 2.3. *Then there are constants* $R_0$ *and* $C_2$ *depending only on* $C_1$ *and* $\mathrm{Max}\ |b_{I,\beta}|$ *such that if* $R > R_0$ *and both* $\bar{Q}L$ *and* $L\bar{Q}$ *are in* $\mathcal{L}_R^0(\bar{V}_\varepsilon^{\mathbb{C}})$ *then* $L$ *is in* $\mathcal{L}_R^M(\bar{V}_\varepsilon^{\mathbb{C}})$, *moreover*

$$(2.9) \qquad \|L\|_{\mathcal{L}_R^M(\overline{V}_\varepsilon^{\mathbb{C}})} \le C_2(\|\overline{Q}L\|_{\mathcal{L}_R^0(\overline{V}_\varepsilon^{\mathbb{C}})} + \|L\overline{Q}\|_{\mathcal{L}_R^0(\overline{V}_\varepsilon^{\mathbb{C}})}).$$

Proof is parallel to that of Métivier [21] Proposition 2.3 or Ōkaji [22] Lemma 7.2 and canl be found in [27]. The following proposition is just a consequence of this lemma.

**Proposition 2.5.** *Let* $\overline{Q}$ *be a self-adjoint operator satisfying* (2.7) *and let* $\overline{K}$ *be the inverse of* $\overline{Q}$ *such that* $\overline{K}\overline{Q} = \overline{Q}\overline{K} = Id$. *Then, if* $R$ *is large enough,* $\overline{K}$ *is in* $\mathcal{L}_R^M(\overline{V}_\varepsilon^{\mathbb{C}})$.

**2.3. Kernel of the inverse.** For an operator $K$ from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$, we denote by $K(\bar{t},\bar{s})$ its distribution kernel.

**Lemma 2.6.** *If* $K$ *is in* $\mathcal{L}_R^M(\overline{V}_\varepsilon^{\mathbb{C}})$ *with* $M \ge d+1$ *then* $K(\bar{t},\bar{s})$ *is in* $L^2(\mathbb{R}^d\times\mathbb{R}^d)$, *moreover there exist constants* $\overline{C}$ *and* $\overline{R}$ *such that for all* $\alpha = (\alpha_+,\alpha_-) \in \mathbb{N}^d\times\mathbb{N}^d$

$$(2.10) \qquad \|(\bar{t}-\bar{s})^{\alpha_-}(\partial_{\bar{t}}+\partial_{\bar{s}})^{\alpha_+}K(\bar{t},\bar{s})\|_{L^2} \le \overline{C}\|K\|_{\mathcal{L}_R^M(\overline{V}_\varepsilon^{\mathbb{C}})}\overline{R}^{|\alpha|}(\alpha_+!)^{1-\rho}(\alpha_-!)^{\rho},$$

*where* $\rho = 1/(1+h)$.

*Proof.* Note that if $K$ and $K^*$ are bounded from $L^2(\mathbb{R}^d)$ into $B^{d+1}(\bar{\eta})$ then $K$ is a Hilbert-Schmidt operator with the continuous kernel such that

$$\|K(\bar{t},\bar{s})\|_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} \le C\|K\|_{d+1,\bar{\eta}}.$$

To prove (2.10) we consider

$$(2.11) \qquad \left((\bar{t}-\bar{s})^{\alpha_-}(\partial_{\bar{t}}+\partial_{\bar{s}})^{\alpha_+}\right)^{1+h}K(\bar{t},\bar{s})$$

$$= \sum(\pm)\bar{t}^{\beta'_-}\bar{s}^{\beta''_-}\partial_{\bar{t}}^{\beta'_+}\partial_{\bar{s}}^{\beta''_+}(\bar{t}-\bar{s})^{\alpha_-}(\partial_{\bar{t}}+\partial_{\bar{s}})^{h\alpha_+}K(\bar{t},\bar{s}),$$

where the sum consists of $2^{h|\alpha_-|+|\alpha_+|}$ terms of the coefficients 1 or $-1$ with the multi-indeces $\beta'_-$, $\beta''_-$, $\beta'_+$, $\beta''_+$ such that $\beta'_-+\beta''_- = h\alpha_-$, $\beta'_++\beta''_+ = \alpha_+$.

Now

$$(2.12) \qquad \bar{t}^{\beta'_-}\bar{s}^{\beta''_-}\partial_{\bar{t}}^{\beta'_+}\partial_{\bar{s}}^{\beta''_+}(\bar{t}-\bar{s})^{\alpha_-}(\partial_{\bar{t}}+\partial_{\bar{s}})^{h\alpha_+}K(\bar{t},\bar{s})$$

is the distribution kernel of

$$T_-^{\beta'_-}T_+^{\beta''_-}(\mathrm{ad}\ T_-)^{\alpha_-}(\mathrm{ad}\ T_+)^{\alpha_+}(K)T_-^{\beta_-}T_+^{\beta''_+};$$

which is bounded from $L^2(\mathbb{R}^d)$ into $B^M(\bar{\eta})$ together with its adjoint. Since $M \geq d+1$ we know (2.12) is a continuous function with $L^2$ norm bounded by

$$C\|K\|_{\mathcal{L}_R^M(\bar{V}_\varepsilon^{\mathbb{C}})}R^{|\alpha_-|+h|\alpha_+|}(|\alpha_-|+h|\alpha_+|)!.$$

Adding up these estimates we have

$$(2.13) \qquad \|\{(\bar{t}-\bar{s})^{\alpha_-}(\partial_{\bar{t}}+\partial_{\bar{s}})^{\alpha_+}\}^{1+h}K(\bar{t},\bar{s})\|_{L^2(\mathbb{R}^d\times\mathbb{R}^d)}$$

$$\leq C\|K\|_{\mathcal{L}_R^M(\bar{V}_\varepsilon^{\mathbb{C}})}\bar{R}^{|\alpha|}(|\alpha_-|+h|\alpha_+|)!$$

provided that $\bar{R} \geq (2R)^h$. Also we have

$$(2.14) \qquad \|K(\bar{t},\bar{s})\|_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} \leq C\|K\|_{\mathcal{L}_R^M(\bar{V}_\varepsilon^{\mathbb{C}})}.$$

Then a simple interpolation argument yields (2.10) in view of the Stirling formula. □

**2.4. Symbol of the inverse.** We write the operator $K$ of kernel $K(\bar{t},\bar{s})$ with a symbol $k = \sigma(K)$ in the way that

$$(2.15) \qquad K(\bar{t},\bar{s}) = (2\pi)^{-d}\int e^{i\langle \bar{t}-\bar{s},\bar{\tau}\rangle}k(\bar{t},\bar{\tau})d\bar{\tau}.$$

That is, $k$ is the distribution on $\mathbb{R}^{2d}$ given by

$$(2.16) \qquad k(z^+,z^-) = \int e^{i\langle u,z^-\rangle}K(z^+,z^++u)du.$$

Here and below we use the notation $z = (z^+,z^-) = (z_1,\cdots,z_d;z_{-1},\cdots,z_{-d}) \in \mathbb{R}^{2d}$.

Since (2.15), (2.16) have a sence as the partial Fourier transform the mapping $\sigma$ is clearly an isomorphism between $L^2(\mathbb{R}^d_t \times \mathbb{R}^d_{\bar{s}})$ and $L^2(\mathbb{R}^{2d}_z)$. Also by the definition of $\sigma$ we have

$$\sigma((\text{ad } T_j)(K)) = \partial_{z_j} \sigma(K).$$

Hence Lemma 2.6 is restated as follows:

**Lemma 2.7.** *Let* $k = k(\bar{\eta}) = \sigma(K(\bar{\eta}))$*: the symbol of* $K(\bar{\eta}) \in \mathcal{L}^M_R(\bar{V}^{\mathbb{C}}_\varepsilon)$ *with* $M \geq d+1$. *Then there exist constants* $\bar{C}, \bar{R}$ *such that for all* $\alpha = (\alpha_+, \alpha_-) \in \mathbb{N}^d \times \mathbb{N}^d$ *and* $\bar{\eta} \in \bar{V}^{\mathbb{C}}_\varepsilon$

$$(2.17) \qquad \|\partial^\alpha_z k(\bar{\eta})\|_{L^2(\mathbb{R}^{2d}_z)} \leq \bar{C} \|K\|_{\mathcal{L}^M_R(\bar{V}^{\mathbb{C}}_\varepsilon)} \bar{R}^{|\alpha|} (\alpha_+!)^{1-\rho} (\alpha_-!)^\rho$$

*where* $\rho = 1/(1+h)$.

Now suppose that $K(\bar{\eta}) \in \mathcal{L}^M_R(\bar{V}^{\mathbb{C}}_\varepsilon)$ $(M \geq d+1)$ depends holomorphically on $\bar{\eta}$. Then we have

**Proposition 2.8.** *Let* $K(\bar{\eta})$ *be as above and let* $k(z, \bar{\eta}) = \sigma(K(\bar{\eta}))(z)$. *Then there exists a constant* $C$ *such that for* $(z, \bar{\eta}) \in \mathbb{R}^{2d} \times \bar{V}^{\mathbb{C}}_{\varepsilon'}$ with $0 < \varepsilon' < \varepsilon$ *and for all* $(\alpha, \beta) = (\alpha_+, \alpha_-, \beta', \beta'') \in \mathbb{N}^d \times \mathbb{N}^d \times \mathbb{N}^{d'} \times \mathbb{N}^{d''}$

$$(2.18) \qquad |\partial^\alpha_z \partial^\beta_{\bar{\eta}} k(z, \bar{\eta})|$$

$$\leq C^{|\alpha|+|\beta|+1} \left(\frac{1}{\varepsilon - \varepsilon'}\right)^{|\beta|} (\alpha_+!)^{1-\rho} (\alpha_-!)^\rho \beta! (1 + |\bar{\eta}'|)^{-|\beta'|},$$

*where* $\rho = 1/(1+h)$.

*Proof.* Recall that

$$\bar{V}^{\mathbb{C}}_\varepsilon = \{\bar{\eta} = (\bar{\eta}', \bar{\eta}'') \in \mathbb{C}^{d'} \times \mathbb{C}^{d''} ;\ |\text{Im } \bar{\eta}'| < \varepsilon(1 + |\text{Re } \bar{\eta}'|),\ |\bar{\eta}'' - \mathring{\eta}''| < \varepsilon\}.$$

Then we use the Cauchy inequality to obtain

$$\|\partial^\beta_{\bar{\eta}} K\|_{\mathcal{L}^M_R(\bar{V}^{\mathbb{C}}_{\varepsilon'})} \leq \|K\|_{\mathcal{L}^M_R(\bar{V}^{\mathbb{C}}_\varepsilon)} \left(\frac{n}{\varepsilon - \varepsilon'}\right)^{|\beta|} \beta! (1 + |\bar{\eta}'|)^{-|\beta'|}.$$

Applying Lamma 2.7 to $\partial^\beta_{\bar{\eta}} K(\bar{\eta})$ we get (2.18) by means of the Sobolev lemma. □

### 3. Parametrix; proof of Theorem I

In Section 2 we have showed that there is the inverse $\bar{K}(\bar{\eta})$ of $\bar{Q}(\bar{\eta}) = q(\bar{t},D_{\bar{t}},\bar{\eta}) = (P^*P)^k(t,D_{\bar{t}},\eta)$ $(2km \geq d+1)$ for $\bar{\eta} \in \bar{V}_\varepsilon^{\mathbb{C}}$ such that

$$q(\bar{t},D_{\bar{t}},\bar{\eta})\bar{K}(\bar{t},\bar{s},\bar{\eta})d\bar{s} = \delta(\bar{t}-\bar{s})d\bar{s}$$

with the kernel

$$\bar{K}(\bar{t},\bar{s},\bar{\eta}) = (2\pi)^{-d}\int e^{i\langle\bar{t}-\bar{s},\bar{\tau}\rangle}\bar{k}(\bar{t},\bar{\tau},\bar{\eta})d\bar{\tau},$$

where $\bar{k}$ satisfies (2.18) in $\mathbb{R}^d_{\bar{t}}\times\mathbb{R}^d_{\bar{\tau}}\times(\bar{V}^{\mathbb{C}}_{\varepsilon'}\cap\mathbb{R}^{d'}_{\bar{\eta}})$ for $0 < \varepsilon' < \varepsilon$.

Now we return to the original variables:

$$t = \bar{t}/\eta_n^\rho,\ \tau = \bar{\tau}\eta_n^\rho,\ \eta' = \bar{\eta}'\eta_n^\rho,\ \eta'' = \bar{\eta}''\eta_n,\ (\eta_n>0)$$

and set

$$\hat{K}(t,s,\eta) = (2\pi)^{-d}\int e^{i\langle t-s,\tau\rangle}\hat{k}(t,\tau,\eta)d\tau,$$

where

$$(3.1)\qquad \hat{k}(t,\tau,\eta) = \eta_n^{-2\rho km}\,\bar{k}(\bar{t},\bar{\tau},\bar{\eta})$$

$$= \eta_n^{-2\rho km}\,\bar{k}(t\eta_n^\rho,\tau/\eta_n^\rho,\eta'/\eta_n^\rho,\eta''/\eta_n).$$

Then in view of (2.4)

$$q(t,D_t,\eta)\hat{K}(t,s,\eta)ds = \delta(t-s)ds$$

for $\eta \in V_\varepsilon = \{(\eta',\eta'')\in\mathbb{R}^n\backslash 0;\ |\eta'|<\varepsilon\eta_n,\ |\eta''-\bar{\eta}''\eta_n|<\varepsilon\eta_n\}$.

Let us introduce a cut off function given by Métivier:

**Lemma 3.1.** *For given two cones* $V_1 \subset\subset V_2 \subset \mathbb{R}^N\backslash 0$ *and* $0 < \rho < 1$ *there exist* $g \in C^\infty(\mathbb{R}^N)$ *and* $C$ *such that*

$$(3.2)\qquad g(\xi) = 0 \quad \textit{for}\ \xi \notin V_2\ \textit{or}\ |\xi| \leq 1$$

$$g(\xi) = 1 \quad \textit{for} \quad \xi \in V_1 \textit{ and } |\xi| \geq 2$$

*and*

$$(3.3) \qquad |\partial_\xi^\alpha g(\xi)| \leq C^{|\alpha|+1}\left(\frac{|\alpha|}{|\xi|}\right)^{\rho|\alpha|}$$

*for all* $\alpha, \xi$ *such that* $|\alpha| \leq |\xi|$. (Lemma 3.1 in [21].)

With $\rho = 1/(1+h)$ and $\xi \in V_1 \subset\subset V_2 = \{\xi=(\tau,\eta)\in\mathbb{R}^d\times\mathbb{R}^n;\ |\tau|<\varepsilon',\ \eta\in V_{\varepsilon'}\}$, we take $g(\xi) = g(\tau,\xi)$ as above and set $k_g(t,\tau,\eta) = \hat{k}(t,\tau,\eta)g(\tau,\eta)$. Then

**Proposition 3.2.** *There exists a constant* $C_0$ *such that*

$$(3.4) \qquad |\partial_\eta^\beta \partial_\tau^{\alpha_-} \partial_t^{\alpha_+} k_g(t,\tau,\eta)| \leq C_0^{|\alpha|+|\beta|+1}(1+|t|)^{|\beta|}(|\alpha_+|^{1-\rho}|\xi|^\rho)^{|\alpha_+|}$$
$$\times\left(\frac{|\alpha_-|}{|\xi|}\right)^{\rho|\alpha_-|}\left(\frac{|\beta'|}{|\xi|^\rho+|\eta'|}+\chi_g(\xi)\left(\frac{|\beta'|}{|\xi|}\right)^\rho\right)^{|\beta'|}\left(\frac{|\beta''|}{|\xi|}\right)^{\rho|\beta''|}$$

*for* $|\alpha_-|+|\beta| \leq |\xi|$, *where* $\xi = (\tau,\eta) = (\tau,\eta',\eta'') \in \mathbb{R}^N$, $(\alpha,\beta) = (\alpha_+,\alpha_-,\beta',\beta'') \in \mathbb{N}^d\times\mathbb{N}^d\times\mathbb{N}^{d'}\times\mathbb{N}^{d''}$, $\rho = 1/(1+h)$ *and* $\chi_g$ *is the characteristic function of the support of* $\nabla_{\eta'} g$.

Now let $K_g = k_g(t,D_t,D_y) = \mathrm{Op}(k_g)$; that is, $K_g$ is the operator defined by the kernel:

$$(3.5) \qquad K_g(t,y,s,w) = (2\pi)^{-N}\int e^{i\langle t-s,\tau\rangle+i\langle y-w,\eta\rangle}k_g(t,\tau,\eta)\,d\tau d\eta.$$

Then we have

$$(3.6) \qquad QK_g = K_g^*Q = g(D_t,D_y) = \mathrm{Op}(g)$$

and the following

**Proposition 3.3.**

$$(3.7) \qquad WF_A(K_g) \subset \{(t,y,t,w;\tau,\eta,-\tau,-\eta)\in T^*(\mathbb{R}^{2N})\backslash 0;\ y''=w'',\ (\tau,\eta)\in\overline{V}_2\}.$$

$$(3.8) \qquad WF_{1+h}(K_g) \subset \{(t,y,t,y;\tau,\eta,-\tau,-\eta)\in T^*(\mathbb{R}^{2N})\backslash 0;\ (\tau,\eta)\in\overline{V}_2\}.$$

*Proof.* By Lemma 3.3 and Remark 3.4 in Métivier [21] we obtain (3.7). Hence to prove (3.8) it suffices to show that $K$ is in $G^{1+h}$ for $y' \neq 0$. Using the vector field $(1/|y'|^2)\langle y',D_{\eta'}\rangle$ for integrating by parts we can prove this as in Case 2 in the proof of Lemma 3.3 in [21]. □

For any set $V$ we write $\mathrm{diag}(V) = \{(\rho,\rho)\in V\times V\}$. We have therefore proved the following theorem; from which Theorem I follows immediately.

**Theorem 3.4.** *Let* $P$ *be an operator of the form* (1.1) *satisfying* (H-1), (H-2) *for* $(\mathring{x},\mathring{\xi}) \in \Sigma$ *and let* $Q = (P^*P)^k$ *with* $2km \geq d+1$. *Then there are a conic neighborhood* $V \subset \mathbb{R}^N\backslash 0$ *of* $\mathring{\xi}$ *and an operator* $K\colon \mathcal{E}'(\mathbb{R}^N) \longrightarrow \mathcal{D}'(\mathbb{R}^N)$ *such that for every* $u \in \mathcal{E}'(\mathbb{R}^N)$

$$(3.9) \qquad WF_A(QKu - u) \cap (\mathbb{R}^N\times V) = \phi,$$

$$(3.10) \qquad WF_A(K^*Qu - u) \cap (\mathbb{R}^N\times V) = \phi$$

*and that*

$$(3.11) \qquad WF'_{1+h}(K) \subset \mathrm{diag}(T^*(\mathbb{R}^N)\backslash 0),$$

*where* $WF'_{1+h}(K) = \{(x,\xi;\tilde{x},\tilde{\xi});\ (x,\tilde{x};\xi,-\tilde{\xi})\in WF_{1+h}(K)\}$.

## 4. Proof of Theorem II

Let $\mathring{\xi} = (0,0,\mathring{\eta}'')$ with $\mathring{\eta}'' \neq 0$. We consider the operator $p'(t,D_t,\mathring{\eta}'')$; which is precisely the same one that was studied by Grušin [10].

From the result of Grušin [10] we can take $c \in \mathbb{C}$ and $0 \neq v \in \mathcal{S}(\mathbb{R}^N)$ such that

$$(4.1) \qquad p'(t,D_t,\mathring{\eta}'')v(t) = -c^m q(\bar{\eta}'),$$

where $\bar{\eta}' \in \mathbb{R}^{d'}$ is fixed with $|\bar{\eta}'| = 1$. Then

$$u_\lambda(t,y) = \exp(i\lambda^\rho c\langle y',\bar{\eta}'\rangle + i\lambda\langle y'',\mathring{\eta}''\rangle)v(\lambda^\rho t), \quad \rho = 1/(1+h)$$

is a solution of $Pu = 0$ for every $\lambda \geq 0$. Hence

$$u(t,y) = \int_0^{+\infty} u_\lambda(t,y)e^{-\lambda^\rho}d\lambda$$

is a $C^\infty$ solution in $U = \{(t,y',y'')\in\mathbb{R}^N;\ |\mathrm{Im}c||y'|<1\}$.

By Lemma 3.7 in Ōkaji [22], $v$ satisfies the estimate

$$|\partial_t^\alpha v(t)| \leq C^{|\alpha|+1}(\alpha!)^{1-\rho}.$$

Hence we have

$$WF_A(u) \subset \{(t,y;0,0,\lambda\mathring{\eta}'')\in T^*(\mathbb{R}^N)\backslash 0;\ \lambda > 0\} \tag{4.2}$$

in the same way as (3.7).

On the other hand, since $v$ is analytic, $\partial_t^\alpha v(0) \neq 0$ for some $\alpha \in \mathbb{N}^d$. Therefore,

$$|\langle\mathring{\eta}'',D_{y''}\rangle^k\partial_t^\alpha u(0,0)| = \int_0^{+\infty}|\mathring{\eta}''|^{2k}\lambda^{\rho|\alpha|+k}|\partial_t^\alpha v(0)|e^{-\lambda^\rho}d\lambda \tag{4.3}$$

$$= \text{const.}\ \Gamma((k+1)/\rho + |\alpha|).$$

This combined with (4.2) implies $(0;0,0,\mathring{\eta}'') \in WF_\nu(u)$ for every $\nu < 1+h$, and proof is now complete. □

## 5. Second microlocalization in Gevrey class

Following Sjöstrand [29] we introduce the Fourier–Bros–Iagolnitzer transform (F.B.I. tr.):

$$T^{(1)}f(z,\lambda) = \int e^{-\lambda(z-x)^2/2}f(x)dx, \quad (f \in \mathcal{S}'(\mathbb{R}^N)) \tag{5.1}$$

associated to $\kappa$: $T^*(\mathbb{R}^N)\backslash 0 \ni (x,\xi) \longmapsto x-i\xi \in \mathbb{C}_z^N$.

$T^{(1)}f$ is defined on $\mathbb{C}_z^N\times\mathbb{R}_\lambda^+$, holomorphic with respect to $z$ and bounded by $Ce^{\lambda|\mathrm{Im}z|^2/2}(\lambda+|y|)^k$ for some $C$, $k$ real.

In terms of the F.B.I. tr. we can characterize the Gevrey wave front set as follows: For $f \in \mathcal{S}'(\mathbb{R}^N)$, $(\mathring{x},\mathring{\xi}) \notin WF_\nu(f)$ if and only if there are constants $C$, $c > 0$ such that

$$(5.2) \qquad |T^{(1)}f(z,\lambda)| \le Ce^{\frac{\lambda}{2}|\mathrm{Im}z|^2 - c\lambda^{1/\nu}} \quad \text{for } |z-(\mathring{x}-i\mathring{\xi})| < c.$$

Let $\Lambda$ be the involutive submanifold of $T^*(\mathbb{R}^N)$:

$$\Lambda = \{(x,\xi)\in T^*(\mathbb{R}^N);\ \xi_1=\cdots=\xi_{d'}=0\} \quad (1\le d'<N),$$

and $\Gamma_0$ be the bicharacteristic leaf pathing through $(\mathring{x},\mathring{\xi}) \in \Lambda$. Then $\Lambda$ and $\Gamma_0$ can be identified with $\kappa(\Lambda) = \{z\in\mathbb{C}^N;\ \mathrm{Im}z'=0\}$ and $\kappa(\Gamma_0) = \{z\in\mathbb{C}^N;\ \mathrm{Im}z'=0,\ z''=\mathring{x}''-i\mathring{\xi}''\}$ respectively, where $z = (z',z'') \in \mathbb{C}^{d'}\times\mathbb{C}^{N-d'}$.

We set $\varphi_\Lambda(z) = |\mathrm{Im}z''|^2/2$; which is the pluri-subharmonic function canonically associated to $\Lambda$. If $\Omega$ is a neighborhood of $\mathring{z} \in \kappa(\Lambda)$, we denote by $H_\Lambda^{\nu,loc}(\Omega)$ the space of holomorphic functions $u(z,\lambda)$ in $\Omega$ with a parameter $\lambda > 0$ such that for all $K \subset\subset \Omega$ and $\varepsilon > 0$ there exists $C_{K,\varepsilon}$ with the estimate:

$$(5.3) \qquad |u(z,\lambda)| \le C_{k,\varepsilon}e^{\lambda\varphi_\Lambda+\varepsilon\lambda^{1/\nu}} \quad \text{for } z \in K,\ \lambda \ge 1.$$

For $\mathring{z} \in \Lambda$ we also use the notation: $u \in H^\nu_{\Lambda,\mathring{z}}$ if there is a neighborhood $\omega_{\mathring{z}}$ of $\mathring{z}$ such that $u \in H^\nu_\Lambda(\omega_{\mathring{z}})$.

If $u \in H_\Lambda^{\nu,\mathrm{loc}}(\Omega)$ we denote by $S^\nu_\Lambda(u)$ the subset in $\Omega$ defined by:

(5.4) $\quad \mathring{z} \notin S^\nu_\Lambda(u)$ if and only if there exist a neighborhood $\omega_{\mathring{z}}$ of $\mathring{z}$ and constants $C$, $c > 0$ such that

$$|u(z,\lambda)| \le Ce^{\lambda\varphi_\Lambda - c\lambda^{1/\nu}} \quad \text{for } z \in \omega_{\mathring{z}},\ \lambda \ge 1.$$

By applying the maximum principle to $z' \longmapsto \lambda^{-1/\nu}(\log|u(z,\lambda)|-\lambda|\mathrm{Im}z''|^2/2)$ it can be seen easily the following two lemmas.

**Lemma 5.1.** *Let $\Gamma_0$ be a bicharacteristic leaf in $\Lambda$ and $\omega$ be a connected open set in $\Gamma_0$ containing $(\mathring{x},\mathring{\xi})$. If $u \in H^\nu_{\Lambda,z}$ for all*

$z \in \kappa(\omega)$ *and* $\kappa(\mathring{x},\mathring{\xi}) = \mathring{x}-i\mathring{\xi} \notin S_\Lambda^\nu(u)$ *then* $\kappa(\omega) \cap S_\Lambda^\nu(u) = \phi$.

**Lemma 5.2.** *Let* $(\mathring{x},\mathring{\xi}) \in \Lambda$, $f \in \mathcal{S}'(\mathbb{R}^N)$. *If* $(\mathring{x},\mathring{\xi}) \notin WF_\nu(f)$ *and* $T^{(1)}f \in H^\nu_{\Lambda,\mathring{x}-i\mathring{\xi}}$ *then* $\mathring{x}-i\mathring{\xi} \notin S_\Lambda^\nu(T^{(1)}f)$.

Let us introduce the F.B.I. tr. of second kind along $\Lambda$ following Lebeau [19]:

$$(5.5)\quad T_\Lambda^{(2)}f(w,\mu,\lambda) = \int e^{-\lambda(w''-x'')^2/2\ -\lambda\mu(w'-x')^2/2}f(x)dx \quad (f\in\mathcal{S}'(\mathbb{R}^N)).$$

Then $T_\Lambda^{(2)}f(w,\mu,\lambda)$ is a holomorphic function with respect to $w \in \mathbb{C}^N$ with the bound:

$$|T_\Lambda^{(2)}f(w,\mu,\lambda)| \le Ce^{\frac{\lambda}{2}|\mathrm{Im}w''|^2 + \frac{\lambda}{2}\mu|\mathrm{Im}w'|^2}(\lambda+|w|)^k.$$

It was shown in [20] and [2] that the relation between $T^{(1)}f$ and $T_\Lambda^{(2)}f$ is

$$(5.6)\qquad T_\Lambda^{(2)}f(w,\mu,\lambda) = \left(\frac{\lambda}{2\pi(1-\mu)}\right)^{\frac{d'}{2}}\int_{\mathbb{R}^{d'}} e^{-\lambda\rho(w'-x')^2/2}T^{(1)}f(x',w'',\lambda)dx',$$

where $\rho = \mu/(1-\mu)$ with an inversion formula:

$$(5.7)\qquad T^{(1)}f(z,\lambda)$$

$$= \frac{1}{2}\left(\frac{1}{2\pi\lambda}\right)^{\frac{d'}{2}}\int_{\mathbb{R}^{d'}_\xi} e^{-\lambda R|\xi'|/2}\left(1-i\frac{\langle\xi',\nabla'\rangle}{\lambda|\xi'|^2}\right)T_\Lambda^{(2)}f\left(z'-i\frac{R\xi'}{|\xi'|},z'',\mu,\lambda\right)\frac{Rd\xi'}{R+|\xi'|},$$

where $\mu = |\xi'|/(R+|\xi'|)$.

Now we define second wave front sets adapted to the Gevrey class. (See also Esser [7].)

**Definition 5.3.** If $1 \le \nu < +\infty$ and $f \in \mathcal{S}'(\mathbb{R}^N)$, *the second wave front set along* $\Lambda$ of $f$, denoted by $WF^{(2)}_{\Lambda,\nu}(f)$, is the subset in $T_\Lambda(T^*(\mathbb{R}^N)\backslash 0)$ defined by the following condition:

$$(5.8)\qquad (\mathring{x},0,\mathring{\xi}'';\mathring{\sigma}') \notin WF^{(2)}_{\Lambda,\nu}(f)$$

if and only if there exist $C, c > 0$, $0 < \mu_0 < 1$ and a decreasing function $o(\lambda)$ with $\lim_{\lambda\to+\infty} o(\lambda) = 0$ such that

$$(5.9) \qquad |T_\Lambda^{(2)} f(w,\mu,\lambda)| \le Ce^{\frac{\lambda}{2}|\mathrm{Im} w''|^2+\frac{\lambda}{2}\mu|\mathrm{Im} w'|^2-c\lambda\mu}$$

for

$$(5.10) \quad 0 < \mu < \mu_0, \ \lambda\mu > o(\lambda)\lambda^{1/\nu}, \ |w'-(\mathring{x}'-i\mathring{\sigma}')|+|w''-(\mathring{x}''-i\mathring{\xi}'')| < c.$$

Using (5.6) and (5.7) we can show the following:

**Lemma 5.4.** *Let* $(\mathring{x},\mathring{\xi}) \in \Lambda$ *and* $f \in \mathscr{S}'(\mathbb{R}^N)$. *Then* $T^{(1)}f \in H^\nu_{\Lambda,\mathring{x}-i\mathring{\xi}}$ *if and only if* $\pi_\Lambda^{-1}(\mathring{x},\mathring{\xi}) \cap WF^{(2)}_{\Lambda,\nu}(f) = \phi$, *where* $\pi_\Lambda : T_\Lambda(T^*(\mathbb{R}^N)\backslash 0) \longrightarrow \Lambda$ *is the canonical projection.*

At last, we introduce the space of partially holomorphic Gevrey functions $G^\nu\mathscr{A}_{x'}$ as follows: $f(x) \in G^\nu\mathscr{A}_{x'}(\Omega)$ if and only if for every comapct set $K \subset\subset \Omega$ there is a constant $C$ such that

$$(5.11) \qquad |\partial_{x'}^{\alpha'}\partial_{x''}^{\alpha''} f(x)| \le C^{|\alpha|+1}\alpha'!(\alpha''!)^\nu \quad \text{for} \quad x \in K.$$

We have

**Lemma 5.5.** *If* $f \in \mathscr{S}'(\mathbb{R}^N)\cap G^\nu\mathscr{A}_{x'}(\Omega)$ *and* $1 \le \nu' < \nu$ *then* $T^{(1)}f \in H^{\nu'}_{\Lambda,z}$ *for every* $z \in \kappa(\pi^{-1}(\Omega)\cap\Lambda)$.

## 6. Proof of Theorem III

As in Section 2 we suppose that $\mathring{x} = 0$, $\mathring{\xi} = (0,0,\mathring{\eta}'') = (0,\cdots,0,1) \in \mathbb{R}^N\backslash 0$ and set $Q = (P^*P)^k$ with $2km \ge d+1$. Here we also introduce the pseudo-differential oprator:

$$(6.1) \qquad \mathrm{Op}(r) = \mathrm{Op}(\eta_n^{2km/(1+h)} e^{-|\eta'|^{2l(1+h)}/\eta_n^{2l}}),$$

where $l$ is a positive integer to be determined. Then $\mathrm{Op}(r)$ has the same quasi-homogeneity in its symbol as $Q$ has.

Consider the operator $Q + \mathrm{Op}(r)$. Then it satisfies (H-2) since $Q$ is non negative self-adjoint operator at $\mathring{\xi}$. We also note that though not being polynomial, $r$ is holomorphic with the uniform bound $O(|\xi|^{2km/(1+h)})$ in a small quasi-homogeneous neighborhood of $\mathring{\xi}$ of the form:

$$V_\varepsilon^{\mathbb{C}} = \{(\eta',\eta'')\in\mathbb{C}^{d'}\times\mathbb{C}^{d''};\ |\mathrm{Im}\eta'|<\varepsilon(|\eta_n|^{1/(1+h)}+|\mathrm{Re}\eta'|),\ |\eta''/\eta_n-\mathring{\eta}''|<\varepsilon\}.$$

Now all the results in Section 2 are remain valid for $Q + \mathrm{Op}(r)$ and we get a symbol $k_g(t,\tau,\eta)$ satisfying (3.4) such that

$$\mathrm{Op}(k_g)^*(Q+\mathrm{Op}(r)) = \mathrm{Op}(g). \tag{6.2}$$

Here $g$ is an arbitrary cut off function satisfying (3.3) for $\rho = 1/(1+h)$ with its support in

$$V_{\varepsilon_0} = \{(\tau,\eta)\in T^*(\mathbb{R}^N)\backslash 0;\ |\tau|<\varepsilon_0\eta_n,\ |\eta'|<\varepsilon_0\eta_n,\ |\eta''/\eta_n-\mathring{\eta}''|<\varepsilon_0\}. \tag{6.3}$$

If $(\mathring{x},\mathring{\xi}) = (0;0,0,\mathring{\eta}'') \in \Sigma$ then the bicharacteristic leaf is $\Gamma_0 = \{(0,y',0;0,0,\mathring{\eta}'');\ y' \in\mathbb{R}^{d'}\}$. For any compact set $F \subset \pi(\Gamma_0\cap W)$ there exist a neighborhood $U \subset\subset O_R = \{x\in\mathbb{R}^N;\ |x|<R\}$ of $F$ and a conic neighborhood $V$ of $\mathring{\xi}$ such that

$$WF_\nu(Pu) \cap \bar{U}\times(\bar{V}\backslash 0) = \phi, \tag{6.4}$$

where $\bar{U}$, $\bar{V}$ denote the closures of $U$, $V$ respectively.

After replacing $u$ by $\varphi u$ with a suitable $\varphi \in C_0^\infty(O_R)$ we can suppose $u \in \mathcal{E}'(O_R)$ with no influence on (6.4).

We fix a conic neighborhood $V_2$ of $\mathring{\xi}$ with $V_2 \subset\subset V\cap V_{\varepsilon_0}$. If we choose another conic neighborhood $V_1$ of $\mathring{\xi}$ sufficiently small then the cut off function $g$ in Lemma 3.1 can be taken in the form: $g(\xi) = g'(\eta',\eta_n)g''(\tau,\eta'')$ so that $\mathrm{supp}\ \nabla_{\eta'}g \subset \{(t,\eta',\eta'');\ |\eta'|>\delta|\xi|\}$ for some $\delta > 0$.

As in Proposition 3.3 one can see the following:

**Proposition 6.1.** *If* $k_g$ *satisfies* (3.4) *with* $\chi_g(\xi) = 0$ *for* $|\eta'| < \delta|\xi|$ $(\delta > 0)$, *then*

$$K_g(t,y,s,w) \in G^{1+h}\mathcal{A}_{y',w'}((\mathbb{R}^N\times\mathbb{R}^N)\backslash\mathrm{diag}(\mathbb{R}^N)). \tag{6.5}$$

*where* $K_g$ *denotes the distribution kernel of* $\mathrm{Op}(k_g)$.

Now we let $g$ be taken as above and write for $u \in \mathcal{E}'(O_R)$

$$\mathrm{Op}(g)u = \mathrm{Op}(k_g)^*Qu + \mathrm{Op}(k_g)^*\mathrm{Op}(r)u \tag{6.6}$$

$$= \mathrm{Op}(k_g)^*Qu + \mathrm{Op}(r)\mathrm{Op}(k_g)^*u.$$

We shall apply the theory of second microlocalization along the involutive submanifold:

$$\Lambda = \{(t,y;\tau,\eta',\eta'')\in T^*(\mathbb{R}^N)\backslash 0;\ \eta'=0\}.$$

Hereafter, we also denote the cordinate in $T^*(\mathbb{R}^N)$ by

$$x' = y',\ x'' = (t,y'') \text{ and } \xi' = \eta',\ \xi'' = (\tau,\eta'')$$

and use the notation in Section 5 without mentioning it.

First we study $\mathrm{Op}(r)\mathrm{Op}(k_g)^*$u, where

$$r(\xi) = \eta_n^{2km/(1+h)}e^{-|\eta'|^{2l(1+h)}/\eta_n^{2l}}$$

was given in (6.1). Now we choose $l$ so that $(1+h)-(1/2l) > \nu$. Then

$$|\eta'|^{2l(1+h)}/\eta_n^{2l} \geq |\eta'| \quad \text{for} \quad |\eta'| \geq \eta_n^{-\varepsilon}\eta_n^{1/\nu},\ \eta_n > 0, \tag{6.7}$$

where $\varepsilon = (1/\nu) - (2l/(2l(1+h)-1)) > 0$. We can see easily the following:

**Lemma 6.2.** *If* $r = O(e^{-c|\eta'|})$, $c > 0$ *for* $|\eta'| \geq \eta_n^{-\varepsilon}\eta_n^{1/\nu}$, $\eta_n > 0$ *then for every* $u \in \mathcal{S}'(\mathbb{R}^N)$

$$WF_{\Lambda,\nu}^{(2)}(\mathrm{Op}(r)u) \cap \pi_\Lambda^{-1}(\Gamma_0) = \phi. \tag{6.8}$$

Since $\mathrm{Op}(k_g)(\mathcal{S}) \subset \mathcal{S}$: equivalently $\mathrm{Op}(k_g)^*(\mathcal{S}') \subset \mathcal{S}'$, (6.8) holds for $\mathrm{Op}(r)\mathrm{Op}(k_g)^*u$. Therefore we have

$$T^{(1)}(\mathrm{Op}(r)\mathrm{Op}(k_g)^*u) \in H_{\Lambda,z}^{\nu} \quad \text{for all} \quad z \in \kappa(\Gamma_0) \tag{6.9}$$

in view of Lemma 5.4.

Next we study $\mathrm{Op}(k_g)^*Qu$. Let $\tilde{g}$ be another cut off function given by Lemma 3.1 with two cones $\tilde{V}_1$, $\tilde{V}_2$ such that

$$V_2 \subset\subset \tilde{V}_1 \subset\subset \tilde{V}_2 = V .$$

Noticing that $WF_\nu(Qu) \subset WF_\nu(Pu)$, we then get by (6.4)

$$(6.10) \qquad WF_\nu(\mathrm{Op}(\tilde{g})Qu) \subset WF_\nu(Pu)\cap(\mathbb{R}^N\times\overline{V}) \subset \pi^{-1}(O_R\backslash U),$$

$$(6.11) \qquad WF_\nu(\mathrm{Op}(1-\tilde{g})Qu) \subset WF_\nu(Pu)\cap(\mathbb{R}^N\times(\mathbb{R}^N\backslash\tilde{V}_1)) \subset O_R\times(\mathbb{R}^N\backslash\overline{V}_2).$$

Hence we can write

$$(6.12) \qquad Qu = \chi_{F_\varepsilon}\mathrm{Op}(\tilde{g})Qu + \chi_{O_R}(1-\chi_{F_\varepsilon})\mathrm{Op}(\tilde{g})Qu + \chi_{O_R}\mathrm{Op}(1-\tilde{g})Qu$$

$$(\underset{\rightarrow}{=} v_1 + v_2 + v_3),$$

where $\chi_B$ denotes the characteristic function of each set $B$ and

$$F_\varepsilon = \{(x',x'')\in\mathbb{R}^N;\ (x',0)\in F,\ |x''|\le\varepsilon\}$$

with $\varepsilon > 0$ so small that $F_\varepsilon \subset U$.

In the following we assume further that

$$(6.13) \qquad F \text{ is convex with an analytic boundary in } \pi(\Gamma_0).$$

By (6.10) we see that

$$WF_\nu(v_1) \subset \{(x,\xi);(x',\xi')\in T^*_{\partial F}(\pi(\Gamma_0)),|\chi''|<\varepsilon,\xi''=0\} \cup \pi^{-1}(\{x;|x''|\ge\varepsilon\}).$$

Hence by (3.7)

$$(6.14) \qquad \mathrm{Op}(k_g)^*v_1 \in G^\nu(\mathrm{Int}(F_\varepsilon)),$$

where $\mathrm{Int}(F_\varepsilon)$ denotes the interior of $F_\varepsilon$.

Since $\mathrm{supp}(v_2) \subset \overline{O}_R\backslash F_\varepsilon$, it follows by Proposition 6.1

$$(6.15) \qquad \mathrm{Op}(k_g)^*v_2 \in G^{1+h_{\mathscr{A}_{x'}}}(\mathrm{Int}(F_\varepsilon)).$$

Thus by Lemma 5.5,

$$(6.16) \qquad T^{(1)}(\mathrm{Op}(k_g)^* v_2) \in H^\nu_{\Lambda,z} \quad \text{for all} \quad z \in \kappa(\pi^{-1}(\mathrm{Int}(F_\varepsilon)\cap\Lambda)).$$

In view of (6.11),

$$WF_\nu(v_3) \subset O_R \times (\mathbb{R}^N \backslash \overline{V}_1) \cup T^*_{\partial O_R}(\mathbb{R}^N).$$

Again by (3.7) this yields

$$(6.17) \qquad \mathrm{Op}(k_g)^* v_3 \in G^\nu(\mathrm{Int}(F_\varepsilon)).$$

Consequently, by (6.9) and (6.14)–(6.17), we have

$$(6.18) \qquad \mathrm{Op}(g)u = u_1 + u_2,$$

where

$$u_1 = \mathrm{Op}(k_g)^*(v_1 + v_3) \in G^\nu(\mathrm{Int}(F_\varepsilon))$$

and

$$u_2 = \mathrm{Op}(k_g)^* v_2 + \mathrm{Op}(r)\mathrm{Op}(k_g)^* u$$

with

$$T^{(1)}(u_2) \in H^\nu_{\Lambda,z} \quad \text{for all} \quad z \in \kappa(\pi^{-1}(\mathrm{Int}(F_\varepsilon))\cap\Gamma_0).$$

Now we apply Lemma 5.1, 5.2 and obtain

(6.19) If $(\mathring{x},\mathring{\xi}) \in \pi^{-1}(\mathrm{Int}(F_\varepsilon)) \cap \Gamma_0$ and $(\mathring{x},\mathring{\xi}) \notin WF_\nu(u_2)$ then $\pi^{-1}(\mathrm{Int}(F_\varepsilon)) \cap \Gamma_0 \cap WF_\nu(u_2) = \phi$.

Because $g \equiv 1$ in the neighborhood $V_1$ of $\mathring{\xi}$,

$$WF_\nu(u_2) \cap \pi^{-1}(\mathrm{Int}(F_\varepsilon)) \cap \Gamma_0 = WF_\nu(u) \cap \pi^{-1}(\mathrm{Int}(F_\varepsilon)) \cap \Gamma_0.$$

Therefore (6.19) implies Theorem III for $\hat{W} = \pi^{-1}(\mathrm{Int}(F_\varepsilon))$.

Since any compact set in $\Gamma_0 \cap W$ can be covered by a finite number of such $\hat{W}$'s we have actually proved Theorem III. □

## 7. Remarks

The problem to determine the Gevrey class in which certain $C^\infty$ hypoelliptic operators still remain hypoelliptic, has its origin in a celebrated examle given by Baouendi-Goulaouic [1]:

$$P_1 = \partial_t^2 + \partial_x^2 + t^2\partial_y^2;$$

which has a solution $u$ of $P_1 u = 0$ in a neighborhood of the origin only belonging to $G^2$.

Deridj-Zuily [5] and Durand [6] have studied Gevrey hypo-ellipticity for second order operators and proved, for example, $G^{1+h+0}$ and $G^{1+h}$ hypoellipticity of the operator (1.6) in Section 1 respectively.

However, as was shown by Parenti-Rodino [24], hypoellipticity does not always imply microlocal one. In this respect, Iwasaki [17] proved among others $G^2$ microhypoellipticity for double characteristic operators. Our Theorem I is an extention of this in some sence, though the operators are much restricted.

Recently, Kajitani-Wakabayashi also studied Gevrey micro-hypoellipticity in [18] but for more general classes of operators and obtained the results including our Theorem I as a spacial case.

However our poof by constructing parametrices reviels how the quasi-homogeneity of operators relate to the lowest order of Gevrey class in which the operators remain hypoelliptic and gives a more precise information on the singularities of solutions (: Proposition 6.1 and Theorem III).

At last, we remark the following: Since $\mathrm{Op}(k_g)$ act on the space of ultra-distributions $(G^{1+h})'$ preserving local $G^{1+h_\mathcal{A}}$ regulalities Theorem I and III are valid for $u \in (G^{1+h})'$ without any change.

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