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Termination for the Direct Sum of Left-Linear Term Rewriting Systems - Preliminary Draft*-

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1. Introduction

We prove the following conjecture [1]:

> *$R_0 \oplus R_1$ is left-linear and complete (complete = confluent + terminating) iff R_0 and R_1 are so.*

Note that $R_0 \oplus R_1$ is confluent iff R_0 and R_1 are so [3]. Clearly, the direct sum of two systems always preserves their left-linearity. It is trivial that if $R_0 \oplus R_1$ is terminating then R_0 and R_1 are so. Thus, in this paper, we shall prove the termination property of $R_0 \oplus R_1$, assuming that R_0 and R_1 are left-linear and complete.

2. Notations and Definitions

Assuming that the reader is familiar with the basic concepts and notations concerning term rewriting systems in [3], we briefly explain notations and definitions for the following discussions.

Let F be a set of function symbols, and let V be a set of variable symbols. By $T(F, V)$, we denote the set of terms constructed from F and V.

Consider disjoint systems R_0 on $T(F_0, V)$ and R_1 on $T(F_1, V)$. Then the direct sum system $R_0 \oplus R_1$ is the term rewriting system on $T(F_0 \cup F_1, V)$. From here on the notation $\rightarrow$ represents the reduction relation on $R_0 \oplus R_1$.

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Lemma 2.1. $R_0 \oplus R_1$ is weakly normalizing, i.e., every term M has a normal form (denoted by $M\downarrow$).

The identity of terms of $T(F_0 \cup F_1, V)$ (or syntactical equality) is denoted by $\equiv$. $\xrightarrow{*}$ is the transitive reflexive closure of $\rightarrow$, $\xrightarrow{+}$ is the transitive closure of $\rightarrow$, $\xrightarrow{\equiv}$ is the reflexive closure of $\rightarrow$, and $=$ is the equivalence relation generated by $\rightarrow$ (i.e., the transitive reflexive symmetric closure of $\rightarrow$). $\xrightarrow{m}$ denotes a reduction of m $(m \geq 0)$ steps.

Definition. A *root* is a mapping from $T(F_0 \cup F_1, V)$ to $F_0 \cup F_1 \cup V$ as follows: For $M \in T(F_0 \cup F_1, V)$,

$$root(M) = \begin{cases} f & \text{if } M \equiv f(M_1, \ldots, M_n), \\ M & \text{if } M \text{ is a constant or a variable.} \end{cases}$$

Definition. Let $M \equiv C[B_1, \ldots, B_n] \in T(F_0 \cup F_1, V)$ and $C \not\equiv \Box$. Then write $M \equiv C[\![B_1, \ldots, B_n]\!]$ if $C[\ ,\ldots,\ ]$ is a context on F_d and $\forall i, root(B_i) \in F_{\bar{d}}$ $(d \in \{0,1\}$ and $\bar{d} = 1 - d)$. Then the set $S(M)$ of the special subterms of M is inductively defined as follows:

$$S(M) = \begin{cases} \{M\} & \text{if } M \in T(F_d, V) \ \ (d = 0 \text{ or } 1), \\ \bigcup_i S(B_i) \cup \{M\} & \text{if } M \equiv C[\![B_1, \ldots, B_n]\!] \ (n > 0). \end{cases}$$

The set of the special subterms having the root symbol in F_d is denoted by $S_d(M) = \{N \mid N \in S(M) \text{ and } root(N) \in F_d\}$.

Let $M \equiv C[\![B_1, \ldots, B_n]\!]$ and $M \xrightarrow{A} N$ (i.e., N results from M by contracting the redex occurrence A). If the redex occurrence A occurs in some B_j, then we write $M \underset{i}{\rightarrow} N$; otherwise $M \underset{o}{\rightarrow} N$. Here, $\underset{i}{\rightarrow}$ and $\underset{o}{\rightarrow}$ are called an inner and an outer reduction, respectively.

Definition. For a term $M \in T(F_0 \cup F_1, V)$, the rank of layers of contexts on F_0 and F_1 in M is inductively defined as follows:

$$rank(M) = \begin{cases} 1 & \text{if } M \in T(F_d, V) \ \ (d = 0 \text{ or } 1), \\ max_i\{rank(B_i)\} + 1 & \text{if } M \equiv C[\![B_1, \ldots, B_n]\!] \ (n > 0). \end{cases}$$

Lemma 2.2. If $M \rightarrow N$ then $rank(M) \geq rank(N)$.

Lemma 2.3. Let $M \rightarrow N$ and $root(M), root(N) \in F_d$. Then there exists a reduction $M \equiv M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_n \equiv N$ $(n \geq 0)$ such that $root(M_i) \in F_d$ for any i.

The set of terms in the reduction graph of M is denoted by $G(M) = \{N \mid M \xrightarrow{*} N\}$. The set of terms having the root symbol in F_d is denoted by $G_d(M) = \{N \mid N \in G(M) \text{ and } root(N) \in F_d\}$.

Definition. A term M is persistent iff $G(M) = G_d(M)$ for some d.

Definition. A term M is erasable iff $M \xrightarrow{*} x$ for some $x \in V$.

From now on we assume that every term $M \in T(F_0 \cup F_1, V)$ has only x as variable occurrences, unless it is stated otherwise. Since $R_0 \oplus R_1$ is left-linear, this variable convention may be assumed in the following discussions without loss of generality. If we need fresh variable symbols not in terms, we use z, z_1, z_2, $\cdots$.

3. Essential Subterms

In this section we introduce the concept of the essential subterms. We first prove the following property:

$$\forall N \in G_d(M)\ \exists P \in S_d(M),\ M \xrightarrow{*} P \xrightarrow{*} N.$$

Lemma 3.1. Let $M \rightarrow N$ and $Q \in S_d(N)$. Then, there exists some $P \in S_d(M)$ such that $P \xrightarrow{\equiv} Q$.

R_e consists of the single rule $e(x) \rhd x$. $\underset{e}{\rightarrow}$ denotes the reduction relation of R_e, and $\underset{e'}{\rightarrow}$ denotes the reduction relation of $R_e \oplus (R_0 \oplus R_1)$ such that if $C[e(P)] \overset{\Delta}{\underset{e'}{\rightarrow}} N$ then the redex occurrence Δ does not occur in P. It is easy to show the confluence property of $\underset{e'}{\rightarrow}$.

Lemma 3.2. Let $C[e(P_1), \cdots, e(P_{i-1}), e(P_i), e(P_{i+1}), \cdots, e(P_p)] \overset{k}{\underset{e'}{\longrightarrow}} e(P_i)$. Then $C[P_1, \cdots, P_{i-1}, e(P_i), P_{i+1}, \cdots, P_p] \overset{k'}{\underset{e'}{\longrightarrow}} e(P_i)$ $(k' \leq k)$.

Let $M \equiv C[P] \in T(F_0 \cup F_1, V)$ be a term containing no function symbol e. Now, consider $C[e(P)]$ by replacing the occurrence P in M with $e(P)$. Assume $C[e(P)] \overset{*}{\underset{e'}{\rightarrow}} e(P)$. Then, by tracing the reduction path, we can also obtain the reduction $M \equiv C[P] \xrightarrow{*} P$ (denoted by $M \xrightarrow[pull]{*} P$) under $R_0 \oplus R_1$. We say that the reduction $M \xrightarrow[pull]{*} P$ pulls up the occurrence P from M.

Example 3.1. Consider the two systems R_0 and R_1:

$$R_0 \quad \begin{cases} F(x) \to G(x,x) \\ G(C,x) \to x \end{cases}$$

$$R_1 \quad \begin{cases} h(x) \to x \end{cases}$$

Then we have the reduction:
$F(e(h(C))) \xrightarrow[e']{} G(e(h(C), e(h(C))) \xrightarrow[e']{} G(h(C), e(h(C))) \xrightarrow[e']{} G(C, e(h(C))) \xrightarrow[e']{} e(h(C))$.
Hence $F(h(C)) \xrightarrow[pull]{*} h(C)$. However, we cannot obtain $F(z) \xrightarrow[pull]{*} z$. Thus, in generally, we cannot obtain $C[z] \xrightarrow[pull]{*} z$ from $C[P] \xrightarrow[pull]{*} P$. □

Lemma 3.3. Let $P \xrightarrow{*} Q$ and let $C[Q] \xrightarrow[pull]{*} Q$. Then $C[P] \xrightarrow[pull]{*} P$.

Lemma 3.4. $\forall N \in G_d(M)\ \exists P \in S_d(M),\ M \xrightarrow[pull]{*} P \xrightarrow{*} N$.

Now, we introduce the concept of the essential subterms. The set $E_d(M)$ of the essential subterms of the term $M \in T(F_0 \cup F_1, V)$ is defined as follows:
$E_d(M) = \{P \mid P \in G(M) \cap S_d(M) \text{ and } \neg\exists Q \in G(M) \cap S_d(M)\ [Q \xrightarrow{+} P]\}$.

The following lemmas are easily obtained from the definition of the essential subterms and Lemma 3.4.

Lemma 3.5. $\forall N \in G_d(M)\ \exists P \in E_d(M),\ P \xrightarrow{*} N$.

Lemma 3.6. $E_d(M) = \phi$ iff $G_d(M) = \phi$.

We say M is deterministic for d if $|E_d(M)| = 1$; M is nondeterministic for d if $|E_d(M)| \geq 2$. The following lemma plays an important role in the next section.

Lemma 3.7 If $root(M\downarrow) \in F_d$ then $|E_d(M)| = 1$, i.e., M is deterministic for d.

4. Termination for the Direct Sum

In this section we will show that $R_0 \oplus R_1$ is terminating. Roughly speaking, termination is proven by showing that any infinite reduction $M_0 \to M_1 \to M_2 \to \cdots$ of $R_0 \oplus R_1$ can be translated into an infinite reduction $M'_0 \to M'_1 \to M'_2 \to \cdots$ of R_d.

We first define the term $M^d \in T(F_d, V)$ for any term M and any d.

Definition. For any M and any d, $M^d \in T(F_d, V)$ is defined by induction on $rank(M)$:

(1) $M^d \equiv M$ if $M \in T(F_d, V)$.

(2) $M^d \equiv x$ if $E_d(M) = \phi$.

(3) $M^d \equiv C[M_1^d, \cdots, M_m^d]$ if $root(M) \in F_d$ and $M \equiv C[\![M_1, \cdots, M_m]\!]$ $(m > 0)$.

(4) $M^d \equiv P^d$ if $root(M) \in F_{\bar{d}}$ and $E_d(M) = \{P\}$. Note that $rank(P) < rank(M)$.

(5) $M^d \equiv C_1[C_2[\cdots C_{p-1}[C_p[x]]\cdots]]$ if $root(M) \in F_{\bar{d}}$, $E_d(M) = \{P_1, \cdots, P_p\}$ $(p > 1)$, and every P_i^d is erasable. Here $P_i^d \equiv C_i[x] \xrightarrow[pull]{*} x$ $(i = 1, \cdots, p)$. Note that $rank(P_i) < rank(M)$ for any i.

(6) $M^d \equiv x$ if $root(M) \in F_{\bar{d}}$, $|E_d(M)| \geq 2$, and not (5).

Note that M^d is not unique if a subterm of M^d is constructed with (5) in the above definition.

Lemma 4.1. $root(M\downarrow) \notin F_d$ iff $M^d\downarrow \equiv x$.

Note. Let $E_d(M) = \{P_1, \cdots, P_p\}$ $(p > 1)$. Then, from Lemma 3.6 and Lemma 4.1, it follows that every P_i is erasable. Hence case (6) can be removed from the definition of M^d.

Lemma 4.2. If $P \in E_d(M)$ then $M^d \xrightarrow{*} P^d$.

We wish to translate directly an infinite reduction $M_0 \to M_1 \to M_2 \to \cdots$ into an infinite reduction $M_0^d \xrightarrow{*} M_1^d \xrightarrow{*} M_2^d \xrightarrow{*} \cdots$. However, the following example shows that $M_i \to M_{i+1}$ cannot be translated into $M_i^d \xrightarrow{*} M_{i+1}^d$ in generally.

Example 4.1. Consider the two systems R_0 and R_1:

$$R_0 \quad \begin{cases} F(C,x) \to x \\ F(x,C) \to x \end{cases}$$

$$R_1 \quad \begin{cases} f(x) \to g(x) \\ f(x) \to h(x) \\ g(x) \to x \\ h(x) \to x \end{cases}$$

Let $M \equiv F(f(C), h(C)) \to N \equiv F(g(C), h(C))$. Then $E_1(M) = \{f(C)\}$ and $E_1(N) = \{g(C), h(C)\}$. Thus $M^1 \equiv f(x)$, $N^1 \equiv g(h(x))$. It is obvious that $M^1 \xrightarrow{*} N^1$ does not hold. □

Now we will consider to translate indirectly an infinite reduction of $R_0 \oplus R_1$ into an infinite reduction of R_d.

We write $M \underset{o}{\equiv} N$ when M and N have the same outermost-layer context, i.e., $M \equiv C[\![M_1, \cdots, M_m]\!]$ and $N \equiv C[\![N_1, \cdots, N_m]\!]$ for some M_i, N_i.

Lemma 4.3. Let $A \underset{i}{\xrightarrow{*}} M$, $M \underset{o}{\to} N$, $A \underset{o}{\equiv} M$, and $root(M), root(N) \in F_d$. Then, for any A^d there exist B and B^d such that

$$\begin{tikzcd}
M \arrow[r, "o"'] & N \\
A \arrow[u, "*", "i"'] \arrow[r, dashed, "o"'] & \exists B \arrow[u, dashed, "i", "*"'] \\
A^d \arrow[r, dashed] & \exists B^d
\end{tikzcd}$$

Proof. Let $A \equiv C[\![A_1, \cdots, A_m]\!]$, $M \equiv C[\![M_1, \cdots, M_m]\!]$, $N \equiv C'[\![M_{i_1}, \cdots, M_{i_n}]\!]$ ($i_j \in \{1, \cdots, m\}$). Take $B \equiv C'[\![A_{i_1}, \cdots, A_{i_n}]\!]$. Then, we can obtain $A \underset{o}{\rightarrow} B$ and $B \overset{*}{\underset{i}{\rightarrow}} N$. From $A^d \equiv C[A_1^d, \cdots, A_m^d]$ and $B^d \equiv C'[A_{i_1}^d, \cdots, A_{i_n}^d]$, it follows that $A^d \rightarrow B^d$. □

Lemma 4.4. Let $M \overset{*}{\rightarrow} N$, $root(N) \in F_d$. Then, for any M^d there exist A ($A \underset{o}{\equiv} N$) and A^d such that

$$
\begin{tikzcd}
M \arrow[rr, "*"] \arrow[rrd, dashed, "*"'] & & N \\
& & \exists A \; (A \underset{o}{\equiv} N) \arrow[u, dashed, "i", "*"'] \\
M^d \arrow[rr, dashed, "*"] & & \exists A^d
\end{tikzcd}
$$

Proof. We will prove the lemma by induction on $rank(M)$. The case $rank(M) = 1$ is trivial by taking $A \equiv N$. Assume the lemma for $rank(M) < k$. Then we will prove the case $rank(M) = k$. We start from the following claim.

Claim. The lemma holds if $M \xrightarrow[i]{*} N$.

Proof of the Claim. Let $M \equiv C[\![M_1, \cdots, M_m]\!] \xrightarrow[i]{*} N \equiv C[N_1, \cdots, N_m]$ where $M_i \xrightarrow{*} N_i$ for every i. We may assume that $N_1 \equiv x, \cdots, N_{p-1} \equiv x$, $root(N_i) \in F_d$ $(p \leq i \leq q-1)$, and $root(N_j) \in F_{\bar{d}}$ $(q \leq j \leq m)$ without loss of generality. Thus $N \equiv C[x, \cdots, x, N_p, \cdots, N_{q-1}, N_q, \cdots, N_m]$. Then, by using the induction hypothesis, every M_i $(p \leq i \leq q-1)$ has A_i $(A_i \underset{o}{\equiv} N_i)$ and A_i^d such that

$$
\begin{tikzcd}
M_i \arrow[rr, "*"] \arrow[drr, dashed, "*"] & & N_i \\
& & \exists A_i \ (A_i \underset{o}{\equiv} N_i) \arrow[u, dashed, "i", "*"'] \\
M_i^d \arrow[rr, dashed, "*"] & & \exists A_i^d
\end{tikzcd}
$$

Now, take $A \equiv C[x, \cdots, x, A_p, \cdots, A_{q-1}, M_q, \cdots, M_m]$. It is obvious that $M \xrightarrow{*} A$. From Lemma 2.3, we can have the reductions $A_i \xrightarrow{*} N_i$ $(p \leq i < q)$ and $M_j \xrightarrow{*} N_j$ $(q \leq j \leq m)$ in which every term has a root symbol in $F_{\bar{d}}$. Thus it follows that $A \underset{i}{\xrightarrow{*}} N$ and $A \underset{o}{\equiv} N$. From Lemma 4.1 and $M_i \downarrow\equiv x$ $(1 \leq i < p)$, $M_i^d \downarrow\equiv x$. Therefore, since

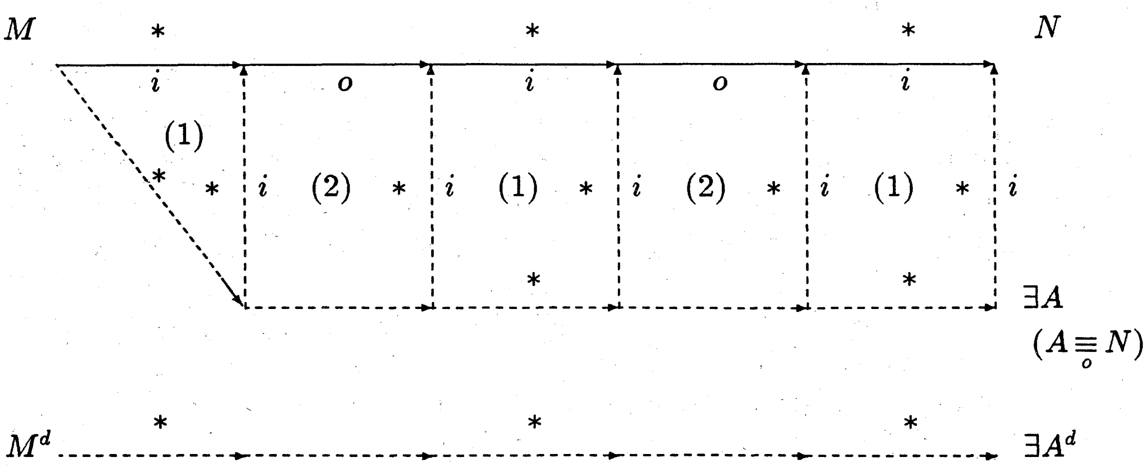
$M^d \equiv C[M_1^d, \cdots, M_{p-1}^d, M_p^d, \cdots, M_{q-1}^d, M_q^d, \cdots, M_m^d]$

and $A^d \equiv C[x, \cdots, x, A_p^d, \cdots, A_{q-1}^d, M_q^d, \cdots, M_m^d]$, it follows that $M^d \xrightarrow{*} A^d$. *(end of the claim)*

Now we will prove the lemma for $rank(M) = k$. Consider two cases.

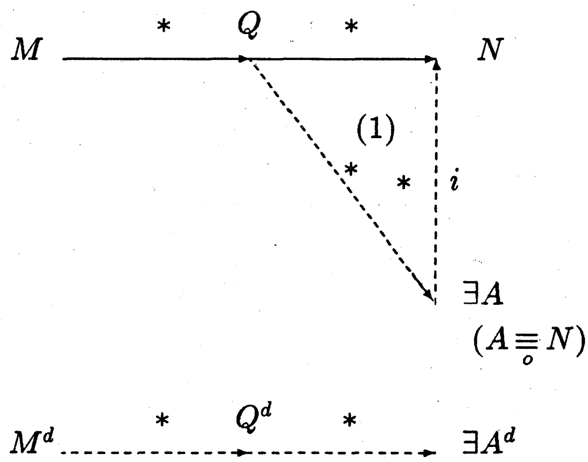
Case 1. $root(M) \in F_d$.

From Lemma 2.3, we may assume that every term in the reduction $M \xrightarrow{*} N$ has a root symbol in F_d. By splitting $M \xrightarrow{*} N$ into $M \underset{i}{\xrightarrow{*}} \underset{o}{\rightarrow} \underset{i}{\xrightarrow{*}} \underset{o}{\rightarrow} \cdots \underset{i}{\xrightarrow{*}} N$ and using the claim for diagram (1) and Lemma 5.1 for diagram (2), we can draw the following diagram:



Case 2. $root(M) \in F_{\bar{d}}$.

Then we have some essential subterm $Q \in E_d(M)$ such that $M \overset{*}{\to} Q \overset{*}{\to} N$. From Lemma 4.2, it follows that $M^d \overset{*}{\to} Q^d$. It is obvious that $rank(Q) < k$. Hence, we can show the following diagram, drawing diagram (1) by the induction hypothesis:



□

Now we can prove the following theorem:

Theorem 4.1. Every term M has no infinite reduction.

Proof. We will prove the theorem by induction on $rank(M)$. The case $rank(M) = 1$ is trivial. Assume the theorem for $rank(M) < k$. Then, we will show the case $rank(M) = k$. Suppose M has an infinite reduction $M \to\to\to \cdots$. From the induction hypothesis, we can have no infinite inner reduction $\underset{i}{\to}\underset{i}{\to}\underset{i}{\to}\cdots$ in this reduction. Thus, $\underset{o}{\to}$ must infinitely appear in the infinite reduction. From the induction hypothesis, all of the terms appearing in this reduction have the same rank; hence, their root symbols are in F_d if $root(M) \in F_d$. Hence, from the discussion for Case 1 in the proof of Lemma 4.4, it follows that M^d has an infinite reduction. This contradicts that R_d is terminating. □

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