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Termination for the Direct Sum of Left-Linear Term Rewriting Systems
- Preliminary Draft*

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1. Introduction

We prove the following conjecture [1]:

\[ R_0 \oplus R_1 \text{ is left-linear and complete (complete = confluent + terminating) iff } R_0 \text{ and } R_1 \text{ are so.} \]

Note that \( R_0 \oplus R_1 \) is confluent iff \( R_0 \) and \( R_1 \) are so [3]. Clearly, the direct sum of two systems always preserves their left-linearity. It is trivial that if \( R_0 \oplus R_1 \) is terminating then \( R_0 \) and \( R_1 \) are so. Thus, in this paper, we shall prove the termination property of \( R_0 \oplus R_1 \), assuming that \( R_0 \) and \( R_1 \) are left-linear and complete.

2. Notations and Definitions

Assuming that the reader is familiar with the basic concepts and notations concerning term rewriting systems in [3], we briefly explain notations and definitions for the following discussions.

Let \( F \) be a set of function symbols, and let \( V \) be a set of variable symbols. By \( T(F, V) \), we denote the set of terms constructed from \( F \) and \( V \).

Consider disjoint systems \( R_0 \) on \( T(F_0, V) \) and \( R_1 \) on \( T(F_1, V) \). Then the direct sum system \( R_0 \oplus R_1 \) is the term rewriting system on \( T(F_0 \cup F_1, V) \). From here on the notation \( \rightarrow \) represents the reduction relation on \( R_0 \oplus R_1 \).

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Lemma 2.1. $R_0 \oplus R_1$ is weakly normalizing, i.e., every term $M$ has a normal form (denoted by $M \downarrow$).

The identity of terms of $T(F_0 \cup F_1, V)$ (or syntactical equality) is denoted by $\equiv$. $\to^*$ is the transitive reflexive closure of $\to$, $\to^+$ is the transitive closure of $\to$, $\equiv$ is the reflexive closure of $\to$, and $=$ is the equivalence relation generated by $\to$ (i.e., the transitive reflexive symmetric closure of $\to$). $\xrightarrow{\to^m}$ denotes a reduction of $m$ ($m \geq 0$) steps.

**Definition.** A root is a mapping from $T(F_0 \cup F_1, V)$ to $F_0 \cup F_1 \cup V$ as follows: For $M \in T(F_0 \cup F_1, V)$,

$$\text{root}(M) = \begin{cases} f & \text{if } M \equiv f(M_1, \ldots, M_n), \\ M & \text{if } M \text{ is a constant or a variable.} \end{cases}$$

**Definition.** Let $M \equiv C[B_1, \ldots, B_n] \in T(F_0 \cup F_1, V)$ and $C \not\equiv \square$. Then write $M \equiv C[B_1, \ldots, B_n]$ if $C[\ldots]$ is a context on $F_d$ and $\forall i, \text{root}(B_i) \in F_{\overline{d}}$ ($d \in \{0,1\}$ and $\overline{d} = 1 - d$). Then the set $S(M)$ of the special subterms of $M$ is inductively defined as follows:

$$S(M) = \begin{cases} \{M\} & \text{if } M \in T(F_d, V) \ (d = 0 \text{ or } 1), \\ \bigcup_i S(B_i) \cup \{M\} & \text{if } M \equiv C[B_1, \ldots, B_n] \ (n > 0). \end{cases}$$

The set of the special subterms having the root symbol in $F_d$ is denoted by $S_d(M) = \{N \mid N \in S(M) \text{ and root}(N) \in F_d\}$.

Let $M \equiv C[B_1, \ldots, B_n]$ and $M \xrightarrow{A} N$ (i.e., $N$ results from $M$ by contracting the redex occurrence $A$). If the redex occurrence $A$ occurs in some $B_j$, then we write $M \xrightarrow{i} N$; otherwise $M \xrightarrow{o} N$. Here, $\xrightarrow{i}$ and $\xrightarrow{o}$ are called an inner and an outer reduction, respectively.

**Definition.** For a term $M \in T(F_0 \cup F_1, V)$, the rank of layers of contexts on $F_0$ and $F_1$ in $M$ is inductively defined as follows:

$$\text{rank}(M) = \begin{cases} 1 & \text{if } M \in T(F_d, V) \ (d = 0 \text{ or } 1), \\ \max_i \{\text{rank}(B_i)\} + 1 & \text{if } M \equiv C[B_1, \ldots, B_n] \ (n > 0). \end{cases}$$

**Lemma 2.2.** If $M \rightarrow N$ then $\text{rank}(M) \geq \text{rank}(N)$.

**Lemma 2.3.** Let $M \rightarrow N$ and $\text{root}(M), \text{root}(N) \in F_d$. Then there exists a reduction $M \equiv M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_n \equiv N \ (n \geq 0)$ such that $\text{root}(M_i) \in F_d$ for any $i$. 

2
The set of terms in the reduction graph of $M$ is denoted by $G(M) = \{N | M \Rightarrow N \}$. The set of terms having the root symbol in $F_d$ is denoted by $G_d(M) = \{N | N \in G(M) \text{ and } \text{root}(N) \in F_d \}$.

**Definition.** A term $M$ is persistent iff $G(M) = G_d(M)$ for some $d$.

**Definition.** A term $M$ is erasable iff $M \Rightarrow x$ for some $x \in V$.

From now on we assume that every term $M \in T(F_0 \cup F_1, V)$ has only $x$ as variable occurrences, unless it is stated otherwise. Since $R_0 \oplus R_1$ is left-linear, this variable convention may be assumed in the following discussions without loss of generality. If we need fresh variable symbols not in terms, we use $z, z_1, z_2, \cdots$.

### 3. Essential Subterms

In this section we introduce the concept of the essential subterms. We first prove the following property:

$$\forall N \in G_d(M) \exists P \in S_d(M), \ M \Rightarrow P \Rightarrow N.$$  

**Lemma 3.1.** Let $M \Rightarrow N$ and $Q \in S_d(N)$. Then, there exists some $P \in S_d(M)$ such that $P \equiv Q$.

$R_e$ consists of the single rule $e(x) \triangleright x$. $\Rightarrow_e$ denotes the reduction relation of $R_e$, and $\Rightarrow_{e'}$ denotes the reduction relation of $R_e \oplus (R_0 \oplus R_1)$ such that if $C[e(P)] \Rightarrow_{e'} N$ then the redex occurrence $\Delta$ does not occur in $P$. It is easy to show the confluence property of $\Rightarrow_{e'}$.

**Lemma 3.2.** Let $C[e(P_1), \cdots, e(P_{l-1}), e(P_l), e(P_{l+1}), \cdots, e(P_p)] \xrightarrow{k} e(P_l)$. Then $C[P_1, \cdots, P_{l-1}, e(P_l), P_{l+1}, \cdots, P_p] \xrightarrow{k'} e(P_l)$ ($k' \leq k$).

Let $M \equiv C[P] \in T(F_0 \cup F_1, V)$ be a term containing no function symbol $e$. Now, consider $C[e(P)]$ by replacing the occurrence $P$ in $M$ with $e(P)$. Assume $C[e(P)] \Rightarrow_{e'} e(P)$. Then, by tracing the reduction path, we can also obtain the
reduction $M \equiv C[P] \xrightarrow{\cdot} P$ (denoted by $M \xrightarrow{\cdot}_{\text{pull}} P$) under $R_0 \oplus R_1$. We say that the reduction $M \xrightarrow{\cdot}_{\text{pull}} P$ pulls up the occurrence $P$ from $M$.

**Example 3.1.** Consider the two systems $R_0$ and $R_1$:

$$
R_0 \begin{cases} 
F(x) \to G(x, x) \\
G(C, x) \to x 
\end{cases}
$$

$$
R_1 \begin{cases} 
h(x) \to x 
\end{cases}
$$

Then we have the reduction:

$F(e(h(C))) \to G(e(h(C)), e(h(C))) \to G(h(C), e(h(C))) \to G(C, e(h(C))) \to e(h(C))$.

Hence $F(h(C)) \xrightarrow{\cdot}_{\text{pull}} h(C)$. However, we cannot obtain $F(z) \xrightarrow{\cdot}_{\text{pull}} z$. Thus, in generally, we cannot obtain $C[z] \xrightarrow{\cdot}_{\text{pull}} z$ from $C[P] \xrightarrow{\cdot}_{\text{pull}} P$. □

**Lemma 3.3.** Let $P \xrightarrow{\cdot} Q$ and let $C[Q] \xrightarrow{\cdot}_{\text{pull}} Q$. Then $C[P] \xrightarrow{\cdot}_{\text{pull}} P$.

**Lemma 3.4.** $\forall N \in G_d(M) \exists P \in S_d(M)$, $M \xrightarrow{\cdot}_{\text{pull}} P \Rightarrow N$.

Now, we introduce the concept of the essential subterms. The set $E_d(M)$ of the essential subterms of the term $M \in T(F_0 \cup F_1, V)$ is defined as follows:

$E_d(M) = \{P \mid P \in G(M) \cap S_d(M) \text{ and } \neg \exists Q \in G(M) \cap S_d(M) [Q \xrightarrow{\cdot} P]\}$.

The following lemmas are easily obtained from the definition of the essential subterms and Lemma 3.4.

**Lemma 3.5.** $\forall N \in G_d(M) \exists P \in E_d(M)$, $P \xrightarrow{\cdot} N$.

**Lemma 3.6.** $E_d(M) = \phi$ iff $G_d(M) = \phi$.

We say $M$ is deterministic for $d$ if $|E_d(M)| = 1$; $M$ is nondeterministic for $d$ if $|E_d(M)| \geq 2$. The following lemma plays an important role in the next section.
Lemma 3.7 If $\text{root}(M \downarrow) \in F_d$ then $|E_d(M)| = 1$, i.e., $M$ is deterministic for $d$.

4. Termination for the Direct Sum

In this section we will show that $R_0 \oplus R_1$ is terminating. Roughly speaking, termination is proven by showing that any infinite reduction $M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots$ of $R_0 \oplus R_1$ can be translated into an infinite reduction $M'_0 \rightarrow M'_1 \rightarrow M'_2 \rightarrow \cdots$ of $R_d$.

We first define the term $M^d \in T(F_d, V)$ for any term $M$ and any $d$.

Definition. For any $M$ and any $d$, $M^d \in T(F_d, V)$ is defined by induction on $\text{rank}(M)$:

(1) $M^d \equiv M$ if $M \in T(F_d, V)$.
(2) $M^d \equiv x$ if $E_d(M) = \phi$.
(3) $M^d \equiv C[M_1^d, \cdots, M_m^d]$ if $\text{root}(M) \in F_d$ and $M \equiv C[M_1, \cdots, M_m]$ ($m > 0$).
(4) $M^d \equiv P^d$ if $\text{root}(M) \in F_d$ and $E_d(M) = \{P\}$. Note that $\text{rank}(P) < \text{rank}(M)$.
(5) $M^d \equiv C_1[C_2[\cdots C_{p-1}[C_p[x]] \cdots]]$ if $\text{root}(M) \in F_d$, $E_d(M) = \{P_1, \cdots, P_p\}$ ($p > 1$), and every $P_i^d$ is erasable. Here $P_i^d \equiv C_i[x] \xrightarrow{\text{pull}} x$ ($i = 1, \cdots, p$). Note that $\text{rank}(P_i) < \text{rank}(M)$ for any $i$.
(6) $M^d \equiv x$ if $\text{root}(M) \in F_d$, $|E_d(M)| \geq 2$, and not (5).

Note that $M^d$ is not unique if a subterm of $M^d$ is constructed with (5) in the above definition.

Lemma 4.1. $\text{root}(M \downarrow) \notin F_d$ iff $M^d \downarrow \equiv x$.

Note. Let $E_d(M) = \{P_1, \cdots, P_p\}$ ($p > 1$). Then, from Lemma 3.6 and Lemma 4.1, it follows that every $P_i$ is erasable. Hence case (6) can be removed from the definition of $M^d$. 

5
Lemma 4.2. If $P \in E_d(M)$ then $M^d \xrightarrow{*} P^d$.

We wish to translate directly an infinite reduction $M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots$ into an infinite reduction $M_0^d \xrightarrow{*} M_1^d \xrightarrow{*} M_2^d \xrightarrow{*} \cdots$. However, the following example shows that $M_i \rightarrow M_{i+1}$ cannot be translated into $M_i^d \xrightarrow{*} M_{i+1}^d$ in generally.

Example 4.1. Consider the two systems $R_0$ and $R_1$:

$R_0 \begin{cases} F(C, x) \rightarrow x \\ F(x, C) \rightarrow x \end{cases}$

$R_1 \begin{cases} f(x) \rightarrow g(x) \\ f(x) \rightarrow h(x) \\ g(x) \rightarrow x \\ h(x) \rightarrow x \end{cases}$

Let $M \equiv F(f(C), h(C)) \rightarrow N \equiv F(g(C), h(C))$. Then $E_1(M) = \{ f(C) \}$ and $E_1(N) = \{ g(C), h(C) \}$. Thus $M^1 \equiv f(x)$, $N^1 \equiv g(h(x))$. It is obvious that $M^1 \xrightarrow{*} N^1$ does not hold. □

Now we will consider to translate indirectly an infinite reduction of $R_0 \oplus R_1$ into an infinite reduction of $R_d$.

We write $M \equiv N$ when $M$ and $N$ have the same outermost-layer context, i.e., $M \equiv C[M_1, \cdots, M_m]$ and $N \equiv C[N_1, \cdots, N_m]$ for some $M_i$, $N_i$.

Lemma 4.3. Let $A \xrightarrow{*} M$, $M \xrightarrow{\circ} N$, $A \equiv M$, and $\text{root}(M), \text{root}(N) \in F_d$. Then, for any $A^d$ there exist $B$ and $B^d$ such that
Proof. Let $A \equiv C[A_1, \ldots, A_m]$, $M \equiv C[M_1, \ldots, M_m]$, $N \equiv C'[M_{i_1}, \ldots, M_{i_n}]$ ($i_j \in \{1, \ldots, m\}$). Take $B \equiv C'[A_{i_1}, \ldots, A_{i_n}]$. Then, we can obtain $A \rightarrow B$ and $B \rightarrow N$. From $A^d \equiv C[A_{i_1}^d, \ldots, A_m^d]$ and $B^d \equiv C'[A_{i_1}^d, \ldots, A_{i_n}^d]$, it follows that $A^d \rightarrow B^d$. □

Lemma 4.4. Let $M \rightarrow N$, root$(N) \in F_d$. Then, for any $M^d$ there exist $A$ ($A \equiv N$) and $A^d$ such that
Proof. We will prove the lemma by induction on \( \text{rank}(M) \). The case \( \text{rank}(M) = 1 \) is trivial by taking \( A \equiv N \). Assume the lemma for \( \text{rank}(M) < k \). Then we will prove the case \( \text{rank}(M) = k \). We start from the following claim.

Claim. The lemma holds if \( M \xrightarrow{\star} N \).

Proof of the Claim. Let \( M \equiv C[M_1, \cdots, M_m] \xrightarrow{\star} N \equiv C[N_1, \cdots, N_m] \) where \( M_i \xrightarrow{\star} N_i \) for every \( i \). We may assume that \( N_1 \equiv x, \cdots, N_{p-1} \equiv x \), root\((N_i) \in F_d \) \((p \leq i \leq q - 1)\), and root\((N_j) \in F_d \) \((q \leq j \leq m)\) without loss of generality. Thus \( N \equiv C[x, \cdots, x, N_p, \cdots, N_{q-1}, N_q, \cdots, N_m] \). Then, by using the induction hypothesis, every \( M_i \) \((p \leq i \leq q - 1)\) has \( A_i \) \((A_i \equiv N_i)\) and \( A_i^d \) such that
Now, take $A \equiv C[x, \cdots, x, A_p, \cdots, A_{q-1}, M_q, \cdots, M_m]$. It is obvious that $M \rightarrow A$. From Lemma 2.3, we can have the reductions $A_i \rightarrow N_i$ ($p \leq i < q$) and $M_j \rightarrow N_j$ ($q \leq j \leq m$) in which every term has a root symbol in $F_d$. Thus it follows that $A \rightarrow N$ and $A \equiv N$. From Lemma 4.1 and $M_i \vdash x$ $(1 \leq i < p)$, $M_i^d \vdash x$. Therefore, since

$$M^d \equiv C[M_1^d, \cdots, M_{p-1}^d, M_p^d, \cdots, M_{q-1}^d, M_q^d, \cdots, M_m^d]$$

and $A^d \equiv C[x, \cdots, x, A_p^d, \cdots, A_{q-1}^d, M_q^d, \cdots, M_m^d]$, it follows that $M^d \rightarrow A^d$. (end of the claim)

Now we will prove the lemma for $\text{rank}(M) = k$. Consider two cases.

Case 1. $\text{root}(M) \in F_d$.

From Lemma 2.3, we may assume that every term in the reduction $M \rightarrow N$ has a root symbol in $F_d$. By splitting $M \rightarrow N$ into $M \rightarrow \cdots \rightarrow N$ and using the claim for diagram (1) and Lemma 5.1 for diagram (2), we can draw the following diagram:
Case 2. \( \text{root}(M) \in F_d \).

Then we have some essential subterm \( Q \in E_d(M) \) such that \( M \Rightarrow Q \Rightarrow N \). From Lemma 4.2, it follows that \( M^d \Rightarrow Q^d \). It is obvious that \( \text{rank}(Q) < k \). Hence, we can show the following diagram, drawing diagram (1) by the induction hypothesis:

Now we can prove the following theorem:
Theorem 4.1. Every term $M$ has no infinite reduction.

Proof. We will prove the theorem by induction on $\text{rank}(M)$. The case $\text{rank}(M) = 1$ is trivial. Assume the theorem for $\text{rank}(M) < k$. Then, we will show the case $\text{rank}(M) = k$. Suppose $M$ has an infinite reduction $M \rightarrow \rightarrow \cdots$. From the induction hypothesis, we can have no infinite inner reduction $\rightarrow \rightarrow \rightarrow \cdots$ in this reduction. Thus, $\rightarrow$ must infinitely appear in the infinite reduction. From the induction hypothesis, all of the terms appearing in this reduction have the same rank; hence, their root symbols are in $F_d$ if $\text{root}(M) \in F_d$. Hence, from the discussion for Case 1 in the proof of Lemma 4.4, it follows that $M^d$ has an infinite reduction. This contradicts that $R_d$ is terminating. $\square$

References