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Extension Theorem of coherent $\underline{\underline{E}}_X$-Modules due to Gabber

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§1. Introduction

The purpose of this note is to prove the following unpublished result due to O. Gabber.

Theorem 1.1 (O. Gabber) Let X be a complex manifold, T^*X the cotangent bundle over X, Ω an open subset of T^*X - {zero section} and Y a conic closed analytic subset Ω with the following condition:

Condition. For any $p \in Y$, there exist a neighbourhood U of p in Ω and $f, g \in \Gamma(U, \underline{\underline{O}}_{T^*X})$ such that $f|Y\cap U = 0$, $g|Y\cap U = 0$ and $\{f, g\}(p) = 0$. (Here $\{f, g\}$ means the Poisson bracket.)

Let $\underline{\underline{M}}$ be a coherent $\underline{\underline{E}}_X$-Module on U and $\underline{\underline{N}}$ a coherent sub-$\underline{\underline{E}}_X(0)$-Module of $\underline{\underline{M}}$. Put $\underline{\underline{N}}' = \{u \in \underline{\underline{M}};\ u|\Omega-Y \in \underline{\underline{N}}|\Omega-Y\}$. Then $\underline{\underline{N}}'$ is also coherent over $\underline{\underline{E}}_X(0)$.

This theorem is a generalization of the fundamental theorem of Sato-Kawai-Kashiwara concerning the involutivity of the characteristic varieties of coherent $\underline{\underline{D}}_X$-Modules (cf. [SKK]). Explain the connection shortly.

Let Ω be the one in Theorem 1.1 and let V be a conic involutive analytic subset of T^*X contained in Ω. Define

$$\underline{\underline{J}}_V = \{ P \in \underline{\underline{E}}_X(1)|\Omega \ ; \ \sigma_1(P)|V = 0 \}$$

and put $\underline{\underline{\mathcal{E}}}_V = \bigcup_{k=0}^{\infty} \underline{\underline{J}}_V^{\,k}$. (Here $\underline{\underline{J}}_V^{\,0} = \underline{\underline{E}}_X(0)$.) Now recall the definition of "regular singularity".

<u>Definition</u>. A coherent $\underline{\underline{E}}_X$-Module $\underline{\underline{M}}$ on Ω has regular singularities (= RS) along V if for any $p \in \Omega$, there exist a neighbourhood U of p in Ω and a coherent $\underline{\underline{E}}_X(0)$-Module $\underline{\underline{L}}$ on U such that $\underline{\underline{M}} = \underline{\underline{E}}_X\underline{\underline{L}}$ and $\underline{\underline{J}}_V\underline{\underline{L}} = \underline{\underline{L}}$.

In general, for a coherent $\underline{\underline{E}}_X$-Module $\underline{\underline{M}}$ on Ω, define

$$IR(\underline{\underline{M}},V) = \{ p \in \Omega ; \ \underline{\underline{M}} \text{ does not have RS along } V \text{ in a neighbourhood } U \text{ of } p \text{ in } \Omega \}.$$

<u>Theorem 1.2</u>. Let Ω, V be as above and $\underline{\underline{M}}$ a coherent $\underline{\underline{E}}_X$-Module on Ω. Then $IR(\underline{\underline{M}},V)$ is an involutive closed analytic subset of Ω and is contained in $\mathrm{Supp}\ \underline{\underline{M}}$.

<u>Sketch of Th. 1.1 $\Rightarrow$ Th. 1.2</u>. It suffices to show the involutivity of $IR(\underline{\underline{M}},V)$. For this purpose, put $Z = IR(\underline{\underline{M}},V)$. Changing Ω with a smaller one, we may assume the following from the first.

(a) $Z \neq 0$.

(b) $\exists\ f,\ g \in \Gamma(\Omega, \underline{\underline{O}}_{T^*X}(0))$ s.t.

$$f|Z = g|Z = 0, \quad \{f, g\}(p) \neq 0.$$

(c) $\exists\ \underline{L}$: a coherent $\underline{\underline{E}}_X(0)$-Module on Ω s.t.

$$\underline{\underline{E}}_X\underline{L} = \underline{M}, \quad Z = \mathrm{Supp}(\underline{\underline{J}}_V\underline{L}/\underline{L}).$$

Put $\underline{L}' = \{u \in \underline{M};\ u|\Omega-Z \in \underline{L}|\Omega-Z\}$. Then, applying Theorem 1.1, we conclude that $\underline{L}'$ is coherent over $\underline{\underline{E}}_X(0)$. Since $\underline{L} \subset \underline{L}'$, the condition (c) implies that $\underline{\underline{E}}_X\underline{L}' = \underline{M}$. On the other hand, by definition, $\underline{\underline{J}}_V\underline{L}' = \underline{L}'$. Then $\underline{M}$ has RS along V and we find that $IR(\underline{M}, V) = \emptyset$. This contradicts the assumption and Theorem 1.2 follows.

Consider the case $V = \emptyset$ in Theorem 1.2. Then $IR(\underline{M}, V) =$ Supp $\underline{M}$ and therefore Supp $\underline{M}$ is involutive. This statement is nothing but the contents of the fundamental theorem of Sato-Kawai-Kashiwara mentioned above.

Theorem 1.1 has an important consequence which we are going to mention (see [KK1, 2] [N]).

<u>Theorem 1.3</u>. Let $\underline{M}$ be a holonomic $\underline{\underline{E}}_X$-Module on Ω and put $\Lambda = \mathrm{Supp}\ \underline{M}$. Then the following conditions (i)-(iv) are mutually equivalent.

(i) $\underline{M}$ has RS along Λ.

(ii) $\underline{M}$ has RS along any conic involutive analytic subset V containing Λ.

(iii) $\underline{M}$ has RS along some conic Lagrangian analytic subset Λ' containing Λ.

(iv) $\underline{M}$ has RS along Λ in an open neighbourhood of a dense open subset of Λ.

To prove Theorem 1.1, we need some preparation on a kind of "non-commutative" algebraic geometry which will be developed in the subsequent sections and therefore the proof of Theorem 1 will be postponed until §4.

Closing this introduction, I give some comments on this note. The outline of the proof of Gabber's theorem was lectured by M. Kashiwara at RIMS in 1981. S. Ishiura, M. Noumi, T. Yano and I learned the contents of the proof based on M. Saito's notes of Kashiwara's lecture. I believe that this note is not comleted without their help. I thank them for their efforts and kindness.

§2. D-rings and MD-rings.

This section is devoted to the preparation to the proof of Theorem 1.1 of §1 which will be done in §4. Hence we first introduce the notion of D-rings and MD-rings which are obtained by abstracting some basic properties of the stalks of the sheaves $\underline{\underline{D}}_X$ and $\underline{\underline{E}}_X$ and next prove some basic properties of them which will be needed in the next section. For example, the ring of left fractions, the homogeneous spectrum, the structure sheaf on the homogeneous spectrum of a D-ring are discussed. Most results of this section are well-known when the ring in question is commutative and therefore the results of this section are in some sense familiar to the experts for the commutative algebra. The only thing we must take care of is the difficulty arised from the non-commutativity of a D-ring.

2.1. The definition of D-ring and MD-ring.

Definition 2.1. Let A be a (not necessarily commutative) ring. Assume that A has a filtration $\{A(n)\}_{n\in\mathbb{Z}}$. Namely $\{A(n)\}_{n\in\mathbb{Z}}$ is a family of sub-$\mathbb{Z}$-modules of A satisfying the following conditions:

$$A(n) \subseteq A(n+1), \quad A = \bigcup_{n\in\mathbb{Z}} A(n), \quad A(m)A(n) \subseteq A(m+n).$$

Then $(A, \{A(n)\}_{n\in\mathbb{Z}})$ or simply A is called a D-ring if the following conditions (i)-(iv) hold for $\{A(n)\}_{n\in\mathbb{Z}}$:

(i) $[A(m), A(n)] \subset A(m+n-1)$ for any $m, n \in \mathbb{Z}$.

(ii) $A(0)$ contains 1 and is left and right Noetherian.

(iii) $gr(A) = \bigoplus_{n\in\mathbb{Z}} gr_n(A)$ $(gr_n(A) = A(n)/A(n-1))$ is a finitely generated $gr_0(A)$-algebra.

(iv) $A_{\langle T\rangle} = \bigoplus_{n\in\mathbb{Z}} A(n)T^n$ is also a left and right Noetherian ring. (Here, T is an indeterminate commuting with each element of A.)

Lemma 2.2. Let A be a D-ring. Then the following statements hold.

(i) A is left and right Noetherian.

(ii) $gr(A)$ is a commutative Noetherian ring.

This lemma is clear from the definition.

Remark 2.3. In the definition of the D-ring, the condition (iv) does not follow from (i)-(iii). In fact, we can construct a ring A which satisfies (i)-(iii) but does not (iv). For example, the following is such a ring.

Let $A = \mathbb{C}[[x]]$ be the ring of formal power series of x and let $\underline{m} = Ax$. Define

$$A(n) = \begin{cases} A & (n \geq 0) \\ \underline{m} & (n < 0). \end{cases}$$

In this case, $\{A(n)\}$ satisfies the conditions (i)-(iii) but $A_{\langle T\rangle}$ is not Noetherian. In fact, $J = \sum_{n=1}^{\infty} A_{\langle T\rangle}xT^{-n}$ is an ideal of $A_{\langle T\rangle}$ but is not finitely generated over $A_{\langle T\rangle}$.

In this section, we will develop "non-commutative algebraic geometry" for a D-ring. In the last part of this section and also in the next section, we restrict our attention to such a ring called an MD-ring which satisfies the conditions stronger than those contained in the definition of a D-ring. For this reason, we now give the definition of an MD-ring.

Definition 2.4. Let A be a (not necessarily commutative) ring with the filtration $\{A(n)\}$. Then $(A, \{A(n)\})$ or simply A is an MD-ring if $\{A(n)\}$ satisfies the the cnditions (i)-(v) below:

(i) $[A(m), A(n)] \subseteq A(m+n-1)$ for any $m, n \in \mathbb{Z}$.

(ii) $gr(A)$ is a finitely generated $gr_0(A)$-module.

(iii) $A(0)$ contains $\mathbb{Q}$ and is left and right Noertherian.

(iv) There exist $u \in A(-1)$ and $v \in A(1)$ such that $uv = vu = 1$.

(v) For any $a \in A(-1)$, $1+a$ is invertible in $A(0)$, that is, there exists an element $b \in A(0)$ such that $(1+a)b = b(1+a) = 1$.

It is not clear whether if $(A, \{A(n)\})$ is an MD-ring, then $(A, \{A(n)\})$ is a D-ring or not. But this is actually true. Namely, the following lemma holds.

Lemma 2.5. Every MD-ring is a D-ring.

Proof. Let $(A, \{A(n)\})$ be an MD-ring. To prove the lemma, it suffices to show that $A_{\langle T\rangle} = \bigoplus_{n\in\mathbb{Z}} A(n)T^n$ is left Noetherian and right Noetherian.

Now show that $A_{\langle T\rangle}$ is left Noetherian. Let $\{I_\nu ; \nu = 1, 2, \ldots\}$ be an increasing sequence of left ideals of $A_{\langle T\rangle}$. We are going to prove that there exists an integer $\nu_0 > 0$

such that $I_\nu = I_{\nu_0}$ for any $\nu \geq \nu_0$. By definition, I_ν is expressed of the form $\bigoplus_{n\in\mathbb{Z}} I_\nu(n)T^n$, where each $I_\nu(n)$ is contained in $A(n)$ and is an $A(0)$-module for any ν and n. Then, by the assumption, $I_\nu(n) \subset I_{\nu+1}(n)$ (ν, n). Fix $u \in A(-1)$ and $v \in A(1)$ such that $uv = vu = 1$. Now define $J(\nu,n) = v^n I_\nu(-n)$ if $n > 0$. Then it follows that each $J(\nu,n)$ is a left ideal of $A(0)$ and that $I_\nu(-n) = u^n J(\nu,n)$. On the other hand, it is easy to see that

$$J(\nu,n) \subset J(\nu+1,n), \quad J(\nu,n) \subset J(\nu,n+1) \quad \text{for any } \nu,\ n > 0.$$

Since $A(0)$ is left Noetherian, these imply the existence of integers ν_0, $n_0 > 0$ such that $J(\nu,n) = J(\nu_0,n_0)$ $(\nu \geq \nu_0,\ n \geq n_0)$. Put $J = J(\nu_0,n_0)$. Then we find that $I_\nu(-n) = u^n J$ $(\nu \geq \nu_0,$ $n \geq n_0)$. By the same reason, there exists a left ideal J' of $A(0)$ such that $I_\nu(n) = v^n J'$ $(\nu \geq \nu_0,\ n \geq n_0)$. Here we change the integers ν_0 and n_0 by the greater ones if necessary. At any rate, we may assume that there exist positive integers ν_0, n_0 and left ideals J, J' of $A(0)$ such that $I_\nu(n) = v^n J'$, $I_\nu(-n) = u^n J$ if $\nu \geq \nu_0$, $n \geq n_0$. Fix an integer n $(|n| < n_0)$. Since if $n > 0$ (resp. $n = 0$, $n < 0$), then $\{u^n I_\nu(n);\ \nu = 1, 2, \ldots\}$ (resp. $\{I_\nu(n);\ \nu = 1, 2, \ldots\}$, $\{v^{-n} I_\nu(-n);\ \nu = 1, 2, \ldots\}$) is an increasing sequence of left ideals of $A(0)$ and since $A(0)$ is left Noetherian, the sequence is stationary. Therefore we conclude that the increasing sequence $\{I_\nu\}$ of left ideals of $A_{\langle T\rangle}$ is also stationary. This means that $A_{\langle T\rangle}$ is left Noetherian. By the same way, we can show that $A_{\langle T\rangle}$ is right Noetherian. q.e.d.

Let $(A, \{A(n)\})$ be a D-ring. For each $n \in \mathbb{Z}$, we denote by σ_n the natural surjection of $A(n)$ to $gr_n(A)$. By definition, $\mathrm{Ker}\, \sigma_n = A(n-1)$. If $a \in A(n) \setminus A(n-1)$, then n is called the order of a and is denoted by ord a. On the other hand, if $a \in \bigcap_n A(n)$, then the order of a is $-\infty$. Now define a surjection σ of A to $gr(A)$ by $\sigma(a) = \sigma_n(a)$ if ord $a = n$. It is clear from the definition that $\sigma(ab) = \sigma(a)\sigma(b)$ for any $a, b \in A$.

An element of $gr(A)$ is homogeneous if it is contained in $gr_m(A)$ for some $m \in \mathbb{Z}$ and in this case m is called the degree of it. An ideal of $gr(A)$ is called homogeneous if it is generated by homogeneous elements of $gr(A)$. It is known that if I is a homogeneous ideal of $gr(A)$, then $I = \bigoplus_m (I \cap gr_m(A))$.

If $a \in A(m)$ and $b \in A(n)$, then define $\{\sigma_m(a), \sigma_n(b)\} = \sigma_{m+n-1}([a, b])$. Since σ_n is surjective for each $n \in \mathbb{Z}$, $\{f, g\}$ is well-defined for any $f \in gr_m(A)$, $g \in gr_n(A)$. Similarly as in the case of $\underline{\underline{E}}_X$, $\{\ ,\ \}$ is called the Poisson bracket on $gr(A)$. The Poisson bracket $\{\ ,\ \}$ extends to a $\mathbb{Z}$-bilinear map of $gr(A) \times gr(A)$ to $gr(A)$ and has the following properties:

(2.1.1) $\{f, f\} = 0$ for any $f \in gr(A)$.

(2.1.2) $\{f, gh\} = \{f, g\}h + g\{f, h\}$ for any $f, g, h \in gr(A)$.

(2.1.3) $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$.

Let $(B, \{B(n)\})$ be another D-ring and let ϕ be an algebra homomorphism of A to B satisfying that $\phi(A(n)) \subseteq B(n)$ $(\forall n \in \mathbb{Z})$ and that $\phi(1) = 1$. Let $\{\ ,\ \}_A$ and $\{\ ,\ \}_B$ be the Poisson brackets on A and B, respectively. Let $\bar{\phi}$ denote the algebra homomorphism of gr(A) to gr(B) induced from ϕ. Then it follows that $\bar{\Phi}(\{f, g\}_A) = \{\phi(f), \phi(g)\}_B$ for any $f, g \in gr(A)$.

We give here some elementary properties of a D-ring.

Lemma 2.6. Let A be a D-ring. Assume that each element of $1 + A(-1)$ is invertible in $A(0)$. If $a \in A$ is such that $\sigma(a)$ is invertible, then a is invertible, that is, there exists a $b \in A$ such that $ab = ba = 1$.

Proof. Assume that $\mathrm{ord}\, a = k$. Since $\sigma_k(a)$ is homogeneous of degree k, the inverse f of $\sigma_k(a)$ is homogeneous of degree $-k$. Take $b_1 \in A(-k)$ such that $\sigma_{-k}(b_1) = f$. This implies that $ab_1 \in A(0)$ and $\sigma_0(ab_1) = 1$. Then $ab_1 = 1+c$ for some $c \in A(-1)$. From the assumption, $1+c$ is invertible and therefore $b = b_1(1+c)^{-1}$ is a right inverse of a. Similarly, there exists a left inverse b' of a. Then $b' = b'(ab) = (b'a)b = b$. q.e.d.

Lemma 2.7. Let A be a D-ring and assume the condition in Lemma 2.6. Let N be a finitely generated left A(0)-module. If $A(-1)N = N$, then $N = 0$.

Proof. From the assumption, $A(-1)$ is contained in the Jacobson radical $\underline{m}$ of $A(0)$. Hence, if $A(-1)N = N$, we have $\underline{m}N = N$. Then Nakayama's lemma implies that $N = 0$. q.e.d.

We are going to give some examples of D-rings.

Example 2.8. Stalks of the sheaf $\underline{\underline{D}}_X$.

Let X be a complex manifold and let $\underline{\underline{D}}_X$ be the sheaf of differential operators on X. As in the provious section, put $D_X(m) = \{P \in \underline{\underline{D}}_X;\ \mathrm{ord}P \leq m\}$ for each $m \in \mathbb{N}$. Now take $x \in X$ and fix it once for all. Then the stalk $\underline{\underline{D}}_{X,x}$ is a D-ring with the filtration $\{A(m)\}_{m\in Z}$, where $A(m) = \underline{\underline{D}}_X(m)_x$ $(m \geq 0)$ and $A(m) = 0$ $(m < 0)$. In this case, $gr(A) = A(0)[\xi_1,\ldots,\xi_n]$ and $A(0)$ is the stalk $\underline{\underline{O}}_{X,x}$.

Example 2.9. Stalks of the sheaf $\underline{\underline{E}}_X$.

Let X be a complex manifold and let T^*X be the cotangent bundle over X. Let $\underline{\underline{E}}_X$ be the sheaf of microdifferential operators on T^*X. As in the previous section, put $\underline{\underline{E}}_X(m) = \{P \in \underline{\underline{E}}_X;\ \mathrm{ord}P \leq m\}$ for each $m \in \mathbb{Z}$. Take $p \in \mathring{T}^*X$ and fix it once for all. Now put $A = \underline{\underline{E}}_{X,p}$ and $A(m) = \underline{\underline{E}}_X(m)_p$ for any $m \in Z$. Then it follows from Theorem 1.10 that $(A, \{A(m)\})$ is an MD-ring. In this case, $gr(A) = gr_0(A)[\zeta,\zeta^{-1}]$ and $gr_0(A)$ is isomorphic to the ring of convergent power series of (2n-1)-variables and $gr_m(A) \simeq gr_0(A)\zeta^m$ for every $m \in \mathbb{Z}$.

Example 2.10. The universal enveloping algebra of a Lie algebra.

Let $\mathfrak{g}$ be a Lie algebra defined over $\mathbb{C}$ and let $U(\mathfrak{g})$ be the universal enveloping algebra over $\mathfrak{g}$. Let $U_k(\mathfrak{g})$ be the linear subspace of $U(\mathfrak{g})$ spanned by the elements of $U(\mathfrak{g})$ whose degrees are $\leq k$. Put $A = U(\mathfrak{g})$, $A(m) = U_m(\mathfrak{g})$ if $m \geq 0$ and $A(m) = 0$ if $m < 0$. Then it is clear that A is a D-ring with the filtration $\{A(m)\}$. In this case $gr(A)$ coincides with the symmetric algebra over $\mathfrak{g}$.

Example 2.11. The Weyl algebra $\mathbb{C}[x_1,\dots,x_n,D_{x_1},\dots,D_{x_n}]$.

Let us consider the polynomial ring $\mathbb{C}[x_1,\dots,x_n]$ of n-variables $x_1,\dots x_n$ and put $D_{x_i} = \frac{\partial}{\partial x_i}$ $(i = 1,\dots,n)$. Then the algebra A generated by $D_{x_1},\dots,D_{x_n}$ over $\mathbb{C}[x_1,\dots,x_n]$ is called the Weyl algebra. Put $A(m) = 0$ if $m < 0$ and $A(0) = \mathbb{C}[x_1,\dots x_n]$. If $m > 0$, then $A(m)$ is inductively defined as follows: $A(m) = A(m-1) + \sum_{k=1}^{n} A(m-1)D_{x_k}$. Then A is a D-ring with the filtration $\{A(m)\}$. In this case, $gr(A) = \mathbb{C}[x_1,\dots,x_n,\xi_1,\dots,\xi_n]$.

2.2. The homogeneous spectrum of gr(A).

In what follows, $(A, \{A(n)\})$ is a D-ring unless otherwise stated.

Definition 2.12. The homogeneous spectrum of $gr(A)$ is the set of all homogeneous prime ideals of $gr(A)$ and is denoted by $Spech(gr(A))$.

Remark 2.13. $Spech(gr(A))$ is analogous to $Proj(S)$ for a graded ring S with a graduation $\{S_m\}_{m\geq 0}$. But there exist some differences between $Spech(gr(A))$ and $Proj\ S$. The most crucial one is this: $gr(A)$ has negative grades in spite that $S_m = 0$ $(m < 0)$.

In the sequel, $Spech(gr(A))$ is denoted by X unless otherwise stated. In order to introduce a topology on X, we first give a lemma. For its proof, see [ ].

Lemma 2.14. For each homogeneous ideal I of $gr(A)$, put $\sqrt{I} = \{f \in gr(A);\ f^n \in I$ for some $n > 0\}$. ($\sqrt{I}$ is called the radical of I.) Then

$$\sqrt{I} = \bigcap_{\underline{p} \in X,\ \underline{p} \supset I} \underline{p}.$$

In particular, $\sqrt{I}$ is also a homogeneous ideal.

For any homogeneous ideal I of $gr(A)$, define $V(I) = \{ \underline{p} \in X;\ \underline{p} \supseteq I \}$ and for any homogeneous element f of $gr(A)$, define $V(f) = V(gr(A)f)$. Then the following properties are easily proved:

(2.2.1.i) $V(0) = X$, $V(1) = \emptyset$.

(2.2.1.ii) Let I and I' be homogeneous ideals of $gr(A)$.

$$\sqrt{I} \subseteq \sqrt{I'} \Longleftrightarrow V(I') \subseteq V(I).$$

$$V(I \cap I') = V(II') = V(I) \cup V(I').$$

(2.2.1.iii) If $\{I_\lambda;\ \lambda \in \Lambda\}$ is a set of homogeneous ideals of $gr(A)$, then $V(\sum_{\lambda \in \Lambda} I_\lambda) = \bigcap_{\lambda \in \Lambda} V(I_\lambda)$.

Now we define a topology on X by taking the subsets of the form $V(I)$ to be the closed subsets. For each homogeneous element f of $gr(A)$, we define $D(f) = \{ \underline{p} \in X;\ f \notin \underline{p} \} = X - V(f)$. Then we find the following:

(2.2.1.iv) $\{ D(f);\ f \in gr(A)$, homogeneous$\}$ forms a basis of open subsets of X.

(2.2.1.v) For each homogeneous element $f \in gr(A)$, $D(f)$ is quasi-compact.

(2.2.1.vi) $D(0) = \emptyset$, $D(1) = X$,

$D(fg) = D(f) \cap D(g)$.

(2.2.1.vii) For any subset Y of X, we define $I(Y) = \bigcap_{\underline{p} \in Y} \underline{p}$.

Then $I(V(I)) = \sqrt{I}$ for each homogeneous ideal I of $gr(A)$.

2.3. Good filtrations on finitely generated left A-modules.

Let M be a finitely generated left A-module. In this subsection, we introduce the notion of a "good filtration" on M. This is used in the definition of the chacteristic variety $Ch(M)$ of M which will be done in the next subsection.

First we recall the definition of filtrations on a left A-module. Let M be a left A-module. Then the sequence $\{M(n)\}_{n \in \mathbb{Z}}$ of sub-$\mathbb{Z}$-modules of M is called a filtration on M if the following conditions hold for $\{M(n)\}$:

(2.3.1) $M(n) \subseteq M(n+1)$, $M = \bigcup_n M(n)$, $A(m)M(n) \subseteq M(m+n)$.

Let N be a sub-A-module of M. Then, putting $N(n) = N \cap M(n)$ ($n \in \mathbb{Z}$), we obtain a filtration $\{N(n)\}$ on N. This is called the induced filtration on N. Similarly, let N be a left A-module and let $\phi: M \to N$ be a surjective A-homomorphism. Then, putting $N(n) = \phi(M(n))$, we obtain a filtration $\{N(n)\}$ on N. This is called the image filtration on N.

If $\{M(n)\}$ is a filtration on M, we define $gr(M) = \bigoplus_n M(n)/M(n-1)$.

<u>Definition 2.15</u>. Let M be a finitely generated left A-module and let $\{M(n)\}_{n\in\mathbb{Z}}$ be a filtration on M. Then $\{M(n)\}$ is called a good filtration on M if $M_{\langle T\rangle} = \bigoplus_n M(n)T^n$ is a finitely generated left $A_{\langle T\rangle}$-module.

<u>Lemma 2.16</u>. Let M be a finitely generated left A-module.

(i) There always exists a good filtration on M.

(ii) Let $\{M(n)\}$ be a good filtration on M. Then $gr(M) = \bigoplus_n M(n)/M(n-1)$ is a finitely generated $gr(A)$-module.

<u>Proof</u>. (i) Let $u_1,\ldots,u_r \in M$ be a system of generators of M over A. Then define $M(n) = \sum_{i=1}^{r} A(n)u_i$ for each $n \in \mathbb{Z}$. It is clear from the definition that $\{M(n)\}$ is a good filtration on M.

(ii) It follows from the definition that $gr(M)$ is identified with $M_{\langle T\rangle}/TM_{\langle T\rangle}$. Since $M_{\langle T\rangle}$ is a finitely generated left $A_{\langle T\rangle}$-module, this implies that $gr(M)$ is a finitely generated $gr(A)$-module.

<u>Remark 2.17</u>. It is clear that good filtrations on M are not unique. In fact if $\{u_1,\ldots, u_r\} \subset M$ is an arbitrary system of generators of M, then as in the proof of Lemma 2.16,(i), we can define a good filtration on M. For the sake of convenience, such a filtration on M is called a good filtration induced from a finite set of generators $\{u_1,\ldots,u_r\}$.

Since the Noetherian condition is stable by the procedure of taking subquotient modules, the next proposition follows from the definition of a good filtration.

Proposition 2.18. Let $0 \longrightarrow M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \longrightarrow 0$ be an exact sequence of left A-modules. Assume that M has a filtration $\{M(n)\}_{n\in\mathbb{Z}}$. Let $\{M'(n)\}_{n\in\mathbb{Z}}$ (resp. $\{M''(n)\}_{n\in\mathbb{Z}}$) be the induced filtrations on M' (resp. the image filtration on M''). Then M is finitely generated over A and $\{M(n)\}$ is a good filtration on M if and only if both M' and M'' are finitely generated over A and $\{M'(n)\}$ (resp. $\{M''(n)\}$ is a good filtration on M' (resp. M'').

Furthermore, in this case, we obtain the exact sequence of finitely generated left $A_{\langle T\rangle}$-modules

$$0 \longrightarrow M'_{\langle T\rangle} \longrightarrow M_{\langle T\rangle} \longrightarrow M''_{\langle T\rangle} \longrightarrow 0$$

and the exact sequence of gr(A)-modules

$$0 \longrightarrow gr(M') \longrightarrow gr(M) \longrightarrow gr(M'') \longrightarrow 0.$$

Definition 2.19. Let M be a finitely generated left A-module and let $\{M(n)\}$ and $\{M'(n)\}$ be good filtrations on M. Then $\{M(n)\}$ and $\{M'(n)\}$ are equivalent if there is an integer $k > 0$ such that $M(n) \subseteq M'(n+k)$ and $M'(n) \subseteq M(n+k)$ hold for each $n \in \mathbb{Z}$.

Lemma 2.20. Any two good filtrations on M are equivalent to each other.

Proof. Let $\{M^i(n)\}_{n\in\mathbb{Z}}$ $(i = 1, 2)$ be two good filtrations on

M. Define $M^i_{\langle T\rangle} = \bigoplus_n M^i(n)T^n$. Then what we must prove is to show the existence of an integer $k > 0$ such that $T^k M^1_{\langle T\rangle} \subseteq M^2_{\langle T\rangle}$ and $T^k M^2_{\langle T\rangle} \subseteq M^1_{\langle T\rangle}$.

First assume that $M^1(n) \subseteq M^2(n)$ for each $n \in \mathbb{Z}$. Put $L_k = \bigoplus_n (M^2(n) \cap M^1(n+k))T^n$. Then $\{L_k;\ k = 1,2,\dots\}$ is an increasing sequence of left sub-$A_{\langle T\rangle}$-modules of $M^2_{\langle T\rangle}$ and $L_0 = M^1_{\langle T\rangle}$, $\bigcup_{k=0}^{\infty} L_k = M^2_{\langle T\rangle}$. It follows from the assumption that $M^2_{\langle T\rangle}$ is a finitely generated left $A_{\langle T\rangle}$-module and that $A_{\langle T\rangle}$ is left noetherian. These imply that $\{L_k\}$ is stationary. Hence there exists an integer $k > 0$ such that $L_k = M^2_{\langle T\rangle}$. This means that $M^2(n) \cap M^1(n+k) = M^2(n)$ $(\forall n \in \mathbb{Z})$, or equivalently that $M^2(n) \subseteq M^1(n+k)$ $(\forall n \in \mathbb{Z})$. Therefore $\{M^1(n)\}$ and $\{M^2(n)\}$ are equivalent.

Next consider the general case. Since $\bigcup_n (M^1(n) \cap M^2(n)) = M$, it follows from Proposition 2.18 that $\{M^1(n) \cap M^2(n)\}$ is a good filtration on M. Therefore applying the previous discussion to the two filtrations $\{M^1(n) \cap M^2(n)\}$ and $\{M^2(n)\}$, we conclude that there exists an integer $k > 0$ such that $M^2(n) \subseteq M^1(n+k) \cap M^2(n+k)$ for each $n \in Z$. Since $M^2(n) \subseteq M^2(n+k)$, it follows that $M^2(n) \subseteq M^1(n+k)$ for each $n \in \mathbb{Z}$. By changing the roles of $M^1(n)$ and $M^2(n)$, we conclude the existence of $k' > 0$ such that $M^1(n) \subseteq M^2(n+k')$ for each $n \in \mathbb{Z}$. Hence $\{M^1(n)\}$ and $\{M^2(n)\}$ are equivalent. q.e.d.

2.4. The characteristic variety of M.

Let M be a finitely generated left A-module. We are going to define the characteristic variety $Ch(M)$ of M.

Definition 2.21. Let $\{M(n)\}_{n\in\mathbb{Z}}$ be a good filtration on M. Define $gr(M) = \bigoplus_n M(n)/M(n-1)$. Then $Ch(M) \equiv V(Ann_{gr(A)}gr(M))$ is called the characteristic variety of M. Here $Ann_{gr(A)}gr(M) = \{a \in gr(A);\ a\cdot gr(M) = 0\}$. (It follows from the definition that $Ann_{gr(A)}(gr(M))$ is a homogeneous ideal.)

As was already shown in Lemma 2.16, there always exists a good filtration on M and therefore $Ch(M)$ is defined for it. Although the definition of $Ch(M)$ depends on the choice of a good filtration, the following holds.

Theorem 2.22. Let M be a finitely generated left A-module. Then the characteristic variety $Ch(M)$ does not depend on the choice of a good filtration on M.

To prove Theorem 2.22, we first give a lemma which is easy to show.

Lemma 2.23. Let $0 \longrightarrow N' \xrightarrow{\phi} N \xrightarrow{\psi} N'' \longrightarrow 0$ be an exact sequence of graded gr(A)-modules. Assume that ϕ and ψ preserve the graduations. Then

$$V(Ann_{gr(A)}N) = V(Ann_{gr(A)}N') \cup V(Ann_{gr(A)}N'').$$

Proof of Theorem 2.22. Let $\{M^1(n)\}_{n\in\mathbb{Z}}$ and $\{M^2(n)\}_{n\in\mathbb{Z}}$ be two good filtrations on M. Define $gr^{(i)}(M) = \bigoplus_n M^i(n)/M(n-1)$ $(i = 1, 2)$. Then it suffices to show the following:

(2.4.1) $\quad V(Ann_{gr(A)} gr^{(1)}(M)) = V(Ann_{gr(A)} gr^{(2)}(M)).$

Define $M_i = \bigoplus_n M^i(n)T^n$ $(i = 1, 2)$. Remark that $M_i/TM_i \simeq gr^{(i)}(M)$ $(i = 1, 2)$. Since $\{M^1(n)\cap M^2(n)\}$ is also a good filtration on M, it suffices to prove (2.4.1) when $M^1(n) \subseteq M^2(n)$ $(\forall n \in \mathbb{Z})$ or equivalently when $M_1 \subseteq M_2$. Hence we assume that $M_1 \subseteq M_2$. Define $L_j = \bigoplus_n (M^2(n)\cap M^1(n+j))T^n$ $(j = 0, 1, 2, \ldots)$. Then $\{L_j;\ j = 0, 1, 2, \ldots\}$ is an increasing sequence of left sub-$A_{\langle T\rangle}$-modules of M_2 and $M_2 = \bigcup_{j=0}^{\infty} L_j$. Therefore by the argument as in the proof of Lemma 2.20, we find that there exists an integer $k > 0$ such that $L_k = M_2$. On the other hand, it follows from the definition of L_j that for each $j \geq 0$, we have

$$TL_j \subseteq TL_{j+1} \subseteq L_j \subseteq L_{j+1}.$$

This implies the following two exact sequences of gr(A)-modules

(2.4.2) $\quad 0 \longrightarrow TL_{j+1}/TL_j \longrightarrow L_j/TL_j \longrightarrow L_j/TL_{j+1} \longrightarrow 0$

(2.4.3) $\quad 0 \longrightarrow L_j/TL_{j+1} \longrightarrow L_{j+1}/TL_{j+1} \longrightarrow L_{j+1}/L_j \longrightarrow 0.$

Applying Lemma 2.23 to these exact sequences, we find that

$$V(Ann_{gr(A)} L_j/TL_j)$$

$$= V(\mathrm{Ann}_{gr(A)}TL_{j+1}/TL_j) \cup V(\mathrm{Ann}_{gr(A)}L_j/TL_{j+1})$$

$$= V(\mathrm{Ann}_{gr(A)}L_{j+1}/L_j) \cup V(\mathrm{Ann}_{gr(A)}L_j/TL_{j+1})$$

$$= V(\mathrm{Ann}_{gr(A)}L_{j+1}/TL_{j+1}).$$

(Here we used the isomorphism $TL_{j+1}/TL_j \simeq L_{j+1}/L_j$.) Therefore it follows that

$$V(\mathrm{Ann}_{gr(A)}L_j/TL_j) = V(\mathrm{Ann}_{gr(A)}L_{j+1}/TL_{j+1}) \quad (\forall\, j \geq 0).$$

In particular

$$V(\mathrm{Ann}_{gr(A)}L_0/TL_0) = V(\mathrm{Ann}_{gr(A)}L_k/TL_k).$$

Since $L_0 = M_1$ and $L_k = M_2$ and since $L_0/TL_0 \simeq gr^{(1)}(M)$ and $L_k/TL_k \simeq gr^{(2)}(M)$, we have shown (2.4.1). q.e.d.

From now on, we are going to show some elementary properties of Ch(M).

<u>Proposition 2.24</u>. Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence of finitely generated left A-modules, then

$$Ch(M) = Ch(M') \cup Ch(M'').$$

This follows from Lemma 2.23 and Proposition 2.18.

<u>Proposition 2.25</u>. Let M be a finitely generated left

A-module and let $u_1,\dots,u_r$ be a system of generators of M over A. Then for any $\underline{p} \in X$, the following conditions are equivalent.

(i) $\underline{p} \notin Ch(M)$

(ii) There exists an $a \in A$ such that $au_i = 0$ $(i = 1,\dots,r)$, $\sigma(a) \notin \underline{p}$.

Proof. The implication "(ii) $\Rightarrow$ (i)" is clear from the definition of $Ch(M)$.

We show the part "(i) $\Rightarrow$ (ii)". By definition, $u = (u_1,\dots u_r)$ is an element of $\bigoplus_{i=1}^{r} Au_j$. Define $N = Au$. Then there exists a left ideal I of A such that $N \simeq A/I$. It follows from Proposition 2.24 that

$$Ch(M) = \bigcup_{i=1}^{r} Ch(Au_i) = Ch(\bigoplus_{i=1}^{r} Au_i) \supset Ch(Au).$$

Now induce the filtrations on I and N from that on A. Then they are good and $gr(N) = gr(A)/gr(I)$. This implies that $Ch(N) = V(gr(I))$. Let $\underline{p} \in X$ be such that $\underline{p} \notin Ch(M)$. Then by the remark above, $\underline{p} \notin Ch(N) = V(gr(I))$. This implies that $\underline{p} \not\supset gr(I)$. Therefore there exists an element a of I such that $\sigma(a) \notin \underline{p}$. It follows from the definition of I that $(au_1,\dots,au_r) = au = 0$. q.e.d.

Proposition 2.26. Let A be a D-ring satisfying that each element of $1+A(-1)$ has an inverse in $A(0)$. If M is a finitely generated left A-module such that $Ch(M) = \emptyset$, then $M = 0$.

Proof. Assuming that $M \neq 0$, we lead a contradiction. Hence take $u \in M$ with $u \neq 0$. Then it follows from Proposition 2.24 that $Ch(Au) = \emptyset$. There exists a left ideal I of A such that $Au \simeq A/I$. Induce the filtration on I from that on A. Then, since $Ch(Au) = V(gr(I))$, it follows that $gr(I) = gr(A)$. Therefore we obtain $A(0) = I \cap A(0) + A(-1)$. Since $1 \in A(0)$, there exist $a \in I \cap A(0)$ and $b \in A(-1)$ such that $1 = a+b$. From the assumption, $1-b$ is invertible in $A(0)$ and therefore $1 = (1-b)^{-1}a \in I \cap A(0)$. This implies $I = A$. Hence $u = 0$. This contradicts the assumption $u \neq 0$. q.e.d.

2.5. The ring of left fractions and the module of left fractions.

The purpose of this subsection is to define the ring of left fractions and the module of left fractions for a non-commutative ring and its module. The reader is refered to [Stenstrom] for the details.

For the present, let A be a ring with the unit 1. Let S be a multiplicative subset of A. This means that $1 \in S$ and that if $s, t \in S$, then $st \in S$. Furthermore, we assume that S satisfies the conditions

(m.1) If $s \in S$ and $a \in A$, there exist $t \in S$ and $b \in A$ such that $bs = ta$.

(m.2) If $as = 0$ with $s \in S$, then $ta = 0$ for some $t \in S$.

Now we define a relation "~" on $S \times A$ as follows:

For $(s_1,a_1), (s_2,a_2) \in S \times A$, $(s_1,a_1) \sim (s_2,a_2)$ if and only if there exist $t_1, t_2 \in A$ such that $t_1 s_1 = t_2 s_2 \in S$ and $t_1 a_1 = t_2 a_2$.

Then it is easy to check that "$\sim$" is an equivalence relation on $S \times A$. Let $S^{-1}A$ denote the quotient space of $S \times A$ by the equivalence relation "$\sim$" and let $s^{-1}a$ denote the equivalence class of $(s,a) \in S \times A$.

For each $(s_1,a_1), (s_2,a_2) \in S \times A$, we define the sum $s_1^{-1}a_1+s_2^{-1}a_2$ and the product $s_1^{-1}a_1 \cdot s_2^{-1}a_2$ as follows:

<u>Sum</u> : By the condition (m.1), there exist $b_1, b_2 \in A$ such that $b_1s_1 = b_2s_2 \in S$. Then

$$s_1^{-1}a_1+s_2^{-1}a_2 \equiv (b_1s_1)^{-1}(b_1a_1+b_2a_2).$$

<u>Product</u> : By the condition (m.1), there exist $b \in A$ and $t \in S$ such that $bs_2 = ta_1$. Then

$$s_1^{-1}a_1 \cdot s_2^{-1}a_2 = (ts_1)^{-1}(ba_2).$$

In this way, $S^{-1}A$ has the ring structure.

<u>Definition 2.27</u>. The ring $S^{-1}A$ is called the ring of left fractions of A by S.

We define a map ϕ of A to $S^{-1}A$ by $\phi(a) = 1^{-1}a$. Then ϕ is a ring homomorphism and has the following properties:

(2.5.1) $\phi(s)$ is invertible for each $s \in S$.

(2.5.2) Each element of $S^{-1}A$ is expressed as the form $\phi(s)^{-1}\phi(a)$ $(s \in S,\ a \in A)$.

(2.5.3) Let $a \in A$. Then $\phi(a) = 0$ if and only if $sa = 0$ for some $s \in S$.

<u>Proposition 2.28</u>. Let B be a ring with $1 \in B$ and let

$\phi: A \to B$ be a ring homomorphism such that $\phi(s)$ is invertible in B for each $s \in S$, there exists a unique homomorphism $\sigma: S^{-1}A \to B$ such that $\sigma \circ \phi = \phi$.

For a proof, see [St].

Proposition 2.29. Let A and S be as above. Then for every left ideal $I' \subset S^{-1}A$ ($I' \neq S^{-1}A$), there exists a left ideal I of A such that $S^{-1}I = I'$ and that $I \cap S = \emptyset$.

Corollary 2.30. Let $\mathfrak{p}$ be a prime ideal of A such that $\mathfrak{p} \cap S = \emptyset$. Then $S^{-1}\mathfrak{p}$ is a prime ideal of $S^{-1}A$. Furthermore, every prime ideal of $S^{-1}A$ is expressed as $S^{-1}\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of A such that $\mathfrak{p} \cap S = \emptyset$.

Proposition 2.29 and Corollary 2.30 are easy to show and therefore we omit their proof (cf. [B]).

Let S be a multiplicative subset of A satisfying (m.1), (m.2) and let M be a left A-module. Then $S^{-1}M$ is defined by the way similar to $S^{-1}A$ and it has the structure of left $S^{-1}A$-module. To explain this, first define an equivalence relation "$\sim$" on $S \times M$ as follows:

> For $(s_1, u_1), (s_2, u_2) \in S \times M$, $(s_1, u_1) \sim (s_2, u_2)$ if and only if there exist $t_1, t_2 \in A$ such that $t_1 s_1 = t_2 s_2 \in S$ and $t_1 u_1 = t_2 u_2$.

Let $S^{-1}M$ denote the quotient space $S \times M/\sim$. The sum of two elements of $S^{-1}M$ and the product of an element of $S^{-1}A$ and that of $S^{-1}M$ are defined by the way similar to the case of $S^{-1}A$.

Hence we do not repeat them. At any rate, $S^{-1}M$ has the structure of left $S^{-1}A$-module.

<u>Definition 2.31</u>. $S^{-1}M$ is called the module of left fractions of M by S.

<u>Proposition 2.32</u>. Let A and S be as above. Then for any left A-module M, we have

$$S^{-1}M \simeq S^{-1}A \otimes_A M$$

Furthermore $S^{-1}A$ is right flat over A, that is, if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of left A-modules, then $0 \to S^{-1}M' \to S^{-1}M \to S^{-1}M'' \to 0$ is an exact sequence of left $S^{-1}A$-modules.

The proof of this proposition is similar to the commutative case. Hence we do not give it (cf. [B]).

2.6. <u>The ring of left fractions of a D-ring</u>.

We return to our situation, namely, let $(A, \{A(n)\}_{n\in\mathbb{Z}})$ be a D-ring. The purpose of this subsection is to study the ring of left fractions of A.

<u>Theorem 2.33</u>. Let $(A, \{A(n)\})$ be a D-ring. Let $\overline{S}$ be a multiplicative subset of $gr(A)$ consisting of homogeneous elements. Then $S = \{s \in A; \sigma(s) \in \overline{S}\}$ is a multiplicative subset of A satisfying the conditions (m.1), (m.2).

Proof. It is clear that S is a multiplicative subset of A.

We are going to show that S satisfies (m.1) and (m.2). For this purpose, put $S(n) = S \cap A(n)$ ($\forall n \in \mathbb{Z}$).

First show that (m.2) holds. Assume that $as = 0$ for $a \in A$ (ord $a = k$), $s \in S$ (ord $s = m$). Define a left A-endomorphism r_s of A by $r_s(b) = bs$. Then $a \in N \equiv \mathrm{Ker}(r_s)$. Also define $N(n) = N \cap A(n)$ ($\forall n \in \mathbb{Z}$) and $gr(N) = \oplus_n N(n)/N(n-1)$. Then $\sigma(s)gr(N) = 0$. In fact, for each $n \in \mathbb{Z}$, since $N(n)s = 0$, it follows that $sN(n) = [s, N(n)] \subset N(m+n-1)$. Therefore $V(\mathrm{Ann}_{gr(A)} gr(N)) \subset V(\sigma(s)) = X-D(\sigma(s))$. On the other hand, since $Aa \subset N$, it follows that $Ch(Aa) \subset V(\sigma(s))$. Define a left A-endomorphism r_a for a as we did for s and put $I = \mathrm{Ker}(r_a)$. Then we obtain an exact sequence

$$0 \longrightarrow I \longrightarrow A \xrightarrow{r_a} A \longrightarrow 0.$$

Define the induced filtration $\{I(n)\}$ on I by $I(n) = I \cap A(n)$ and put $gr(I) = \oplus_n I(n)/I(n-1)$. Since $Ch(Aa) = V(gr(I)) \subset V(\sigma(s))$, it follows from (2.1.1.vii) that $\sigma(s)^n = \sigma(s^n) \in gr(I)$ for $n >> 0$. Now fix such an n. Then $s^n \in I(mn) + A(mn-1)$ and there exist $t \in I(mn)$ and $b \in A(mn-1)$ such that $s^n = t + b$. Since $\sigma(t) = \sigma(s^n) = \sigma(s)^n \in \overline{S}$, we have $t \in S$. Moreover $ta = 0$. Hence we find that (m.2) holds for S.

Next show (m.1). Take $a \in A$ (ord $a = k$) and $s \in S$ (ord $s = m$). Define the image filtration on $N \equiv A/As$. By definition, $Ch(N) = V(\sigma(s))$. Next define $M = Aa/(Aa \cap As)$ and $M(n) = A(n)a/(A(n) \cap As)$ ($\forall n \in \mathbb{Z}$). Since $M \simeq (Aa+As)/As$, M is regarded as a left sub-A-module of N and $\{M(n)\}$ is a filtration on M induced from that on N. Then it follows from Proposition 2.24 that $Ch(M) \subseteq Ch(N) = V(\sigma(s))$. This implies that there exists $p > 0$ such that $\sigma(s)^p gr(M) = 0$. This means that $s^p M(n) \subseteq M(n+mp-1)$ $(\forall n \in \mathbb{Z})$. Now we consider the case $n = 0$. Since $M(mp-1) \simeq (A(mp-1)a+As)/As$, it follows that $s^p a \in A(mp-1)a+As$. Then there exist $b \in A$ and $c \in A(mp-1)$ such that $s^p a = ca+bs$, or equivalently, that $(s^p-c)a = bs$. Since $\sigma(s^p-c) = \sigma(s^p) = \sigma(s)^p \in \bar{S}$, we conclude that $(s^p-c)a = bs \in As \cap Sa$. Hence (m.1) is shown. q.e.d.

Let $\bar{S}$ be a multiplicative subset of $gr(A)$ consisting of homogeneous elements and put $S = \{s \in A;\ \sigma(s) \in \bar{S}\}$. Then in virtue of Theorem 2.33 and the discussion in subsection 2.5, the ring of left fraction $S^{-1}A$ is well-defined.

For each element $s^{-1}a \in S^{-1}A$ $(s \in S,\ a \in A)$, define $ord(s^{-1}a) = ord(a) - ord(s)$. Then $ord(s^{-1}a)$ is independent of the choice of the representatives. Noting this, we define

$$(S^{-1}A)(n) = \{s^{-1}a \in S^{-1}A;\ ord(s^{-1}a) \leq n\} \quad (\forall n \in \mathbb{Z}).$$

Then $\{(S^{-1}A)(n)\}_{n\in\mathbb{Z}}$ is a filtration on $S^{-1}A$ and it is easy to show that

$$[(S^{-1}A)(m),\ (S^{-1}A)(n)] \subseteq (S^{-1}A)(m+n-1) \quad (\forall m,\ n \in \mathbb{Z}).$$

As in the case of A, we put $gr(S^{-1}A) = \bigoplus_n gr_n(S^{-1}A)$, where $gr_n(S^{-1}A) = (S^{-1}A)(n)/(S^{-1}A)(n-1)$ and denote by σ_n^S the natural surjection of $(S^{-1}A)(n)$ to $gr_n(S^{-1}A)$. Then we obtain the surjection σ^S of $S^{-1}A$ to $gr(S^{-1}A)$ by putting $\sigma^S(s^{-1}a) = \sigma_n^S(s^{-1}a)$ when $ord(s^{-1}a) = n$. On the other hand, since $\overline{S}$ is a multiplicative subset of $gr(A)$, $\overline{S}^{-1}gr(A)$ is well-defined. By direct calculation, we find that $\sigma^S(s^{-1}a) = \sigma(s)^{-1}\sigma(a) \in \overline{S}^{-1}gr(A)$ when $s \in S$, $a \in A$. This implies a natural isomorphism $gr(S^{-1}A) \simeq \overline{S}^{-1}gr(A)$.

Theorem 2.34. Let A, $\overline{S}$ and S be as in Theorem 2.33. Then the fllowing statements hold.

(i) $(S^{-1}A, \{(S^{-1}A)(n)\}_{n\in\mathbb{Z}})$ is a D-ring.

(ii) There exists a natural isomorphism $gr(S^{-1}A) \simeq \overline{S}^{-1}gr(A)$.

(iii) Each element of $1 + (S^{-1}A)(-1)$ is invertible in $(S^{-1}A)(0)$.

Proof. From the remark before the theorem, (ii) follows. On the other hand, (iii) is clear from the definition. In fact, each element of $1 + (S^{-1}A)(-1)$ is expressed of the form $t^{-1}t'$ with $t, t' \in S$ such that $\sigma(t) = \sigma(t')$. Then $t'^{-1}t$ is its inverse.

Hence it suffices to show that $S^{-1}A$ is actually a D-ring. We are going to prove the conditions (ii)-(iv) of Definition 2.1 hold for $S^{-1}A$.

The condition (ii).

It is clear that $(S^{-1}A)(0)$ contains 1. Hence it suffices to show that $(S^{-1}A)(0)$ is left and right Noetherian.

Let I be a left ideal of $(S^{-1}A)(0)$. Define

$$\tilde{I}(n) = \{a \in A;\ s^{-1}a \in I \in (S^{-1}A)(n) \text{ for some } s \in S\}$$

and $\tilde{I} = \bigoplus_{n \in \mathbb{Z}} \tilde{I}(n)T^n$. Then $\tilde{I}$ is a left ideal of $A_{\langle T \rangle}$. If I' is another left ideal of $(S^{-1}A)(0)$, we also define $\tilde{I}'$ similarly. By definition, $I = I'$ is equivalent to $\tilde{I} = \tilde{I}'$. Since $A_{\langle T \rangle}$ is left Noetherian, this implies that every increasing sequence of left ideals of $(S^{-1}A)(0)$ is stationary. Therefore $(S^{-1}A)(0)$ is left Noetherian. Similarly, we can show that $(S^{-1}A)(0)$ is right Noetherian.

<u>The condition (iii)</u>.

Since $gr(S^{-1}A) \simeq \bar{S}^{-1}gr(A)$ is commutative, this is well-known (cf. [B]).

<u>The condition (iv)</u>.

Define $\tilde{S} = \bigcup_{k \in \mathbb{Z}} (S \cap A(k))T^k$. Then $\tilde{S}$ is a multiplicative subset of $A_{\langle T \rangle}$ satisfying the conditions (m.1), (m.2) and therefore $\tilde{S}^{-1}A_{\langle T \rangle}$ exists. On the other hand, it follows that

$$\tilde{S}^{-1}A_{\langle T \rangle} = \bigoplus_{k} (S^{-1}A)(k)T^k = (S^{-1}A)_{\langle T \rangle}.$$

Since $A_{\langle T \rangle}$ is left and right Noetherian, this equality implies that so is $(S^{-1}A)_{\langle T \rangle}$.

We have thus shown that $S^{-1}A$ is a D-ring. q.e.d.

We now give examples of the ring of left fractions of A which will be used later.

<u>Example 2.35</u>. The case where $\bar{S} = \{f^n;\ n \geq 0\}$ ($f \in gr(A)$, homogeneous).

For the sake of convenience, denote by $\overline{S}_f$, S_f and A_f, the multiplicative sets $\overline{S}$, S and $S^{-1}A$, respectively.

In particular consider the case where $\overline{S} = 1$ (that is, $f = 1$). In this case, $S = 1 + A(-1)$. By definition, we find that

$$(S^{-1}A)(n) \simeq S^{-1}A(n) \quad \text{for any} \quad n \in Z,$$
$$gr(S^{-1}A) = gr(A).$$

If each element of $1 + A(-1)$ has its inverse in $A(0)$, then $S^{-1}A = A$.

Example 2.36. The case where $\overline{S} = gr(A)-\underline{p}$ ($\underline{p}$ is a homogeneous prime ideal of $gr(A)$).

In this case, $\overline{S} = \{f \in gr(A);\ f \notin \underline{p}\}$. For the sake of convenience, put $A_{\underline{p}} = S^{-1}A$.

Lemma 2.37. Let $f \in gr(A)$ be homogeneous and let $\underline{p}$ be a homogeneous prime ideal of $gr(A)$. Assume that $f \notin \underline{p}$. Then $(A_f)_{(A_f - S_f^{-1}\underline{p})} \simeq A_{\underline{p}}$.

Proof. It is not clear whether $\tilde{S} = A_f - S_f^{-1}\underline{p}$ is a multiplicative subset of A_f or not. But this follows from the primeness of $\underline{p}$. On the other hand, it is easy to check that $\tilde{S}$ satisfies the conditions (m.1) and (m.2). The rest of the claim is shown by an argument similar to the commutative case. q.e.d.

Let S and $\overline{S}$ be as in Theorem 2.33 and let M be a left A-module. Then the module of left fractions $S^{-1}M$ is well-defined.

As in Examples 2.35 and 2.36, we can define M_f and $M_{\underline{p}}$ for a homogeneous element f of $gr(A)$ and for a homogeneous prime ideal $\underline{p}$ of $gr(A)$, respectively.

<u>Lemma 2.38</u>. Let M be a left A-module and let $f, g \in gr(A)$ be homogeneous. Consider an element $s^{-1}u \in M_f$ with $s \in S_f$ ($\sigma(s) = f^m$) and $u \in M$. Then for any $t, t' \in S_g$ such that $\sigma(t) = \sigma(t') = g^m$, we have $(ts)^{-1}tu = (t's)^{-1}t'u$ in M_{fg}.

<u>Proof</u>. It follows from (m.1) that there exist $p \in S_g$ and $p \in A$ such that $pt = p't' \in S_g$. If $\sigma(p) = g^n$, we take a $q \in S_f$ such that $\sigma(q) = f^n$. Then

$$qpts = qp't's \in S_{fg}, \quad qptu = qp't'u.$$

This means that $(ts)^{-1}tu = (t's)^{-1}t'u$ in M_{fg}. q.e.d.

This lemma implies the following proposition.

<u>Proposition 2.39</u>. Let M, f, g be as in the previous lemma. Define a map $\dot{\phi}_{f,fg}$ of M_f to M_{fg} as follows. Let $s^{-1}u$ be an element of M_f. If $\sigma(s) = f^m$, then $\phi_{f,fg}(s^{-1}u) = (ts)^{-1}tu$ for some $t \in S_g$ such that $\sigma(t) = g^m$. Then $\phi_{f,fg}$ is an A_f-homomorphism.

The following proposition is also shown by an argument similar to the case of commutative rings. Hence we do not give here its proof.

Proposition 2.40. Let M, f, g be as above.

(i) $(M_f)_g \simeq M_{fg}$.

(ii) $M_f \simeq M_{f^m}$ for any $m \in \mathbb{N}_+$.

Let M, f and g be as above. Assume that $D(f) \supset D(g)$. Then $g^m = fh$ for som $h \in gr(A)$, homogeneous and $m > 0$. Then it follows from Propositions 2.39 and 2.40 that there exists a natural left A_f-homomorphism $\phi_{f,g}$ of M_f to M_g. The following relations are consequences of Propositions 2.39 and 2.40.

(2.6.1) If $D(f) = D(g)$, then $M_f \simeq M_g$.

(2.6.2) If $D(f) \supset D(g) \supset D(h)$, then $\phi_{f,h} = \phi_{g,h} \circ \phi_{f,g}$.

Proposition 2.41. Let M be a left A-module and let $f \in gr(A)$ be homogeneous. If $u \in M$ satisfies the condition that $u = 0$ in $M_{\underline{p}}$ for each $\underline{p} \in D(f)$, then $u = 0$ in M_f.

Proof. Define $I = \mathrm{Ann}_A u$ and introduce the induced filtration on I from A. Show that $V(gr(I)) \subset V(f)$. Take $\underline{p} \in D(f) = X - V(f)$. Since $u = 0$ in $M_{\underline{p}}$, there exists a $t \in A$ such that $tu = 0$ and that $\sigma(t) \notin \underline{p}$. Since $\sigma(t) \in gr(I)$, we find that $\underline{p} \notin V(gr(I))$. Hence $V(gr(I)) \subset V(f)$. This combined with (2.1.vii) implies that $f^m \in gr(I)$ for some $m > 0$. Then there exists a $t \in I$ such that $\sigma(t) = f^m$. This means that $u = 0$ in M_f. q.e.d.

Corollary 2.42. Let M and f be as in the proposition. If

$M_{\underline{p}} = 0$ for each $\underline{p} \in D(f)$, then $M_f = 0$.

This is an easy consequence of Proposition 2.41.

2.7. Sheaves on Spech(gr(A)).

Let $(A, \{A(n)\})$ be a D-ring and let X denote Spech(gr(A)). We are going to define sheaves on X associated to A and left A-modules. In particular, the sheaf of rings $\tilde{A}$ associated to A is regarded as the structure sheaf on X in comparison with the case of commutative rings.

Since gr(A) is Noetherian, it follows from the definition of the topology on X that every open subset of X is expressed of the form $D(f)$ for some homogeneous $f \in gr(A)$. Noting this, we define a presheaf $A^{\wedge}$ on X as follows. For any homogeneous $f \in gr(A)$, put $A^{\wedge}(D(f)) = A_f$. If $D(f) \supset D(g)$, then the restriction map $\rho_{f,g}: A^{\wedge}(D(f)) \rightarrow A^{\wedge}(D(g))$ is defined by $\rho_{f,g} = \phi_{f,g}$ (cf. subsection 2.6). Let $\tilde{A}$ be the sheafication of $A^{\wedge}$.

<u>Theorem 2.43</u>. The following properties hold for the sheaf $\tilde{A}$.

(0) $\tilde{A}$ is a sheaf of rings on X.

(i) $\tilde{A}_{\underline{p}} \simeq A_{\underline{p}}$ for each $\underline{p} \in X$.

(ii) $\Gamma(D(f), \tilde{A}) \simeq A_f$ for each $f \in gr(A)$, homogeneous.

(iii) $\Gamma(X, \tilde{A}) \simeq S^{-1}A$, where $S = 1 + A(-1)$.

<u>Proof</u>. The claims (0) and (i) are nearly obvious from the definition. On the other hand, (iii) is a special case of (ii).

We are going to prove (ii). Define a map $\theta_f : A_f \to \Gamma(D(f), \tilde{A})$ by the correspondence: If $u \in A_f$, then $\theta_f(u) \in \Gamma(D(f), \tilde{A})$ is defined by $\theta_f(u) = u$ in $A_{\underline{p}}$ for each $\underline{p} \in D(f)$. Then what we must prove is the bijectivety of θ_f.

First show the injectivity of θ_f. Take $u \in A_f$ and assume that $\theta_f(u) = 0$. This means that $u = 0$ in $A_{\underline{p}}$ for any $\underline{p} \in D(f)$. Then Proposition 2.41 implies that $u = 0$.

Next show the surjectivity of θ_f. Take $v \in \Gamma(D(f), \tilde{A})$. Then there exist $f_1, \dots, f_m \in gr(A)$, homogeneous and $u_i \in A_{f_i}$ for each i such that $D(f) = \bigcup_{i=1}^{m} D(f_i)$ and that $\rho_{f,f_i}(v) = u_i$. For any i, j, consider the elements $\rho_{f_i,f_if_j}(u_i)$ and $\rho_{f_j,f_if_j}(u_j)$ of $A_{f_if_j}$. It follows from the definition that $\rho_{f_i,f_if_j}(u_i) = \rho_{f_j,f_if_j}(u_j)$ in $A_{\underline{p}}$ for any $\underline{p} \in D(f_if_j)$. Then Proposition 2.41 implies that $\rho_{f_i,f_if_j}(u_i) = \rho_{f_j,f_if_j}(u_j)$. Now we need a lemma.

<u>Lemma 2.44</u>. Define

$$H = \{(z_i) \in \prod_{i=1}^{m} A_{f_i} ;\ \phi_{f_i,f_if_j}(z_i) = \phi_{f_j,f_if_j}(z_j)\ (\forall\, i, j)\}$$

$$K = \{(z_i) \in \prod_{i=1}^{m} A_{f_i} ;\ z \in A_f \ \text{s.t.}\ z_i = \phi_{f,f_i}(z)\ (\forall\, i)\} .$$

Then H and K are left A_f-modules and $H = K$.

The proof of this lemma will be given later and we continue the proof of (iii). By the argument before the lemma, we find that $(u_i) \in H$. It follows from Lemma 2.44 that there exists a $u \in A_f$ such that $u_i = \phi_{f,f_i}(u)$ $(= \rho_{f,f_i}(u))$ for each i. This implies that $\theta_f(u) = v$. Hence θ_f is surjective.

We have thus shown Theorem 2.43. q.e.d.

Proof of Lemma 2.44. It is clear from the definition that H and K are left A_f-modules and that $K \subset H$.

In virtue of Corolarry 2.42, it suffices to show that $K_{\underline{p}} = H_{\underline{p}}$ for each $\underline{p} \in D(f)$. Take $\underline{p} \in D(f)$ and fix it once for all. By definition, we have the following.

$$H_{\underline{p}} = \{(z_i) \in \prod_{i=1}^{m} (A_{f_i})_{\underline{p}};\ \phi_{f_i, f_i f_j}(z_i) = \phi_{f_j, f_i f_j}(z_j) \text{ in } (A_{f_i f_j})_{\underline{p}}\ (\forall i, j)\}$$

$$K_{\underline{p}} = \{(z_i) \in \prod_{i=1}^{m} (A_{f_i})_{\underline{p}};\ z \in A_{\underline{p}} \text{ s.t. } z_i = \phi_{f, f_i}(z_i) \text{ in } (A_{f_i})_{\underline{p}}\ (\forall i)\}.$$

Here we wrote $(A_{f_i})_{\underline{p}}$ for $(A_{f_i})_{S_{f_i}^{-1}\underline{p}}$, etc. We may assume without loosing generality that $\underline{p}$ is contained in $D(f_1)$. Then it follows that $(A_{f_1})_{\underline{p}} = A_{\underline{p}}$, $(A_{f_1 f_i})_{\underline{p}} = (A_{f_i})_{\underline{p}}$ for each i. Take $(z_i) \in H_{\underline{p}}$. Then put $z = z_1 \in (A_{f_1})_{\underline{p}} = A_{\underline{p}}$. It follows from the assumption that $\phi_{f_1, f_1 f_i}(z_1) = \phi_{f_i, f_1 f_i}(z_i)$ in $(A_{f_1 f_i})_{\underline{p}}$. Since $\phi_{f_i, f_1 f_i} = id_{(A_{f_i})_{\underline{p}}}$, we conclude that $z_i = \phi_{f, f_i}(z)$ in $(A_{f_i})_{\underline{p}}$ $(= (A_{f_1 f_i})_{\underline{p}})$. Then $(z_i) \in K_{\underline{p}}$ and therefore the lemma is shown. q.e.d.

Remark 2.45. Assume that each element of $1 + A(-1)$ is invertible in $A(0)$. Then Theorem 2.43, (iii) is rewritten in the following form.

(iii') $\Gamma(X, \tilde{A}) \simeq A$.

Similar to the construction of $\tilde{A}$, we can also define a subsheaf $\tilde{A}(n)$ of $\tilde{A}$ for each $n \in \mathbb{Z}$ as follows. For each homogeneous element f of $gr(A)$, put $A^{\wedge}(n)(D(f)) = (A_f)(n)$. It follows from the definition that for homogeneous $f, g \in gr(A)$ such that $D(f) \supset D(g)$, $\phi_{f,g}:A_f \to A_g$ preserves the filtrations, that is, $\phi_{f,g}((A_f)(m)) \subset \phi_{f,g}((A_g)(m))$ $(\forall m \in \mathbb{Z})$. Hence $A^{\wedge}(n)$ defines a presheaf. Let $\tilde{A}(n)$ be the sheafication of $A^{\wedge}(n)$. From the definition, each $\tilde{A}(n)$ is a subsheaf of $\tilde{A}$.

Theorem 2.46.

$\{\tilde{A}(n)\}_{n\in\mathbb{Z}}$ is a filtration on $\tilde{A}$, namely, the following relations hold:

$$\tilde{A}(m)\tilde{A}(n) \subset \tilde{A}(m+n), \quad \tilde{A} = \bigcup_n \tilde{A}(n).$$

Furthermore,

$$[\tilde{A}(m), \tilde{A}(n)] \subset \tilde{A}(m+n-1).$$

$$\tilde{A}(n)_{\underline{p}} \simeq (A_{\underline{p}})(n) \quad \text{for each } \underline{p} \in X.$$

Theorem 2.47. Define $gr(\tilde{A}) = \bigoplus_n \tilde{A}(n)/\tilde{A}(n-1)$.

(i) $gr(\tilde{A})$ is the sheaf on X associated to $gr(A)$.

(ii) $(gr(\tilde{A}))_{\underline{p}} \simeq gr(A_{\underline{p}})$ for each $\underline{p} \in X$.

(iii) $\Gamma(D(f), gr(\tilde{A})) \simeq gr(A_f)$ for each $f \in gr(A)$, homogeneous.

(iv) $\Gamma(X, gr(\tilde{A})) \simeq gr(A)$.

Theorem 2.46 follows from the argument before it except the last eqaulity whose proof is similar to that of Theorem 2.43. On the other hand, since $gr(A)$ is commutative, Theorem 2.47 is proved by an standard argument for the case of commutative rings (cf. [H

Let M be a left A-module. Then as in the case of A, we can define a sheaf $\tilde{M}$ of left $\tilde{A}$-modules. If M has a filtration $\{M(n)\}_{n\in\mathbb{Z}}$, then the sheaves $\tilde{M}(n)$ ($\forall\, n \in \mathbb{Z}$) and the sheaf $gr(\tilde{M})$ associated to $gr(M) = \underset{n}{\oplus}\, M(n)/M(n-1)$ are defined similarly. We do not repeat the definition of them.

Definition 2.48. Let M be a left A-module. Then $\tilde{M}$ is called the left $\tilde{A}$-Module on X associated to M.

Similar to Theorem 2.43, we also obtain the following theorem for a left A-module. Hence we omit its proof.

Theorem 2.49. Let M be a left A-module. Then the following hold for the left $\tilde{A}$-Module $\tilde{M}$.

(i) $\tilde{M}_{\underline{p}} \simeq M_{\underline{p}}$ for each $\underline{p} \in X$.

(ii) $\Gamma(D(f), \tilde{M}) \simeq M_f$ for each $f \in gr(A)$, homogeneous.

(iii) $\Gamma(X, \tilde{M}) \simeq S^{-1}M$, where $S = 1 + A(-1)$.

Corollary 2.50. If M is a left A-module, then $\tilde{M} = \tilde{A} \otimes_A M$.

This follows from Theorem 2.49 and Proposition 2.32.

Recall that for each sheaf $\underline{\underline{S}}$ on X, its support $Supp(\underline{\underline{S}})$ is defined by $Supp(\underline{\underline{S}}) = \{\underline{p} \in X;\ \underline{\underline{S}}_{\underline{p}} \neq 0\}$.

Proposition 2.51. Let M be a finitely generated left

A-module. Then $Ch(M) = Supp(\tilde{M})$.

This is a direct consequence of Proposition 2.25 and Theorem 2.43, (i).

Remark 2.52. Since A is Noetherian, $\tilde{A}$ is a coherent sheaf of rings on X. Hence if M is a finitely generated left A-module, then $\tilde{M}$ is a coherent $\tilde{A}$-Module. In this way, we can develop the general theory of coherent sheaves for a D-ring. But this will be not necessary in the subsequent discussions, we stop the arguments at this stage.

2.8. MD-rings.

In this subsection, we always assume that $(A, \{A(n)\})$ is an MD-ring (cf. Definition 2.4). Then, by definition, there exist $u \in A(-1)$ and $v \in A(1)$ such that $uv = vu = 1$. Define $\xi = \sigma(v) \in gr_1(A)$. Then $\sigma(u) = \xi^{-1} \in gr_{-1}(A)$. By definition, we obtain

$$A(n) = A(0)v^n, \quad A(-n) = A(0)u^n \quad (\forall n > 0).$$

This implies that $gr(A) \simeq gr_0(A)[\xi, \xi^{-1}]$. Then the natural inclusion $gr_0(A) \hookrightarrow gr(A)$ induces a homeomorphism of $Spech(gr(A))$ to $Spec(gr_0(A))$ defined by $\underline{p} \longrightarrow \underline{p}(0) = \underline{p} \cap gr_0(A)$. Its inverse is defined by $\underline{p}(0) \longrightarrow \bigoplus_n \underline{p}(0)\xi^n$. Now put $X = Spech(gr(A))$ and $X_0 = Spec(gr_0(A))$. In the sequel we frequently identify X with X_0 by the above homeomorphism.

For each homogeneous element f of $gr(A)$, we find that $D(f) = D(\xi f) = D(\xi^{-1} f)$. Noting this, we can take $\{D(f);\ f \in gr_0(A)\}$ as a basis of open subsets of $X \simeq X_0$.

If $f \in gr_0(A)$, then S_f is contained in $A(0)$ and therefore $(A_f)(n) \simeq S_f^{-1}(A(n))$ holds for each $n \in \mathbb{Z}$.

Definition 2.53. Let N be a left $A(0)$-module. Then $\tilde{N} = \tilde{A}(0) \otimes_{A(0)} N$ is called the left $\tilde{A}(0)$-Module on X associated to N.

Theorem 2.54. Let N be a left $A(0)$-module. Then the following properties hold for the left $\tilde{A}(0)$-Module $\tilde{N}$.

(i) $\tilde{N}_{\underline{p}(0)} \simeq (A_{\underline{p}(0)})(0) \otimes_{A(0)} N$ for any $p(0) \in X_0$.

(ii) $\Gamma(D(f), \tilde{N}) \simeq N_f$ for each $f \in gr_0(A)$.

(iii) $\Gamma(X_0, \tilde{N}) \simeq N$.

Since the proof is similar to that of Theorem 2.43, we omit it.

Remark 2.55. If N is a left A-module, then $\tilde{A}(0) \otimes_{A(0)} N$ coincides with $\tilde{A} \otimes_A N$.

Proposition 2.56. If N is a left $A(0)$-module such that $\mathrm{Supp}(\tilde{N}) = \emptyset$, then $N = 0$.

Proof. Since the procedure of left fractions and that of inductive limits are commute to each other, we may assume from the first that N is finitely generated over $A(0)$.

Define $\bar{N} = N/A(-1)N$. Then $\bar{N}$ is a $gr_0(A)$-module. Let $(\bar{N})^{\sim}$

be the $gr_O(\tilde{A})$-Module on $Spec(gr_O(A))$ associated to $\bar{N}$. Since $Supp((\bar{N})^{\sim}) \subset Supp(\tilde{N})$, it follows that $Supp((\bar{N})^{\sim}) = \emptyset$. Then $\bar{N} = 0$ or equivalently, $N = A(-1)N$. We conclude from Lemma 2.7 that $N = 0$. q.e.d.

§3. Gabber's theorem on the characteristic variety.

The purpose of this section is to prove a theorem on MD-rings which is an algebraic version of Theorem 1.14.

3.1. A theorem on non-commutative rings.

Theorem 3.1 (O. Gabber [G]). Let B be a left and right Noetherian ring containing $\mathbb{Q}$ and satisfying the following conditions (i) and (ii):

(i) There exists an element $\omega \in B$ such that ω commutes with every element of B, that $\omega^2 = 0$ and that $\bar{B} = B/\omega B$ is a commutative local ring.

(ii) There exist $f, g \in B$ such that $\omega = [f, g]$ and that $f\omega B$ and $g\omega B$ are contained in the maximal ideal of $\bar{B}$.

Under the condition, if Q is a left B-module of finite length such that $\omega Q = \{ u \in Q; \omega u = 0 \}$, then $Q = 0$.

Proof. (1) Let $\rho: B \to \bar{B}$ be the canonical projection and let $\bar{\underline{m}}$ be the maximal ideal of $\bar{B}$. Define $\underline{m} = \rho^{-1}(\bar{\underline{m}})$. The assumption that $\bar{B}$ is commutative is equivalent to saying that $[B, B]$ is contained in the left and right ideal ωB of B. On the other hand, since $\omega^2 = 0$, ωB is contained in the center of B. In fact,

(3.1.1) $\quad [\omega B, B] \subset \omega[B, B] \subset \omega(\omega B) = 0.$

It follows from the assumption that $\bar{Q} \equiv Q/\omega Q$ is isomorphic to ωQ. Then we find that $\bar{Q}$ has the structure of $\bar{B}$-module of finite length.

(2) We now show that B is assumed to be left Artinian.

Since $\overline{Q}$ is a B-module, consider its annihilator ideal $I \equiv \mathrm{Ann}_B(\overline{Q}) = \{a \in B;\ a\overline{Q} = 0\}$ of B. On the other hand, since Q is a finitely generated B-module, so is $\overline{Q}$. Taking a system $\{u_1,\ldots,u_N\}$ of generators of $\overline{Q}$ over B, we define a mapping $B \ni a \to (au_i)_{i=1}^N \in \overline{Q}^N$. By (3.1.1), I is the kernel of this mapping and we obtain an injection $B/I \to \overline{Q}^N$. Since I is a bi-ideal of B, this is a left B/I-homomorphism. On the other hand, $\overline{Q}^N$ is of finite length over B and is also of finite length over B/I. Therefore B/I is left Artinian. This combined with that I/I^2 is finitely generated over B/I implies that I/I^2 is also Artinian. Then in virtue of the exact equence $0 \to I/I^2 \to B/I^2 \to B/I \to 0$ we find that B/I^2 is also Artinian. Since $IQ \subset wQ$, it follows that

$$I^2Q \subset IwQ \subset w^2Q = 0.$$

Hence Q is regarded as a left B/I^2-module. Replacing B with B/I^2, we may assume from the first that B is left Artinian.

(3) Before continuing the proof, we need a lemma.

<u>Lemma 3.2 (Cohen)</u>. Let C be a commutative Artinian local $\mathbb{Q}$-algebra and let $\underline{m}$ be its maximal ideal. Then there exists a subfield F of C such that $C = F \oplus \underline{m}$.

<u>Proof</u>. The totality of the subalgebras R of C such that $R \cap m = \{0\}$ is an inductively ordered set. Therefore there exists

a maximal element R of this set containing $\mathbb{Q}$ (by Zorn's lemma). Let $\pi: C \to C/\underline{m}$ be the natural projection. Since $R \cap \underline{m} = \{0\}$, $\pi|_R: R \to C/\underline{m}$ is injective. This implies that R is an integral domain.

(a) R is a field.

Take an $a \in R$ $(a \neq 0)$. Since $a \notin \underline{m}$, a is invertible in C. Consider the ring $R_a \equiv R[a^{-1}]$. Let $b = a^{-n}c \in R_a$ $(c \in R)$ and assume that $\pi(b) = 0$. Then $\pi(c) = 0$. This implies that $c = 0$ and therefore $b = 0$. Hence we find that $R_a \cap \underline{m} = \{0\}$. The maximality of R implies that $a^{-1} \in R_a = R$.

(b) $R + \underline{m} = C$.

First note that there exists an isomorphism $\pi: R \simeq \pi(R)$.

Let $x \in C$. Assume first that $\pi(x)$ is algebraically independent over the field $\pi(R)$. Then we have an isomorphism between the polynomial rings

$$R[x] \simeq \pi(R)[\pi(x)].$$

It follows from the maximal condition on R that $x \in R[x] = R$.

Let $x \in C$ and next assume that $\pi(x)$ is algebraic over $\pi(R)$. Let $f(X)$ be the minimal polynomial of $\pi(x)$ over $\pi(R)$:

$$f(X) = X^m + a_1X^{m-1} + \dots + a_m$$
$$a_1, \dots, a_m \in R.$$

(Here we idenitified a_i with $\pi(a_i)$.) Define inductively $x_k \in C$ $(k = 1,2,3,\dots)$ such that

$$(3.1.2)_k \quad \begin{array}{l} x_k \equiv x \pmod{m} \\ f(x_k) \equiv 0 \pmod{m^k}. \end{array}$$

First put $x_1 = x$. Assuming the existence of x_k satisfying $(3.1.2)_k$, we prove the existence of x_{k+1} satisfying $(3.1.3)_{k+1}$. Let $f'(X)$ be the differential of $f(X)$. Since R is of characteristic 0, it follows that $\pi(f'(x_k)) = f'(\pi(x_k)) = f'(\pi(x)) \neq 0$. Namely, $f'(x_k) \notin \underline{m}$ and therefore $f'(x_k)$ is invertible in C. Put $y = -f'(x_k)^{-1}f(x_k)$. Then by the hypothesis of induction, $f(x_k) \in \underline{m}^k$. Hence $y \in \underline{m}^k$ and $y^j \in \underline{m}^{k+1}$ $(j > 1)$. Now put $x_{k+1} = x_k + y$. Then $x_{k+1} \equiv x_k \equiv x \pmod{\underline{m}}$. Taking the Taylor expansion, we find that

$$f(x_{k+1}) \equiv f(x_k) + f'(x_k)y \equiv 0 \pmod{\underline{m}^{k+1}}.$$

We have thus shown the existence of x_{k+1} satisfying $(3.1.3)_{k+1}$. On the one hand, since C is an Artinian ring, it follows that $\underline{m}^k = 0$ for a sufficiently large k. Take such a k and fix it once for all. Then $f(x_k) = 0$ and we have $R(x_k) \simeq \pi(R)(\pi(x))$. Since $R \subset R(x_k)$, the maximality of R implies that $x_k \in R(x_k) = R$. Therefore $x \in R + \underline{m}$.

Putting $R = F$, we obtain the lemma. q.e.d.

(4) Since $\bar{B} = B/\omega B$ is a commutative Artinian local $\mathbb{Q}$-algebra, we apply Lemma 3.2 to $\bar{B}$. Then it follows that there exists a subfield F of $\bar{B}$ sch that $\bar{B} = F \oplus \bar{\underline{m}}$. Since $\bar{B}$ is Artinian, we find that $\bar{B}$, $\bar{Q}$ and $\bar{\underline{m}}^k/\bar{\underline{m}}^{k+1}$ are finite dimensional vector spaces

over F.

Take an F-basis $\bar{e}_1,\dots,\bar{e}_r$ of $\bar{Q}$ (we denote by $\bar{e}_i$ the class of $e_i \in Q$ in $\bar{Q}$) as follows:

$$r_1 < r_2 < \cdots < r_k = r, \quad \bar{\underline{m}}^k = 0$$

$$\bar{e}_{r_i+1},\dots, \bar{e}_{r_{i+1}} \in \bar{\underline{m}}^i\bar{Q} \text{ form an F-basis of } \bar{\underline{m}}^i\bar{Q}/\bar{\underline{m}}^{i+1}\bar{Q}$$
$$(o \leq i \leq k-1).$$

Then it follows that

$$\bar{\underline{m}}^i\bar{Q} = \sum_{j>r_i} F\bar{e}_j \quad (0 \leq i \leq k-1).$$

Now fix i and consider fe_i. By definition, there exist $F^0_{ji} \in B$ such that $f\bar{e}_i = \sum_j F^0_{ji}\bar{e}_j$. Then $fe_i - \sum_j F^0_{ji}e_j \in \omega Q$. As was noted before, ωQ is identified with $\bar{Q}$. Hence there exist $F^1_{ji} \in B$ such that $fe_i - \sum_j F^0_{ji}e_j = \sum_j F^1_{ji}\omega e_j$. By the same reason, there exist G^0_{ji}, $G^1_{ji} \in B$ such that $ge_i = \sum_j G^0_{ji}e_j + \sum_j G^1_{ji}\omega e_j$. Hence we obtain $r\times r$ matrices $F^0 = (F^0_{ji})$, $F^1 = (F^1_{ji})$, $G^0 = (G^0_{ji})$, $G^1 = (G^1_{ji})$. These matrices are so taken that the following conditions hold.

i) The entries of $\rho(F^0)$ $(= (\rho(F^0_{ji})))$, $\rho(F^1)$, $\rho(G^0)$, $\rho(G^1)$ are contained in F.

ii) F^0 and G^0 are lower triangular matrices whose diagonals are zero.

In fact, since $f, g \in \underline{m}$, if $r_\nu < i \leq r_{\nu+1}$, then $\bar{e}_i \in \bar{\underline{m}}^\nu\bar{Q}$ and $f\bar{e}_i \in \bar{\underline{m}}^{\nu+1}\bar{Q} = \sum_{j>r_{\nu+1}} F\bar{e}_j$. Therefore we may take

$$f\bar{e}_i = \sum_{j>r_{\nu+1}} \rho(F^0_{ji})\bar{e}_j, \quad \rho(F^0_{ji}) \in F,$$

$$F^0_{ji} = 0 \quad (j \leq r_{\nu+1}).$$

This implies that F^0 satisfies ii). Since $fe_i - \sum_j F^0_{ji}e_j \in wQ \simeq \bar{Q}$ and since we_j $(j = 1,\dots,r)$ form an F-basis of wQ, it follows that F^1 satisfies i).

By direct calculation, we find that

$$\begin{aligned} fge_i &= f(\sum_\nu G^0_{\nu i}e_\nu + \sum_\nu G^1_{\nu i}we_\nu) \\ &= \sum_\nu (G^0_{\nu i}fe_\nu + G^1_{\nu i}wfe_\nu + [f, G^0_{\nu i}]e_\nu) \\ &= \sum_{\nu,j}(G^0_{\nu i}F^0_{j\nu}e_j + G^0_{\nu i}F^1_{j\nu}we_j + G^1_{\nu i}F^0_{j\nu}we_j) + \sum_j [f, G^0_{ji}]e_j \\ &= \sum_{\nu,j}(G^0_{\nu i}F^0_{j\nu}e_j + wF^1_{j\nu}G^0_{\nu i}e_j + wF^0_{j\nu}G^1_{\nu i}e_j) + \sum_j [f, G^0_{ji}]e_j. \end{aligned}$$

(Here we used (3.1.1).). Now put

$$A_{ji} = \sum_\nu (G^0_{\nu i}F^0_{j\nu} - F^0_{\nu i}G^0_{j\nu})$$

$$[f, G^0_{ji}] - [g, F^0_{ji}] = wH_{ji}.$$

Note that (A_{ji}) and (H_{ji}) are lower triangular matrices whose diagonals are zero and that $\rho(A_{ij}) \in F$. Then it follows that

$$\begin{aligned} we_i &= fge_i - gfe_i \\ &= \sum_j A_{ji}e_j + \sum ([F^1, G^0] + [F^0, G^1] + H)_{ji}we_j. \end{aligned}$$

This implies that

$$\sum \rho(A_{ji})\bar{e}_j = 0.$$

Since $\{\bar{e}_i\}$ are linearly independent and since $\rho(A_{ji}) \in F$, we have $\rho(A_{ji}) = 0$. Then there exists an $A'_{ji} \in B$ such that $A_{ji} = wA'_{ji}$. Put $B = (B_{ji})$ with $B_{ji} = A'_{ji} + H_{ji}$. Then

$$we_i = \sum_j (B + [F^0, G^1] + [F^1, G^0])_{ji} e_j .$$

This combined with $\bar{Q} \simeq wQ$ implies that

$$e_i \equiv \sum_j (B + [F^0, G^1] + [F^1, G^0])_{ji} e_j \quad (\mathrm{mod}\ wQ).$$

Now we show that there exists a lower triangular matrix $B' = (B'_{ji})$ $(B'_{ji} \in F)$ such that $B'_{ii} = 0$ $(\forall i)$ and that

$$\sum_j B_{ji}\bar{e}_j = \sum_j B'_{ji}\bar{e}_j .$$

Let $r_\nu < i \leq r_{\nu+1}$. Then it follows from the definition of B_{ji} that

$$\sum_j B_{ji}\bar{e}_j = \sum_{\mu=\nu+1}^{k-1} \Big(\sum_{r_\mu < j \leq r_{\mu+1}} B_{ji}\bar{e}_j \Big),$$

where $B_{ji}\bar{e}_j \in \bar{\underline{m}}^{\mu}\bar{Q}$. On the other hand, since $\bar{e}_i \in \bar{\underline{m}}^{\nu}\bar{Q}$, it follows that

$$\sum_{\mu=\nu+1}^{k-1} \Big(\sum_{r_\mu < j < r_{\mu+1}} B_{ji}\bar{e}_j \Big) \in \sum_{\mu>\nu} \bar{m}^{\mu}\bar{Q} \subset \bar{m}^{\mu+1}\bar{Q} = \sum_{j>r_{\nu+1}} F\bar{e}_j .$$

This assures the existence of the matrix B' with the required properties.

From the definition, we find that

$$\bar{e}_i = \sum_j (B' + [\rho(F^0), \rho(G^1)] + [\rho(F^1), \rho(G^0)])_{ji}\bar{e}_j.$$

It follows from the freeness of the basis $(\bar{e}_j)$ that

$$1 = B' + [\rho(F^0), \rho(G^1)] + [\rho(F^1), \rho(G^0)].$$

Taking the traces of both sides of this equation, we find that $r = 0$. This is a contradiction. This means that $\bar{Q} = 0$, that is, $wQ = 0$. Since $\bar{Q} = Q/wQ$, we conclude that $Q = 0$. q.e.d.

3.2. Involutive subsets of Spech(gr(A)).

Let $(A, \{A(n)\})$ be an MD-ring and put $X = \mathrm{Spech}(\mathrm{gr}(A))$. Following Definition 1.6, we introduce the notion of an involutive closed subset of X.

Definition 3.3. Let Y be a closed subset of X. Then Y is called involutive if $\{I(Y), I(Y)\} \subseteq I(Y)$, where $I(Y)$ is the defining ideal of Y.

Let Y be a closed subset of X. Then define an increasing sequence $\{I_k\}_{k\geq 0}$ of homogeneous ideals of $\mathrm{gr}(A)$ as follows:

$$I_0 = I(Y), \quad I_{k+1} = \sqrt{I_k + \{I_k, I_k\}} \quad (k > 0).$$

Since $\mathrm{gr}(A)$ is Noetherian, this sequence is stationary and therefore there exists an integer $N > 0$ such that $I_n = I_N$ for any $n > N$. Define $I = I_N$. Then, by defnition, $\sqrt{I} = I$ and $\{I, I\} \subseteq I$. In particular, $V(I)$ is involutive. Since $V(I)$ depends only on Y, we denote it by $R(Y)$ in the sequel.

Lemma 3.4. The closed subset $R(Y)$ has the following properties.

(i) $R(Y)$ is involutive.

(ii) Let Z be an involutive closed subset of X contained in Y. Then Z is contained in $R(Y)$.

This lemma is clear from the arguments above.

The following theorem is fundamental in the subsequent discussions.

Theorem 3.5. Let A be an MD-ring. Let M be a finitely generated left A-module and let N be a sub-$A(0)$-module of M. Define

$$\Omega = \{\, \underline{p} \in X;\ \tilde{N} \text{ is finitely generated over } \tilde{A}(0) \text{ in a neighbourhood of } \underline{p} \,\}.$$

Then $Z = X-\Omega$ is an involutive closed subset of X.

Proof. It is clear from the definition that Z is a closed subset of X. Hence it suffices to show that Z is involutive.

Let $u_1, \ldots, u_r$ be a generators of M over A, that is, $M = \sum_j Au_j$. Define $L = \sum_j A(0)u_j$. Then clearly, $M = AL$.

Furthermore define $N(j) = N \cap A(j)L$. Then $\{N(j)\}_{j\in\mathbb{N}}$ is an increasing sequence of sub-$A(0)$-modules of N such that $N = \bigcup_j N(j)$.

The following two lemmas are easy to show.

Lemma 3.6. Let $\mathfrak{p}$ be a homogeneous prime ideal of $gr_0(A)$. Then $N_{\mathfrak{p}}$ is a finitely generated $A(0)_{\mathfrak{p}}$-module if and only if the sequence $\{N(j)_{\mathfrak{p}}\}$ is stationary.

Lemma 3.7. Let $w \in A(-1)$ be invertible. Then for every $j \in N$, the map $\ell_w : N(j)/N(j-1) \longrightarrow N(j-1)/N(j-2)$ defined by $u \to wu$ is an injective $gr_0(A)$-homomorphism.

We return to the proof of Theorem 3.5. Assuming that Z is not involutive, we lead a contradiction. Then by the assumption, there exist $f, g \in I(Z)$ such that $h = \{f, g\} \notin I(Z)$. We may assume that f and g are homogeneous. Furthermore we may also assume from the definition of an MD-ring that f and g are both homogeneous of degree 0. Hence it follows that $h \in gr_{-1}(A)$. By the assumption, there exists $\mathfrak{p} \in Z$ such that $h \notin \mathfrak{p}$. Replacing X with $D(h)$, Z with $Z \cap D(h)$, A with A_h, M with M_h and N with $(A_h)(0)\otimes_{A(0)}N$, we may assume that h is invertible in $gr(A)$. Hence we may assume from the first that there exist f, $g \in I(Z) \cap gr_0(A)$ such that $h = \{f, g\} \in gr_{-1}(A)$ is invertible.

Let $(N(j)/N(j-1))^{\sim}$ denote the $gr_0(\tilde{A})$-Module on $X_0 = \mathrm{Spec}(gr_0(A))$ associated to the $gr_0(A)$-module $N(j)/N(j-1)$ and define $Z_j = \mathrm{Supp}((N(j)/N(j-1))^{\sim})$. It follows from Lemma 3.7 that $Z_j \subset Z_{j-1}$ for every $j \in \mathbb{N}$. Hence $\{Z_j\}_{j\in\mathbb{N}}$ is a decreasing sequence of closed subsets of X_0. Since $gr_0(A)$ is Noetherian, this is stationary, namely, there exists an integer $j_0 > 0$ such that $Z_j = Z_{j_0}$ $(\forall j \geq j_0)$. On the other hand, we find that $Z = \bigcap_j Z_j$. In fact, it follows from Lemma 3.6 that $\underline{p} \in \Omega$ if and only if $(N(j)/N(j-1))_{\underline{p}} = 0$ $(j \geq \exists j_1)$. This is equivalent to saying that $\underline{p} \notin Z_j$ $(j \geq j_1)$.

Let $\underline{p}$ be a generic point of Z. This means that there exists a neighbourhood U of $\underline{p}$ such that $Z \cap U = \overline{\{\underline{p}\}} \cap U$. Now localize A, M and others at $\underline{p}$. Namely, we replace A with $A_{\underline{p}}$, M with $M_{\underline{p}}$, N with $(A_{\underline{p}})(0) \otimes_{A(0)} N$, L with $(A_{\underline{p}})(0) \otimes_{A(0)} L$, X with $\mathrm{Spec}(gr_0(A_{\underline{p}}))$ and Z with $\{\underline{p}\}$. Hence we proceed with the discussion by assuming that $Z = \{\underline{p}\}$ and that $gr_0(A)$ is a local ring with the maximal ideal $\underline{p}$.

In virtue of the above discussion, we assume that $\underline{p}$ is the unique closed point of X and that $\mathrm{Supp}((N(j)/N(j-1))^{\sim}) = \{\underline{p}\}$ $(j \geq j_0 > 0)$. Since $N(j)/N(j-1)$ is a finitely generated $gr_0(A)$-module, we find that $\underline{p}^r(N(j)/N(j-1)) = 0$ for some $r > 0$. Since $gr_0(A)$ is a local ring with the maximal ideal $\underline{p}$, it follows that $gr_0(A)/\underline{p}^r$ is a $gr_0(A)$-module of finite length. These imply that $N(j)/N(j-1)$ is of finite length as a $gr_0(A)$-module. Let r_j denote the length of $N(j)/N(j-1)$ as a $gr_0(A)$-module. Then it follows from Lemma 3.7 that $\{r_j;\ j \geq j_0\}$ is a decreasing sequence of positive integers and therefore is stationary. Hence the $gr_0(A)$-homomorphism $\ell_w: N(j)/N(j-1) \longrightarrow N(j-1)/N(j-2)$ is bijective for a sufficiently large j. Now take $j \gg 0$ such that this is actually bijective and fix it. Define $B = A(0)/A(-2)$, $\bar{w} \in B$ (the class of w), $Q = N(j)/N(j-2)$. Then the assumptions of Theorem 3.1 hold for B, $\bar{w}$, Q. In fact, we have the following:

(a) $\bar{\omega}^2 = 0$, $\bar{\omega}$ is contained in the center of B and $B/\bar{\omega}B \simeq gr_0(A)$ is a commutative local ring with the maximal ideal $\underline{m} = \underline{p}/\bar{\omega}B$.

(b) Let $\bar{f}$ and $\bar{g}$ be the classes of f and g in B, respectively. Then $\bar{f}, \bar{g} \in \underline{p}$ and $[\bar{f}, \bar{g}] = \bar{\omega}$.

(c) $\mathrm{Ker}(Q \xrightarrow{\bar{\omega};} Q) = \bar{\omega}Q$.

( ) $\mathrm{Ker}(Q \xrightarrow{\bar{\omega};} Q) \simeq \{u \in N(j);\ \omega u \in N(j-2)\}/N(j-2)$

$$\simeq N(j-1)/N(j-2).$$

On the other hand,

$$\bar{\omega}Q = (\omega N(j)+N(j-2))/N(j-2) = N(j-1)/N(j-2).$$

Hence applying Theorem 3.1, we conclude that $Q = 0$. This means that $N(j)/N(j-1) = 0$ and therefore contradicts that $\mathrm{Supp}((N(j)/N(j-1))^{\sim}) = \{\underline{p}\}$.

We have thus shown that Z is involutive. q.e.d.

Corollary 3.8. Let A be an MD-ring. Let M be a finitely generated left A-module and let N be a finitely generated sub-$A(0)$-module of M. Let Z be a closed subset of X such that $R(Z) = \emptyset$. Define

$$N' = \{\ u \in M;\ \phi_{\underline{p}}(u) \in (A_{\underline{p}})(0)\otimes_{A(0)}N \text{ for any } \underline{p} \in X-Z\ \}.$$

Here $\phi_{\underline{p}}$ is the natural homomorphism of M to $M_{\underline{p}}$. Then N' is a finitely generated $A(0)$-module.

Proof. As in Theorem 3.5, we define a subset Ω of X by

$$\Omega = \{\ \underline{p} \in X;\ \exists\, f \in gr_0(A) \text{ s.t. } (1)\ \underline{p} \in D(f)$$

(2) $(N')_f$ is a finitely generated left $(A_f)(0)$-module }.

We now show that $\Omega = X$. In fact, if $\underline{p} \in X-Z$, then it is clear that $N'_{\underline{p}(0)} = N_{\underline{p}(0)}$, where $\underline{p}(0) = \underline{p} \cap gr_0(A)$. This implies that $\underline{p} \in \Omega$. Hence $X-\Omega$ is contained in Z. Theorem 3.5 shows that $X-\Omega$ is involutive. But $R(Z) = \emptyset$. Hence, in virtue of Lemma 3.4, we find that $\Omega = X$.

Since X is quasi-compact and since $X = \Omega$, we may assume that there exist a finite number of elements of $gr_0(A)$, say $f_1,\ldots,f_r$ such that $X = \bigcup_{i=1}^{r} D(f_i)$ and that $(N')_{f_i}$ is a finitely generated left $(A_{f_i})(0)$-module for each i. Furthermore we also assume that there exist $u_1,\ldots,u_m \in N'$ such that $(N')_{f_i} = \sum_{k=1}^{m} (A_{f_i})(0)u_k$. Now define $N'' = \sum_k A(0)u_k$. Then N'' is a finitely generated sub-$A(0)$-module of N'. On the other hand, from the assumption, we have $(N'/N'')_{f_i} = 0$ $(1 \leq i \leq r)$. Then it follows from Proposition 2.56 that $N' = N''$.

We have thus shown Corollary 3.8. q.e.d.

§4. The characteristic variety of a coherent $\underline{E}_X$-Module.

In this section, we complete the proof of Theorem 1.1 and discuss the properties of the characteristic variety a little more. The notation introduced in §1 are used without any comment.

4.1. An extension theorem for a coherent $\underline{E}_X$-Module.

Throughout this section, Ω denotes an open subset of $\mathring{T}^*X$ unless otherwise stated.

Lemma 4.1. Let $\underline{M}$ be a coherent $\underline{E}_X$-Module defined on Ω. Assume that there exists a system $\{u_1, \ldots, u_m\}$ of generators of $\underline{M}$ over $\underline{E}_X$ defined on Ω. Define $\underline{M}(k) = \sum_{i=1}^{m} \underline{E}_X(k)u_i$ for each $k \in \mathbb{Z}$. Then $\mathrm{Supp}(\underline{M}) = \mathrm{Supp}(\underline{M}(0)/\underline{M}(-1))$.

Proof. It is clear from the definition that $\mathrm{Supp}(\underline{M}) \supset \mathrm{Supp}(\underline{M}(0)/\underline{M}(-1))$. We are going to prove the converse inclusion. Let $p \notin \mathrm{Supp}(\underline{M}(0)/\underline{M}(-1))$. Since $\underline{M}(-1) = \underline{E}_X(-1)\underline{M}(0)$, it follows that $\underline{E}_X(-1)_p\underline{M}(0)_p = \underline{M}(-1)_p \subset \underline{M}(0)_p$. Then Nakayama's Lemma implies that $\underline{M}(0)_p = 0$. Since $\underline{M} = \underline{E}_X\underline{M}(0)$, we conclude that $\underline{M}_p = 0$, whence $p \notin \mathrm{Supp}(\underline{M})$. q.e.d.

This lemma implies the following.

Proposition 4.2. Let $\underline{M}$ be a coherent $\underline{E}_X$-Module defined on Ω. Then its support $\mathrm{Supp}(\underline{M})$ is a closed analytic subset of Ω.

Proof. The question being local, we may assume from the first

that $\underline{M}$ is generated by sections of $\Gamma(\Omega, \underline{M})$, say, $\underline{M} = \sum_{i=1}^{m} \underline{E}_X u$ holds on Ω for some $u_1, \ldots, u_m \in \Gamma(\Omega, \underline{M})$. Then $\underline{M}(0) = \sum_{i=1}^{m} \underline{E}_X(0)u_i$ is coherent over $\underline{E}_X(0)$. Furthermore, $\underline{M}(0)/\underline{M}(-1)$ is coherent over $\underline{O}_{T^*X}$. Hence $\mathrm{Supp}(\underline{M}(0)/\underline{M}(-1))$ is a closed analytic subset of Ω. Then Lemma 4.1 implies the proposition.
q.e.d.

Definition 4.3. Let $\underline{M}$ be a coherent $\underline{E}_X$-Module defined on Ω. Then a coherent sub-$\underline{E}_X(0)$-Moudle $\underline{L}$ of $\underline{M}$ is called a lattice of $\underline{M}$ if $\underline{M} = \underline{E}_X\underline{L}$ holds on Ω.

The next theorem due to O. Gabber plays a fundamental role in the proof of Theorem 1.1 as well as in the definition of a system with regular singularities ~~which will be developed in the subsequent sections.~~ (cf. §1) Before state the theorem, we need a preparation. Let Y be a closed analytic subset of Ω. Then Y is called homogeneous if there exists a homogeneous closed analytic subset $\tilde{Y}$ of $\mathring{T}^*X$ such that $Y = \tilde{Y} \cap \Omega$.

Theorem 4.4 (O. Gabber). Let Y be a homogeneous closed analytic subset of Ω. Let $j: \Omega - Y \hookrightarrow \Omega$ be the natural projection. Assume that there exist $f, g \in \Gamma(\Omega, \underline{O}_{T^*X}(0))$ such that $f|Y = g|Y = 0$ but $\{f, g\}(p) \neq 0$ for any $p \in Y$.

Let $\underline{M}$ be a coherent $(\underline{E}_X|\Omega)$-Module and let $\underline{N}$ be a lattice of $\underline{M}$. Then $\underline{M} \cap j_* j^{-1}\underline{N}$ is a coherent $\underline{E}_X(0)$-Module on Ω.

Proof. (1) Define $\underline{N}' = \underline{M} \cap j_* j^{-1}\underline{N}$. Then it is easy to show that $E_X(0)$ acts on $\underline{N}'$ and therefore $\underline{N}'$ is an $\underline{E}_X(0)$-Module.

Furthermore, define an increasing sequence $\{\underline{N}_k;\ k = 0,1,2,\ldots\}$ of left $\underline{E}_X(0)$-Modules, inductively, as follows:

$$\underline{N}_0 = \underline{N},\quad \underline{N}_k = \underline{E}_X(1)\underline{N}_{k-1} \cap j_* j^{-1}\underline{N}_{k-1} \quad (\ k > 0).$$

Note that $\underline{N}_k = \underline{N}$ on $\Omega - Y$ for every $k > 0$.

(2) Each $\underline{N}_k$ is a coherent $\underline{E}_X(0)$-Module.

<u>Proof</u>. We prove this by the induction on k. It is clear from the definition that $\underline{N}_0$ is coherent over $\underline{E}_X(0)$. Hence assuming that $\underline{N}_k$ is coherent over $\underline{E}_X(0)$, we show that so is $\underline{N}_{k+1}$. Since

$$0 \longrightarrow \underline{N}_k \longrightarrow \underline{N}_{k+1} \longrightarrow \underline{N}_{k+1}/\underline{N}_k \longrightarrow 0$$

is an exact sequence of $\underline{E}_X(0)$-Modules, it suffices to show that $\underline{N}_{k+1}/\underline{N}_k$ is coherent over $\underline{E}_X(0)$, or equivalently, is coherent over $\underline{O}_{T^*X}(0)$. By definition, we have

$$\underline{N}_{k+1}/\underline{N}_k = \{\ u \in \underline{E}_X(1)\underline{N}_k/\underline{N}_k;\ \mathrm{supp}\ u \subset Y\ \}.$$

Let $\underline{I}$ be the defining Ideal of Y in $\underline{O}_{T^*X}$. Then it follows from the Nullstellen Satz of Hilbert that

$$(4.1.1)\qquad \underline{N}_{k+1}/\underline{N}_k = \bigcup_{d=0}^{\infty} \underline{E}_d,$$

where $\underline{E}_d = \{\ u \in \underline{E}_X(1)\underline{N}_k/\underline{N}_k;\ (I\cap\underline{O}_{T^*X}(0))^d u = 0\ \}$. Since $\underline{O}_{T^*X}(0)$ is a Noetherian sheaf of rings and since each $\underline{E}_d$ is coherent over $\underline{\underline{O}}_{T^*X}(0)$, it follows that the increasing sequence $\{\underline{\underline{E}}_d\}$ is locally stationary. This combined with (4.1.1) implies that $\underline{\underline{N}}_{k+1}/\underline{\underline{N}}_k$ is coherent over $\underline{\underline{O}}_{T^*X}(0)$ on Ω. Hence we conclude that $\underline{\underline{N}}_{k+1}$ is coherent over $\underline{\underline{E}}_X(0)$. q.e.d.

(3) Define $S_k = \mathrm{Supp}(\underline{\underline{N}}_k/\underline{\underline{N}}_{k-1})$. Since each $\underline{\underline{N}}_k/\underline{\underline{N}}_{k-1}$ is coherent over $\underline{\underline{O}}_{T^*X}(0)$, it follows that $\{S_k;\ k = 1,2,\dots\}$ is a sequence of closed analytic subsets of Ω and that each S_k is contained in Y. On the other hand, it is clear from the definition that $S_k \subseteq S_{k-1}$. Hence $\{S_k\}$ is locally stationary. This implies that $S = \bigcap_{k=1}^{\infty} S_k$ is a closed analytic subset of Ω. Since $S \subset Y$, the definition of Y implies that S does not contain any involutive closed subset.

(4) $S = \emptyset$ or equivalently, the increasing sequence $\{\underline{\underline{N}}_k\}$ is locally stationary.

<u>Proof</u>. Assuming that $S \neq \emptyset$, we lead a contradiction. Thus take a $p \in S$ and fix it once for all.

Now put $A = \underline{\underline{E}}_{X,p}$, $A(m) = \underline{\underline{E}}_X(m)_p$, $M = \underline{\underline{M}}_p$, $N = \underline{\underline{N}}_p$, $N_k = \underline{\underline{N}}_{k,p}$ ($\forall\, k \in \mathbb{N}$) and $I = \underline{\underline{I}}_Y$, where $\underline{\underline{I}}$ is the defining Ideal of Y in $\underline{\underline{O}}_{T^*X}$. As was noted before, A is an MD-ring, M is a finitely generated left A-module and N is a finitely generated $A(0)$-module contained in M. Put $Z = V(I)$. Since the germs of f and g at p are contained in I but $\{f, g\}$ is invertible, it follows that $R(Z) = \emptyset$. On the other hand, it follows that $\{N_k\}$ is an increasing sequence of finitely generated left $A(0)$-modules.

We are going to show that each N_k is contained in the left

A(O)-module N' defined by

$$N' = \{u \in M;\ \phi_{\underline{p}}(u) \in (A_{\underline{p}})(0)N \text{ for any } \underline{p} \in \mathrm{Spech}(gr(A))-Z\},$$

where $\phi_{\underline{p}}$ is the homomorphism of M to $M_{\underline{p}}$ defined by $u \to 1^{-1}u$ for each $u \in M$. For this purpose, define a left Ideal $\underline{\underline{J}}$ of $\underline{\underline{E}}_X(0)|\Omega$ by $\underline{\underline{J}} = \{ P \in \underline{\underline{E}}_X(0)|\Omega;\ \sigma_0(P) \in \underline{\underline{I}} \cap \underline{\underline{O}}_{T^*X}(0)\}$ and also define $J = \underline{\underline{J}}_p$. Since Y is homogeneous, it follows that $\sigma_0(\underline{\underline{J}}) = \underline{\underline{I}} \cap (\underline{\underline{O}}_{T^*X}(0)|\Omega)$ and that I is a homogeneous ideal of A. On the other hand, it follows from the definition of N_k that $u \in N_k$ if and only if $u \in A(1)N_{k-1}$ and $J^m u \in N_{k-1}$ for some $m \in N_+$. Let $\underline{p} \in \mathrm{Spech}(gr(A))$ be such that $\underline{p} \notin Z$. Then there exists a homogeneous element h of I such that $h \notin \underline{p}$. Take a $P \in J$ such that $\sigma(P) = h$. Then for each $u \in N_k$, it follows that $P^m u \in N_{k-1}$ for some $m \in \mathbb{N}_+$. This implies that $u \in (N_{k-1})_{\underline{p}} = N_{\underline{p}}$. Hence it follows from the definition of N' that u is contained in N'.

Since $R(Z) = \emptyset$, it follows from Corollary 3.8 that N' is a finitely generated left A(O)-module. Since $\{N_k\}$ is an increasing sequence, this implies that this sequence is stationary. Then $N_{k-1} = N_k$ for a sufficiently large k. Hence $(\underline{\underline{N}}_k/\underline{\underline{N}}_{k-1})_p = 0$, or equivalently, $p \notin S_k$ for $k >> 0$. This contradicts the assumption that $p \in S$.

(5) Define $\underline{\underline{N}}'' = \bigcup_{k=0}^{\infty} \underline{\underline{N}}_k$. Then it follows from (4) that $\underline{\underline{N}}''$ is a coherent $\underline{\underline{E}}_X(0)$-Module. On the other hand, the equality $\underline{\underline{N}}'' = \underline{\underline{E}}_X(1)\underline{\underline{N}}'' \cap j_* j^{-1}\underline{\underline{N}}''$ follows from the definition of $\underline{\underline{N}}_k$.

(6) $\underline{\underline{N}}'' = \underline{\underline{N}}'$.

Proof. It is clear from the definition that $\underline{N}'' \subseteq \underline{N}'$. Hence it suffices to show that $\underline{N}' \subseteq \underline{N}''$. Since $\underline{N} \subseteq \underline{N}''$ and since $\underline{M} = \underline{E}_X\underline{N}$, it follows that $\underline{M} = \underline{E}_X\underline{N}'' = \bigcup_k \underline{E}_X(k)\underline{N}''$. This implies that $\underline{N}' = \bigcup_{k \geq 0} (\underline{N}' \cap \underline{E}_X(k)\underline{N}'')$. Therefore if we show that $\underline{N}' \cap \underline{E}_X(k)\underline{N}'' = \underline{N}' \cap \underline{E}_X(k-1)\underline{N}''$ for every $k \geq 1$, we conclude that $\underline{N}' = \underline{N}' \cap \underline{E}_X(0)\underline{N}'' \subseteq \underline{N}''$. Take $p \in \Omega$ and fix it once for all. Assume that $k \geq 1$ and take $u \in (\underline{N}' \cap \underline{E}_X(k)\underline{N}'')_p$. We may assume that u is a section on a small neighbourhood U of p. It follows from [SKK] that there exists $P \in \Gamma(U, \underline{E}_X(1))$ which is invertible at p. Then $P^{1-k}u \in \Gamma(U, \underline{E}_X(1)\underline{N}'')$. On the other hand, since $u|(U-Y) \in \Gamma(U-Y, \underline{N})$ and since $P^{1-k} \in \underline{E}_X(0)$, it follows that $(P^{1-k}u)|(U-Y) \in \Gamma(U-Y, \underline{N})$. These combined with (5) imply that $P^{1-k}u \in \underline{N}''_p$. Hence $u \in (\underline{E}_X(k-1)\underline{N}'')_p$ and therefore $(\underline{N}' \cap \underline{E}_X(k)\underline{N}'')_p \subseteq (\underline{N}' \cap \underline{E}_X(k-1)\underline{N}'')_p$. This holds for every $p \in \Omega$. Hence $\underline{N}' \cap \underline{E}_X(k)\underline{N}'' \subseteq \underline{N}' \cap \underline{E}_X(k-1)\underline{N}''$. We have thus shown that $\underline{N}' \subseteq \underline{N}''$ and therefore that $\underline{N}' = \underline{N}''$.

(7) It follows from (5) and (6) that $\underline{N}'$ is a coherent $\underline{E}_X(0)$-Module.

We have thus proved the theorem completely. q.e.d.

4.2. Proof of Theorem 1.1

Theorem 1.1 is a consequence of Theorem 4.4.

Theorem 1.1 (repeat). Let $\underline{M}$ be a coherent $\underline{E}_X$-Module defined on Ω. Then its support $\mathrm{Supp}(\underline{M})$ is an involutive closed analytic subset of Ω.

Proof. Define $Y = Ch(\underline{M})$. Then it follows from Proposition 4.2 that Y is a closed analytic subset of Ω.

Assuming that Y is not involutive, we lead a contradiction. Then there exist $p \in Y$ and holomorphic functions f, g defined in a neighbourhood of p such that $f|Y = g|Y = 0$ but $\{f, g\}(p) \neq 0$. Then replacing Ω with a small neighbourhood of p, we may assume from the first that $\{f, g\}(q) \neq 0$ for any $q \in \Omega$. We may also assume that there exists a lattice $\underline{L}$ of $\underline{M}$ defined on Ω. If $j: \Omega \setminus Y \hookrightarrow \Omega$ is the natural inclusion, it is clear from the definition that $\underline{M} \cap j_* j^{-1} \underline{L}$ coincides with $\underline{M}$ itself. Then Theorem 4.4 implies that $\underline{M}$ is a coherent $\underline{E}_X(0)$-Module. Take $q \in Y$ and fix it once for all. Noting that Y is homogeneous, we find from [SKK] that there exists an invertible element $E \in \underline{E}_X(-1)_q$. Hence $\underline{M}_q = EE^{-1}\underline{M}_q \subset E\underline{M}_q \subset \underline{E}_X(-1)_q \underline{M}_q$. Then Nakayama's Lemma implies that $\underline{M}_q = 0$. This contadicts that $q \in Y = Ch(\underline{M})$. q.e.d.

Theorem 4.4 is slightly generalized as follows.

Theorem 4.5 (O. Gabber). Let Y be a homogeneous closed analytic subset of Ω. Let $j: \Omega - Y \hookrightarrow \Omega$ be the natural inclusion. Assume that for each $p \in Y$, there exist holomorphic functions f, g defined in a neighbourhood of p such that $f|Y = g|Y = 0$ but $\{f, g\}(p) \neq 0$.

Let $\underline{M}$ be a coherent $\underline{E}_X$-Module defined on Ω and let $\underline{N}$ be a coherent sub-$\underline{E}_X(0)$-Module of $\underline{M}$ defined on Ω. Then

$\underline{N}' = \underline{M} \cap j_* j^{-1} \underline{N}$ is also a coherent $\underline{E}_X(0)$-Module.

Proof. The question being local, we may replace Ω with a small open subset of it if necessary. Take a $p \in Y$ and fix it once for all. Then we may assume that there exist holomorphic functions f, g defined on Ω such that $f|Y = g|Y = 0$ but $\{f, g\}$ is invertible on Ω.

We show that $\underline{N}' \subset \underline{E}_X \underline{N}$ on Ω. In fact, take an element $u \in \underline{N}'$. Then we may assume that $u \in \Gamma(U, \underline{M})$ for some open subset U of Ω. Since the $\underline{E}_X$-Module $\underline{L} = (\underline{E}_X u + \underline{E}_X \underline{N})/\underline{E}_X \underline{N}$ defined on U is coherent over $\underline{E}_X$, it follows from Theorem 1.1 (proved just before!) that $\mathrm{Supp}(\underline{L})$ is involutive. On the other hand, since $u|(U-Y)$ is contained in $\Gamma(U-Y, \underline{N})$, it follows that $\mathrm{Supp}(\underline{L}) \subset U \cap Y$. Noting that Y does not contain any involutive subset, we find that $\mathrm{Supp}(\underline{L}) = \emptyset$, or equivalently that $\underline{E}_X u + \underline{E}_X \underline{N} = \underline{E}_X \underline{N}$ on U. Hence $u \in \underline{E}_X \underline{N}$ and we conclude that $\underline{N}' \subset \underline{E}_X \underline{N}_X$.

From the discussion above, we may replace $\underline{M}$ with the sub-$\underline{E}_X$-Module $\underline{E}_X \underline{N}$ of it. Then $\underline{N}$ is a lattice of $\underline{M}$. We find from Theorem 4.4 that $\underline{N}'$ is coherent over $\underline{E}_X(0)$. q.e.d.

References

[B] N. Bourbaki; Commutative Algebra.

[G] O. Gabber; The integrability of characteristic variety, Amer. J. Math., 103 (1981), 445-468.

[H] R. Hartshorne; Algebraic Geometry, GTM, Springer Verlag.

[KK1] M. Kashiwara and T. Kawai: On holonomic systems of micro-differential equations III - Systems with regular singularities - Publ. RIMS, Kyoto Univ., 17 (1981) 813-979.

[KK2] M. Kashiwara and T. Kawai: On holonomic systems with regular singularities, in "Seminar on Micro-local Analysis", Ann. of Math. Studies, Princeton Univ. Press.

[N] M. Noumi: Holonomic systems and their minimal extensions I, Advanced Studies in Pure Math., 4 (1984), 209-221.

[SKK] M. Sato, M. Kashiwara and T. Kawai: Microfunctions and pseudo-differential equations, Lecture Notes in Math., 287, Springer Verlag, 264-524.

[St] B. Stenström: Rings of Quotients, Springer-Verlag.