NOTES ON P-VALENTLY BAZILEVIĆ FUNCTIONS

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ABSTRACT

The object of the present paper is to improve the former results for p-valently Bazilević functions which were recently proved by the author and others.

I. INTRODUCTION

Let $A_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\})$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$. A function $f(z)$ belonging to $A_p$ is said to be p-valently starlike if and only if it satisfies the condition

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in E).$$

We denote by $S_p^\ast$ the subclass of $A_p$ consisting of functions which are p-valently starlike in $E$.

A function $f(z)$ belonging to the class $A_p$ is said to be p-valently Bazilević of type $\beta$ and order $\gamma$ if there exists a function $g(z)$ belonging to $S_p^\ast$ such that

$$\Re\left\{\frac{zf'(z)}{f(z)^{1-\beta} g(z)^{\beta}}\right\} > \gamma \quad (z \in E)$$

for some $\beta (\beta > 0)$ and $\gamma (0 \leq \gamma < p)$.

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Also we denote by \( B_p(\beta, \gamma) \) the subclass of \( A_p \) consisting of all \( p \)-valently Bazilevič functions of type \( \beta \) and order \( \gamma \) in \( \mathbb{E} \). The concept of Bazilevič functions was first introduced by Bazilevič [6]. Thomas [8] has called a function in the class \( B_1(\beta, 0) \) a Bazilevič function of type \( \beta \). Further, Nunokawa [3] has proved that a function \( f(z) \) in the class \( B_p(\beta, 0) \) is \( p \)-valent in the unit disk \( \mathbb{E} \).

In particular, the class \( B_p(\beta, \gamma) \) for \( g(z) = f(z) \) is called the class of \( p \)-valently starlike functions of order \( \gamma \). Further, we note that the class \( B_p(\beta, 0) \) for \( g(z) = f(z) \) is equivalent to \( S_p^* \).

Let \( A_p(\alpha, \beta) \) be the subclass of \( A_p \) consisting of functions which satisfy the condition

\[
(1.4) \quad \text{Re} \left\{ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left[ 1 + \frac{zf''(z)}{f'(z)} \right] \right\} > \beta \quad (z \in \mathbb{E})
\]

for some real \( \alpha \) and \( \beta \).

The class \( A_1(\alpha, 0) \) when \( p = 1 \) and \( \beta = 0 \) was introduced by Mocanu [2], and was studied by Miller, Mocanu and Reade [1], and Sakaguchi and Fukui [7].

Let \( C_p(\alpha, \beta) \) be the subclass of \( A_p \) consisting of functions satisfying the condition

\[
(1.5) \quad \text{Re} \left\{ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left[ 1 + \frac{zf''(z)}{f'(z)} \right] \right\} < \beta \quad (z \in \mathbb{E})
\]

for some real \( \alpha \) and \( \beta \) (\( \beta > p \)).

The class \( C_p(\alpha, \beta) \) was recently introduced by Nunokawa and Owa [4].

2. SOME PROPERTIES

In order to derive our results, we need the following lemmas.

**Lemma I** ([5]). If a function \( f(z) \) belongs to the class \( A_p(\alpha, \beta) \) with \( \alpha > 0 \) and \( 0 \leq -\beta/\alpha \leq 1/2 \), then \( f(z) \in B_p(1/\alpha, 2^{2\beta/\alpha}) \), and therefore \( f(z) \) is \( p \)-valent in the unit disk \( \mathbb{E} \).
LEMMA 2 ([4]), Let a function \( f(z) \) belong to the class \( C_p(\alpha, \beta) \) with \( \alpha \neq 0, \beta > p, \) and \( |\beta/\alpha| \leq 1/2. \) Then \( f(z) \) is \( p \)-valent in the unit disk \( \mathbb{D}. \) Moreover, if \( 0 \leq -\beta/\alpha \leq 1/2, \) then \( f(z) \in B_p(1/\alpha, 2^{2\beta/\alpha}). \)

Applying the above lemmas, we prove

THEOREM I. If a function \( f(z) \) belongs to the class \( A_p(\alpha, \beta) \) with \( \alpha > 0 \) and \( 0 \leq -\beta/\alpha \leq 1/2, \) then \( f(z) \in B_p(1/\alpha, p^{2\beta/\alpha}). \)

PROOF. For a function \( f(z) \) in the class \( A_p, \) we define the function \( g(z) \) by

\[
(2.1) \quad g(z) = f(z)^{1/p}
= z + g_2 z^2 + g_3 z^3 + \ldots
\]

Then \( g(z) \) is in the class \( A_1, \) and satisfies

\[
(2.2) \quad \frac{zf'(z)}{f(z)} = p \frac{zg'(z)}{g(z)}
\]

and

\[
(2.3) \quad 1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{zg''(z)}{g'(z)} + (p - 1) \frac{zg'(z)}{g(z)}
\]

It follows from (2.2) and (2.3) that

\[
(2.4) \quad (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left[ 1 + \frac{zf''(z)}{f'(z)} \right]
= (p - \alpha) \frac{zg'(z)}{g(z)} + \alpha \left[ 1 + \frac{zg''(z)}{g'(z)} \right].
\]

Therefore, we have

\[
(2.5) \quad f(z) \in A_p(\alpha, \beta) \iff \Re \left\{ \left[ 1 - \frac{\alpha}{p} \right] \frac{zg'(z)}{g(z)} + \frac{\alpha}{p} \left[ 1 + \frac{zg''(z)}{g'(z)} \right] \right\} > \frac{\beta}{p}
\]
\[ \iff g(z) \in A_1(a/p, \beta/p). \]

Applying Lemma 1 for \( p = 1 \), we see that

\[ g(z) \in A_1(a/p, \beta/p) \implies g(z) \in B_1(p/a, 2^{2\beta/a}). \]

It follows that

\[ f(z) \in A_p(a, \beta) \implies g(z) \in B_1(p/a, 2^{2\beta/a}) \]

\[ \iff \Re \left\{ \frac{z g'(z)}{g(z) 1-p/a h(z)^p/a} \right\} > 2^{2\beta/a} \quad (h(z) \in S_1^*) \]

\[ \iff \Re \left\{ \frac{z f'(z)}{f(z) 1-1/\alpha h(z)^p/1/\alpha} \right\} > p^{2\beta/a} \quad (h(z)^p \in S_1^*) \]

\[ \iff f(z) \in B_p(1/\alpha, p^{2\beta/a}). \]

This completes the assertion of Theorem 1.

**Remark.** Noting \( p^{2\beta/a} \geq 2^{2\beta/a} \), we see that

\[ B_p(1/\alpha, p^{2\beta/a}) \subseteq B_p(1/\alpha, 2^{2\beta/a}). \]

Thus Theorem 1 is the improvement of Lemma 1 by Nunokawa, Owa, Saitoh, Yaguchi and Lee [5].

Taking \( p = 1 \) in Theorem 1, we have

**Corollary 1.** If \( f(z) \in A_1(a, \beta) \) with \( a > 0 \) and \( 0 \leq -\beta/a \leq 1/2 \), then \( f(z) \in B_1(1/a, 2^{2\beta/a}) \).

Using the same manner as in the proof of Theorem 1, we have

**Theorem 2.** If a function \( f(z) \) belongs to the class \( C_p(a, \beta) \) with
\( \alpha \neq 0, \beta > p, \text{ and } 0 \leq -\beta/\alpha \leq 1/2, \text{ then } f(z) \in B_p^{1/(\alpha, p2^{2\beta}/\alpha)}. \)

Finally, letting \( p = 1 \) in Theorem 2, we have

**Corollary 2.** If \( f(z) \in C_1^{1}(\alpha, \beta) \) with \( \alpha \neq 0, \beta > 1, \text{ and } 0 \leq -\beta/\alpha \leq 1/2, \)
then \( f(z) \in B_1^{1/(\alpha, 2^{2\beta}/\alpha)}. \)

**Remark 2.** We note that Theorem 2 is the improvement of Lemma 2 due to Nunokawa and Owa [4].

**References**


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