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On Quasi-Hadamard Product of Certain $p$-Valent Functions with Negative Coefficients

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1. INTRODUCTION

In [1], Kumar showed some results for the quasi-Hadamard product of certain univalent functions with negative coefficients.

In the present note, we show that Kumar's results [1] are generalized to the case of certain $p$-valent functions with negative coefficients.

Let $A(p)$ be the class of analytic and $p$-valent function $f(z)$ of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (a_{p+n} \geq 0, \ p \in \mathbb{N})$$

in the unit disk $U = \{ z : |z| < 1 \}$.

Let $T^*(p, \alpha)$ and $C(p, \alpha)$ denote the subclasses of $A(p)$ which satisfy $\Re\left( \frac{zf'(z)}{f(z)} \right) > \alpha$ and $\Re\left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha$, for $0 \leq \alpha < p$, respectively.

Clearly the function in $T^*(p, \alpha)$ and $C(p, \alpha)$ are $p$-valent starlike function and $p$-valent convex function of order $\alpha$, respectively.

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For these classes, Owa has obtained the following results in [2].

**Lemma 1.** A function $f(z)$ is in the class $T_p^\#(p, \alpha)$ if and only if

$$
\sum_{n=1}^{\infty} (p + n - \alpha) a_{p+n} \leq p - \alpha.
$$

The result is sharp.

**Lemma 2.** A function $f(z)$ is in the class $C(p, \alpha)$ if and only if

$$
\sum_{n=1}^{\infty} (p + n)(p + n - \alpha) a_{p+n} \leq p(p - \alpha).
$$

The result is sharp.

Let $A_0(p)$ denote the class of analytic and $p$-valent function $f(z)$ of the form

$$
f(z) = a_p z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (a_p > 0, a_{p+n} \geq 0, p \in \mathbb{N})
$$

in the unit disk $U$.

Furthermore, let $T_0^\#(p, \alpha)$ and $C_0(p, \alpha)$ be the subclasses of $A_0(p)$ as follows:

$$
T_0^\#(p, \alpha) = \left\{ f(z) \in A_0(p) : \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (0 \leq \alpha < p) \right\}
$$
and

\[ C_0(p, \alpha) = \left\{ f(z) \in A_0(p) : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \ (0 \leq \alpha < p) \right\}. \]

For these classes, by Lemma 1 and Lemma 2, we easily obtain the following theorems, respectively.

**THEOREM 1.** A function \( f(z) \) in the class \( T_0^*(p, \alpha) \) if and only if

\[ \sum_{n=1}^{\infty} (p + n - \alpha)a_{p+n} \leq (p - \alpha)a_p. \]

**THEOREM 2.** A function \( f(z) \) is in the class \( C_0(p, \alpha) \) if and only if

\[ \sum_{n=1}^{\infty} (p + n)(p + n - \alpha)a_{p+n} \leq p(p - \alpha)a_p. \]

We now introduce the subclass \( S_0(k, p, \alpha) \) of the class \( A_0(p) \) as follows.

A function \( f(z) \) belongs to the class \( S_0(k, p, \alpha) \) if and only if

\[ \sum_{n=1}^{\infty} \left(\frac{p + n}{p}\right)^k (p + n - \alpha)a_{p+n} \leq (p - \alpha)a_p, \]

where \( k \) is any real number.

Evidently, \( S_0(0, p, \alpha) = T_0^*(p, \alpha) \) and \( S_0(1, p, \alpha) = C_0(p, \alpha) \).

From now on, let the functions of the class \( A_0(p) \) be the following forms:
\[ f_1(z) = a_{p,i}z^p - \sum_{n=1}^{\infty} a_{p+n,i}z^{p+n} \quad (a_{p,i} > 0, a_{p+n,i} \geq 0) \]

and

\[ g_j(z) = b_{p,j}z^p - \sum_{n=1}^{\infty} b_{p+n,j}z^{p+n} \quad (b_{p,j} > 0, b_{p+n,j} \geq 0), \]

respectively.

Let us define the quasi-Hadamard product \( f_1 \ast g_j(z) \) of the functions \( f_1(z) \) and \( g_j(z) \) by

\[ f_1(z) \ast g_j(z) = a_{p,i}b_{p,j}z^p - \sum_{n=1}^{\infty} a_{p+n,i}b_{p+n,j}z^{p+n}. \]

2. RESULTS

Consequently, we have the following theorems. We can prove these theorems by using the same way as Kumar [1].

**THEOREM 4.** Let the functions \( f_1(z) \) belong to the classes \( T_0^*(p,a_i) \) for each \( i = 1, 2, 3, \ldots, m \), respectively. Then the quasi-Hadamard product \( f_1 \ast f_2 \ast f_3 \ast \cdots \ast f_m(z) \) belongs to the class \( S_0(m-1,p,\beta) \), where \( \beta = \max(a_1, a_2, a_3, \ldots, a_m) \).

**THEOREM 4.** Let the functions \( f_1(z) \) belong to the classes \( C_0(p,a_i) \) for each \( i = 1, 2, 3, \ldots, m \), respectively. Then the quasi-Hadamard product \( f_1 \ast f_2 \ast f_3 \ast \cdots \ast f_m(z) \) belongs to the class \( S_0(2m-1,p,\beta) \), where \( \beta = \max(a_1, a_2, a_3, \ldots, a_m) \).
THEOREM 5. Let the functions $f_1(z)$ belong to the classes $T_0^*(p,a_i)$ for each $i = 1, 2, 3, \ldots, m$ and for each $j = 1, 2, \ldots, q$, let the functions $g_j(z)$ belong to the classes $C_0(p, \beta_j)$, respectively. Then the quasi-Hadamard product

$$f_1 \ast f_2 \ast f_3 \ast \cdots \ast f_m \ast g_1 \ast g_2 \ast g_3 \ast \cdots \ast g_q(z)$$

belongs to the class $S_0(m+2q-1, p, \gamma)$, where $\gamma = \max\{a_1, a_2, a_3, \ldots, a_m, \beta_1, \beta_2, \beta_3, \ldots, \beta_q\}$.

THEOREM 6. Let the functions $f_i(z)$ belong to the class $C_0(p, \alpha)$ for each $i = 1, 2, 3, \ldots, m$ and let $0 \leq \alpha \leq r_0$, where $r_0$ is a root of the equation $(p + 1)^m(p - mr) - p(p - r)^m = 0$ in the interval $(0, \frac{p}{m})$. Then the quasi-Hadamard product

$$f_1 \ast f_2 \ast f_3 \ast \cdots \ast f_m(z)$$

belongs to the class $S_0(m-1, p, ma)$.

REMARK. If we put $p = 1$ in these theorems, we have the Kumar's results [1].

REFERENCES