

ON STARLIKE, CONVEX AND CLOSE-TO-CONVEX FUNCTIONS OF COMPLEX ORDER

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I. INTRODUCTION

Let  $\mathcal{A}$  denote the class of functions of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk  $U = \{z: |z| < 1\}$ . A function  $f(z)$  belonging to the class  $\mathcal{A}$  is said to be starlike of complex order  $b$  ( $b \neq 0$ , complex) if and only if  $f(z)/z \neq 0$  ( $z \in U$ ) and

$$(1.2) \quad \operatorname{Re} \left\{ 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} > 0 \quad (z \in U).$$

We denote by  $S_0^*(b)$  the subclass of  $\mathcal{A}$  consisting of functions which are starlike of complex order  $b$ . A function  $f(z)$  in  $\mathcal{A}$  is said to be convex of complex order  $b$  ( $b \neq 0$ , complex) if and only if  $f'(z) \neq 0$  ( $z \in U$ ) and

$$(1.3) \quad \operatorname{Re} \left\{ 1 + \frac{1}{b} \frac{zf''(z)}{f'(z)} \right\} > 0 \quad (z \in U).$$

Also we denote by  $K_0(b)$  the subclass of  $\mathcal{A}$  consisting of functions which are convex of complex order  $b$ . Note that  $f(z) \in K_0(b)$  if and only if  $zf'(z) \in S_0^*(b)$ .

REMARK 1. The class  $S_0^*(b)$  was introduced by Nasr and Aouf [3], and the class  $K_0(b)$  was introduced by Nasr and Aouf [4].

REMARK 2. Letting  $b = 1 - \alpha$ , we observe that  $S_0^*(1-\alpha) = S^*(\alpha)$  and  $K_0(1-\alpha) = K(\alpha)$ , where  $S^*(\alpha)$  and  $K(\alpha)$  denote the classes of starlike functions and convex functions of order  $\alpha$ , respectively.

A function  $f(z)$  belonging to the class  $\Lambda$  is said to be close-to-convex of complex order  $b$  ( $b \neq 0$ , complex) if and only if there exists a function  $g(z) \in S_0^*(1)$  satisfying the condition

$$(1.4) \quad \operatorname{Re} \left\{ 1 + \frac{1}{b} \left[ \frac{f'(z)}{g'(z)} - 1 \right] \right\} > 0 \quad (z \in U).$$

We denote by  $C_0(b)$  the subclass of  $\Lambda$  consisting of functions which are close-to-convex of complex order  $b$ . We note that  $C_0(1-\alpha) = C(\alpha)$ , where  $C(\alpha)$  is the class of close-to-convex functions of order  $\alpha$ .

## 2. FORMER RESULTS

**THEOREM A** (Nasr and Aouf [3]), A function  $f(z)$  is in the class  $S_0^*(b)$  if and only if there exists a probability measure  $\mu(t)$  ( $0 \leq t < 2\pi$ ) such that

$$(2.1) \quad f(z) = z \exp \left\{ \int_0^{2\pi} -2b \log(1 - ze^{-it}) d\mu(t) \right\}.$$

**THEOREM B** (Nasr and Aouf [3]), If a function  $f(z)$  is in the class  $S_0^*(b)$  with  $\operatorname{Re}(b) > 0$ , then

$$(2.2) \quad |a_n| \leq \frac{1}{(n-1)!} \prod_{m=0}^{n-2} |2b + m| \quad (n \geq 2).$$

The equality in (2.2) is attained for the function

$$(2.3) \quad f(z) = \frac{z}{(1-z)^b} = z + \sum_{n=2}^{\infty} \prod_{m=0}^{n-2} \left( \frac{2b+m}{m+1} \right) z^n.$$

**THEOREM C** (Nasr and Aouf [3]), If a function  $f(z)$  is in the class  $S_0^*(b)$ , then

$$(2.4) \quad |a_3 - \mu a_2^2| \leq |b| \max(1, |4b\mu - 2b - 1|),$$

where  $\mu$  is a complex number.

**THEOREM D** (Nasr and Aouf [3]), If a function  $f(z)$  is in the class

$S_0^*(b)$ , then

$$(2.5) \quad \left| \frac{zf'(z)}{f(z)} \right| \leq 1 + \frac{2|b|r}{1-r} \quad (|z| = r < 1).$$

The equality in (2.5) is attained for the function

$$(2.6) \quad f(z) = \frac{z}{(1-z)^{2b}}.$$

### 3. SOME RESULTS

In order to derive our results, we have to recall here the following lemma.

LEMMA I (Miller and Mocanu [2]). Let  $q(z)$  be univalent in the unit disk  $\mathbb{U}$ ,  $\theta(w)$  and  $\phi(w)$  be analytic in the domain  $\mathbb{D}$  containing  $q(\mathbb{U})$ , and  $\phi(w) \neq 0$  for  $w \in q(\mathbb{U})$ . Set

$$Q(z) = zq'(z)\phi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + Q(z).$$

Suppose that

(i)  $Q(z)$  is starlike (univalent) in  $\mathbb{U}$  with  $Q(0) = 0$ ,  $Q'(0) \neq 0$ ;

and

$$(ii) \quad \operatorname{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} = \operatorname{Re} \left\{ \frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)} \right\} > 0 \quad (z \in \mathbb{U}).$$

If  $p(z)$  is analytic in  $\mathbb{U}$  with  $p(0) = q(0)$ ,  $p(\mathbb{U}) \subset \mathbb{D}$ , and

$$(3.1) \quad \theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z),$$

then  $p(z) \prec q(z)$ , and  $q(z)$  is the best dominant of the subordination (3.1).

REMARK 3. The univalent function  $q(z)$  is said to be a dominant of the differential subordination

$$(3.2) \quad \Psi(p(z), zp'(z)) \prec h(z)$$

if  $p(z) \prec q(z)$  for all  $p(z)$  satisfying (3.2). In particular, if  $\tilde{q}(z)$  is a dominant of (3.2) and  $\tilde{q}(z) \prec q(z)$  for all dominants  $q(z)$  of (3.2), then  $\tilde{q}(z)$  is said to be the best dominant of (3.2).

**THEOREM I** (Obradović, Aouf and Owa [5]), If a function  $f(z)$  is in the class  $S_0^*(b)$ , then

$$(3.3) \quad \left\{ \frac{f(z)}{z} \right\}^a \prec \frac{1}{(1-z)^{2ab}},$$

where  $a$  is a complex number,  $a \neq 0$ , and either  $|2ab + 1| \leq 1$  or  $|2ab - 1| \leq 1$ . The function  $1/(1-z)^{2ab}$  is the best dominant of the differential subordination (3.3).

**PROOF.** Letting  $q(z) = 1/(1-z)^{2ab}$ ,  $\theta(w) = 1$  and  $\phi(w) = 1/abw$  in Lemma 1, we see that  $Q(z) = 2z/(1-z)$  and  $h(z) = (1+z)/(1-z)$ . Therefore,  $Q(z)$  is starlike (univalent) in the unit disk  $\mathbb{U}$ ,  $Q(0) = 0$ ,  $Q'(0) = 2 \neq 0$ , and

$$\operatorname{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} = \operatorname{Re} \left\{ \frac{1}{1-z} \right\} > 0 \quad (z \in \mathbb{U}).$$

Thus, the conditions (i) and (ii) in Lemma 1 are satisfied.

Also, we see that  $q(z)$  is univalent in  $\mathbb{U}$  by Royster [6]. We define the function  $p(z)$  by  $p(z) = (f(z)/z)^a$  for  $f(z) \in S_0^*(b)$ . Then  $p(z)$  is analytic in  $\mathbb{U}$ ,  $p(z) = 1 + p_1z + p_2z^2 + \dots$ , and  $p(z) \neq 0$  for  $0 < |z| < 1$ . Since

$$(3.4) \quad \begin{aligned} \theta(p(z)) + zp'(z)\phi(p(z)) &= 1 + \frac{1}{ab} \frac{zp'(z)}{p(z)} \\ &= 1 + \frac{1}{b} \left[ \frac{zf'(z)}{f(z)} - 1 \right], \end{aligned}$$

$f(z) \in S_0^*(b)$  implies that

$$(3.5) \quad 1 + \frac{1}{b} \left[ \frac{zf'(z)}{f(z)} - 1 \right] \prec \frac{1+z}{1-z} = h(z).$$

Consequently, with the help of Lemma 1, we observe that

$$(3.6) \quad 1 + \frac{1}{b} \left\{ \frac{zf'(z)}{f(z)} - 1 \right\} \prec \frac{1+z}{1-z}$$

$$\Rightarrow \left( \frac{f(z)}{z} \right)^a \prec \frac{1}{(1-z)^{2ab}}.$$

This completes the assertion of Theorem 1.

Taking  $a = -1/2b$  in Theorem 1, we have

COROLLARY I. If a function  $f(z)$  is in the class  $S_0^*(b)$ , then

$$(3.7) \quad \left( \frac{z}{f(z)} \right)^{1/2b} \prec 1 - z,$$

and

$$(3.8) \quad \left| \left( \frac{z}{f(z)} \right)^{1/2b} - 1 \right| < |z| \quad (z \in U).$$

REMARK 4. If  $b = 1 - \alpha$  ( $0 \leq \alpha < 1$ ), Corollary 1 becomes that

$$f(z) \in S^*(\alpha) \Rightarrow \left| \left( \frac{z}{f(z)} \right)^{1/2(1-\alpha)} - 1 \right| < |z| \quad (z \in U).$$

This is the former result given by Todorov [8].

COROLLARY 2. If a function  $f(z)$  is in the class  $K_0(b)$ , then

$$(3.9) \quad (f'(z))^a \prec \frac{1}{(1-z)^{2ab}},$$

where  $a$  is a complex number,  $a \neq 0$ , and either  $|2ab + 1| \leq 1$  or  $|2ab - 1| \leq 1$ .

Next, we show

THEOREM 2 (Obradović, Aouf and Owa [5]), Let a function  $f(z)$  be

in the class  $S_0^*(b)$ , and  $a$  be a complex number,  $a \neq 0$ ,  $0 < 2ab \leq 1$ . Then

$$(3.10) \quad \operatorname{Re} \left\{ \frac{f(z)}{z} \right\}^a > \frac{1}{2^{2ab}} \quad (z \in \mathbb{U})$$

and

$$(3.11) \quad \left| \left( \frac{f(z)}{z} \right)^{-a} - 2^{2ab-1} \right| < 2^{2ab-1} \quad (z \in \mathbb{U}).$$

The estimates are best possible.

PROOF. Define the function  $g(\theta) = \cos(\mu\theta) - \cos^\mu\theta$  for  $0 < \mu \leq 1$  and  $-\pi/2 \leq \theta \leq \pi/2$ . Then we see that  $g(\theta)$  is an even function of  $\theta$  and

$$g'(\theta) = \mu \{ \cos^{\mu-1}\theta \sin\theta - \sin(\mu\theta) \}$$

$$\geq \mu \{ \sin\theta - \sin(\mu\theta) \}$$

$$\geq 0$$

for  $0 < \mu \leq 1$  and  $0 \leq \theta \leq \pi/2$ . This proves that  $g(\theta) \geq g(0) = 0$ , that is, that

$$\cos(\mu\theta) - \cos^\mu\theta \geq 0$$

for  $0 < \mu \leq 1$  and  $-\pi/2 \leq \theta \leq \pi/2$ .

Letting  $\mu = 2ab$  and  $\theta = \phi/2 - \pi/2$  ( $0 \leq \phi \leq 2\pi$ ), we obtain that

$$\begin{aligned} \operatorname{Re} \left\{ \frac{1}{(1 - e^{i\phi})^{2ab}} \right\} &= \left( 2 \sin \frac{\phi}{2} \right)^{-\mu} \cos \{ \mu(\phi/2 - \pi/2) \} \\ &= (2 \cos \theta)^{-\mu} \cos(\mu\theta). \end{aligned}$$

Therefore, using Theorem 1, we have

$$\operatorname{Re} \left\{ \frac{f(z)}{z} \right\}^a > \operatorname{Re} \left\{ \frac{1}{(1 - e^{i\phi})^{2ab}} \right\} \geq \frac{1}{2^{2ab}} \quad (0 \leq \phi \leq 2\pi).$$

This completes the first half of the theorem.

The second half of the theorem follows from the above and the

result given by Wilken and Feng [10].

Further, taking the function  $f(z) = z/(1-z)^{2b}$ , we see that the results of the theorem are sharp.

In order to derive our next results, we need the following lemma.

LEMMA 2 (Jack [1]). Let  $w(z)$  be regular in the unit disk  $U$  and such that  $w(0) = 0$ . If  $|w(z)|$  attains its maximum value on the circle  $|z| = r$  at a point  $z_0$ , then we have

$$z_0 w'(z_0) = k w(z_0),$$

where  $k$  is real and  $k \geq 1$ .

With the aid of the above lemma, we prove

THEOREM 3. If a function  $f(z)$  belonging to  $A$  satisfies

$$(3.12) \quad \left| \frac{f'(z)}{g'(z)} - 1 \right|^\alpha \left| \frac{zf''(z)}{g'(z)} - \frac{zf'(z)g''(z)}{g'(z)^2} \right|^\beta < |b|^{\alpha+\beta} \quad (z \in U)$$

for some  $\alpha \geq 0$ ,  $\beta \geq 0$ , and  $g(z) \in S_0^*(1)$ , then  $f(z) \in C_0(b)$ .

PROOF. Defining the function  $w(z)$  by

$$(3.13) \quad w(z) = \frac{1}{b} \left( \frac{f'(z)}{g'(z)} - 1 \right)$$

for  $f(z)$  belonging to  $A$  and  $g(z)$  belonging to  $S_0^*(1)$ , we see that  $w(z)$  is regular in the unit disk  $U$  and  $w(0) = 0$ . Noting that

$$(3.14) \quad b z w'(z) = \frac{zf''(z)}{g'(z)} - \frac{zf'(z)g''(z)}{g'(z)^2},$$

we know that (3.12) can be written

$$(3.15) \quad |b w(z)|^\alpha |b z w'(z)|^\beta < |b|^{\alpha+\beta}.$$

Suppose that there exists a point  $z_0 \in U$  such that

$$(3.16) \quad \max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then, Lemma 2 leads to

$$(3.17) \quad |bw(z_0)|^\alpha |bz_0 w'(z_0)|^\beta = |b|^{\alpha+\beta} k^\beta \quad (k \geq 1) \\ \geq |b|^{\alpha+\beta}$$

which contradicts our condition (3.12). Therefore, we conclude that  $|w(z)| < 1$  for all  $z \in U$ , that is, that

$$\left| \frac{1}{b} \left( \frac{f'(z)}{g'(z)} - 1 \right) \right| < 1 \quad (z \in U).$$

This implies that

$$\operatorname{Re} \left\{ 1 + \frac{1}{b} \left( \frac{f'(z)}{g'(z)} - 1 \right) \right\} > 0 \quad (z \in U),$$

which proves that  $f(z) \in C_0(b)$ .

Letting  $g(z) = z \in S_0^*(1)$  in Theorem 3, we have

**COROLLARY 3.** If a function  $f(z)$  belonging to  $A$  satisfies

$$(3.18) \quad |f'(z) - 1|^\alpha |zf''(z)|^\beta < |b|^{\alpha+\beta} \quad (z \in U)$$

for some  $\alpha \geq 0$ ,  $\beta \geq 0$ , then  $f(z) \in C_0(b)$ .

**THEOREM 4.** If a function  $f(z)$  belonging to  $A$  satisfies

$$(3.19) \quad \left| a \left( \frac{zf'(z)}{f(z)} - 1 \right) + (1-a) \frac{z^2 f''(z)}{f(z)} \right| < |b| \quad (z \in U)$$

for  $0 \leq a \leq 1$ ,  $|b| \leq 1$ , then  $f(z) \in S_0^*(b)$ .

**PROOF.** Let



$$(3.20) \quad w(z) = \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right)$$

for  $f(z) \in \mathcal{A}$ . Then the function  $w(z)$  is regular in the unit disk  $\mathbb{U}$  and  $w(0) = 0$ . Making use of the logarithmic differentiations in both sides of (3.20), we have

$$(3.21) \quad \frac{zf''(z)}{f'(z)} = bw(z) + \frac{bzw'(z)}{1 + bw(z)}$$

It follows that

$$(3.22) \quad a \left( \frac{zf'(z)}{f(z)} - 1 \right) + (1-a) \frac{z^2 f''(z)}{f(z)} \\ = bw(z) \left\{ 1 + (1-a)bw(z) + (1-a) \frac{zw'(z)}{w(z)} \right\},$$

or

$$(3.23) \quad \left| bw(z) \left\{ 1 + (1-a) \left( bw(z) + \frac{zw'(z)}{w(z)} \right) \right\} \right| < |b|.$$

Assume that there exists a point  $z_0 \in \mathbb{U}$  such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then we can write  $w(z_0) = e^{i\theta}$ . Therefore, using Lemma 2, we see that

$$(3.24) \quad \left| bw(z_0) \left\{ 1 + (1-a) \left( bw(z_0) + \frac{z_0 w'(z_0)}{w(z_0)} \right) \right\} \right| \\ = |b| |1 + (1-a)(k + be^{i\theta})| \\ \geq |b| |1 + (1-a)(1 - |b|)| \\ \geq |b|$$

which contradicts (3.23). Thus we have

$$(3.25) \quad |w(z)| = \left| \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right| < 1 \quad (z \in U).$$

Noting that (3.25) implies

$$\operatorname{Re} \left\{ 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} > 0 \quad (z \in U),$$

we complete the proof of Theorem 4.

Further, we derive

**THEOREM 5.** If a function  $f(z)$  belonging to  $A$  satisfies

$$(3.26) \quad \left| \frac{f'(z)}{g'(z)} - 1 \right|^\alpha \left| \frac{zf''(z)}{f'(z)} - \frac{zg''(z)}{g'(z)} \right|^\beta < |b|^\alpha \left( \frac{|b|}{1 + |b|} \right)^\beta$$

for some  $\alpha \geq 0$ ,  $\beta \geq 0$ ,  $g(z) \in S_0^*(1)$ , and for all  $z \in U$ , then  $f(z) \in C_0(b)$ .

**PROOF.** Defining the function  $w(z)$  by (3.13), we have

$$(3.27) \quad \frac{zf''(z)}{f'(z)} - \frac{zg''(z)}{g'(z)} = \frac{bzw'(z)}{1 + bw(z)}.$$

Therefore, the condition (3.26) leads to

$$(3.28) \quad |bw(z)|^\alpha \left| \frac{bzw'(z)}{1 + bw(z)} \right|^\beta < |b|^\alpha \left( \frac{|b|}{1 + |b|} \right)^\beta.$$

Suppose that there exists a point  $z_0 \in U$  such that

$$(3.29) \quad \max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then letting  $w(z_0) = e^{i\theta}$ , and applying Lemma 2, we obtain that

$$(3.30) \quad |bw(z_0)|^\alpha \left| \frac{bz_0 w'(z_0)}{1 + bw(z_0)} \right|^\beta = |b|^\alpha \left| \frac{bkw(z_0)}{1 + bw(z_0)} \right|^\beta \\ \geq |b|^\alpha \left( \frac{|b|}{1 + |b|} \right)^\beta$$

which contradicts our condition (3.26). Thus we see that  $f(z) \in C_0(b)$ .

Taking  $g(z) = z \in S_0^*(1)$  in Theorem 5, we have

COROLLARY 4. If a function  $f(z)$  belonging to  $A$  satisfies

$$(3.31) \quad |f'(z) - 1|^\alpha \left| \frac{zf''(z)}{f'(z)} \right|^\beta < |b|^\alpha \left( \frac{|b|}{1 + |b|} \right)^\beta \quad (z \in U)$$

for some  $\alpha \geq 0$ ,  $\beta \geq 0$ , then  $f(z) \in C_0(b)$ .

#### 4. COEFFICIENT INEQUALITIES

In this section, we consider the coefficient inequalities of functions to be in the classes  $S_0^*(b)$ ,  $K_0(b)$  and  $C_0(b)$ .

THEOREM 6. If a function  $f(z)$  belonging to  $A$  satisfies

$$(4.1) \quad \sum_{n=2}^{\infty} (n + |b| - 1) |a_n| \leq |b| \quad (b \text{ is complex, } b \neq 0),$$

then  $f(z) \in S_0^*(b)$ .

PROOF. It is easy to see that the condition (4.1) implies that

$$(4.2) \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{\sum_{n=2}^{\infty} n a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} a_n z^{n-1}} \right| < \frac{\sum_{n=2}^{\infty} n |a_n|}{1 - \sum_{n=2}^{\infty} |a_n|} \leq |b|.$$

Thus we have  $f(z) \in S_0^*(b)$ .

Similarly, we have

THEOREM 7. If a function  $f(z)$  belonging to  $A$  satisfies

$$(4.3) \quad \sum_{n=2}^{\infty} n(n + |b| - 1) |a_n| \leq |b| \quad (b \text{ is complex, } b \neq 0),$$

then  $f(z) \in K_0(b)$ .

REMARK 5. Letting  $b = 1 - \alpha$  ( $0 \leq \alpha < 1$ ) in Theorem 6 and Theorem 7, we have the corresponding results by Silverman [7].

THEOREM 8. If a function  $f(z)$  belonging to  $A$  satisfies

$$(4.4) \quad \sum_{n=2}^{\infty} n |a_n| \leq |b| \quad (b \text{ is complex, } b \neq 0),$$

then  $f(z) \in C_0(b)$ .

Now, we introduce the subclasses  $S_1^*(b)$ ,  $K_1(b)$  and  $C_1(b)$  of  $A$  consisting of functions  $f(z)$  which satisfy the coefficient inequalities (4.1), (4.3) and (4.4), respectively.

REMARK 6. It follows from the definitions of  $S_1^*(b)$ ,  $K_1(b)$  and  $C_1(b)$  that

$$\begin{aligned} \text{(i) } f(z) \in S_1^*(b) &\implies |a_n| \leq \frac{|b|}{n + |b| - 1} && (n \geq 2), \\ \text{(ii) } f(z) \in K_1(b) &\implies |a_n| \leq \frac{|b|}{n(n + |b| - 1)} && (n \geq 2), \\ \text{(iii) } f(z) \in C_1(b) &\implies |a_n| \leq \frac{|b|}{n} && (n \geq 2). \end{aligned}$$

Next, we prove

THEOREM 9. If a function  $f(z)$  is in the class  $S_1^*(b)$ , then

$$(4.5) \quad |z| - \frac{|b|}{1+|b|} |z|^2 \leq |f(z)| \leq |z| + \frac{|b|}{1+|b|} |z|^2$$

for  $z \in U$ . If  $0 < |b| \leq 1$ , then

$$(4.6) \quad 1 - \frac{2|b|}{1+|b|} |z| \leq |f'(z)| \leq 1 + \frac{2|b|}{1+|b|} |z|$$

for  $z \in U$ . The estimates in (4.5) and (4.6) are sharp.

PROOF. Noting that

$$(4.7) \quad \sum_{n=2}^{\infty} |a_n| \leq \frac{|b|}{1+|b|}$$

for  $f(z) \in S_1^*(b)$ , we have

$$(4.8) \quad |f(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} |a_n| \geq |z| - \frac{|b|}{1+|b|} |z|^2$$

and

$$(4.9) \quad |f(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} |a_n| \leq |z| + \frac{|b|}{1+|b|} |z|^2.$$

Further, if  $0 < |b| \leq 1$ , then we have

$$(4.10) \quad \frac{1+|b|}{2} \sum_{n=2}^{\infty} n|a_n| \leq \sum_{n=2}^{\infty} (n+|b|-1)|a_n| \leq |b|,$$

or

$$(4.11) \quad \sum_{n=2}^{\infty} n|a_n| \leq \frac{2|b|}{1+|b|}.$$

Thus we obtain

$$(4.12) \quad |f'(z)| \geq 1 - |z| \sum_{n=2}^{\infty} n|a_n| \geq 1 - \frac{2|b|}{1+|b|} |z|$$

and

$$(4.13) \quad |f'(z)| \leq 1 + |z| \sum_{n=2}^{\infty} n|a_n| \leq 1 + \frac{2|b|}{1+|b|} |z|.$$

Finally, taking the function

$$(4.14) \quad f(z) = z - \frac{|b|}{1+|b|} z^2,$$

we see that the estimates in (4.5) and (4.6) are sharp.

Using the same manner as in the proof of Theorem 9, we have

**THEOREM IO.** If a function  $f(z)$  is in the class  $K_1(b)$ , then

$$(4.15) \quad |z| - \frac{|b|}{2(1+|b|)} |z|^2 \leq |f(z)| \leq |z| + \frac{|b|}{2(1+|b|)} |z|^2$$

and

$$(4.16) \quad 1 - \frac{|b|}{1+|b|} |z| \leq |f'(z)| \leq 1 + \frac{|b|}{1+|b|} |z|$$

for  $z \in \mathbb{U}$ . The estimates in (4.15) and (4.16) are sharp for the function

$$(4.17) \quad f(z) = z - \frac{|b|}{2(1+|b|)} z^2.$$

**THEOREM II.** If a function  $f(z)$  is in the class  $C_1(b)$ , then

$$(4.18) \quad |z| - \frac{|b|}{2} |z|^2 \leq |f(z)| \leq |z| + \frac{|b|}{2} |z|^2$$

and

$$(4.19) \quad 1 - |b||z| \leq |f'(z)| \leq 1 + |b||z|$$

for  $z \in \mathbb{U}$ . The estimates in (4.8) and (4.9) are sharp for the function

$$(4.20) \quad f(z) = z - \frac{|b|}{2} z^2.$$

Finally, we give

CONJECTURE. Let a function  $f(z) \in A$  be defined by

$$(4.21) \quad f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0).$$

Then

(i)  $f(z) \in S_0^*(b)$  if and only if

$$\sum_{n=2}^{\infty} (n + |b| - 1) a_n \leq |b|.$$

(ii)  $f(z) \in K_0(b)$  if and only if

$$\sum_{n=2}^{\infty} n(n + |b| - 1) a_n \leq |b|.$$

(iii)  $f(z) \in C_0(b)$  with  $g(z) = z \in S_0^*(1)$  if and only if

$$\sum_{n=2}^{\infty} n a_n \leq |b|.$$

REMARK 7. The above conjectures (i), (ii) and (iii) are true for  $b = 1 - \alpha$  ( $0 \leq \alpha < 1$ ).

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