数理解析研究所講究録
670 巻 1988 年 129-162

The Kazhdan-Lusztig polynomials arising in the modular representation theory of reductive algebraic groups

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1. Lusztig's conjecture. A prime objective of the modular representation theory of reductive algebraic groups is to find a character formula for their simple modules. All the modules considered in this survey are rational.

(1.1) Let G be a simply connected simple algebraic group over an algebraically closed field K of characteristic $p > 0$ split over $\mathbb{F}_p$. Let B be a split Borel subgroup of G, T a split maximal torus of B, and F the Frobenius endomorphism of (G, B, T). We denote by R the root system of G relative to T, by R^+ the positive system of R determined by B, by Δ the simple system of R^+, and put $X(T) = \mathrm{Hom}(T, GL_1)$. We write the group operation on $X(T)$ additively:

$$(\lambda + \mu)(t) = \lambda(t)\mu(t) \quad \forall \lambda, \mu \in X(T) \text{ and } t \in T,$$

and define a partial order $\geq$ on $X(T)$ by

$$\lambda \geq \mu \quad \text{iff} \quad \lambda - \mu \in \mathbb{Z}R^+.$$

For a T-module M, as T is diagonizable, M admits the weight space decomposition:

(1) $$M = \coprod_{\lambda \in X(T)} M_\lambda \quad \text{with} \quad M_\lambda = \{m \in M \mid tm = \lambda(t)m \quad \forall t \in T\}.$$

We call $\lambda \in X(T)$ a weight of M iff $M_\lambda \neq 0$.

Let $\mathbb{Z}[X(T)]$ be the group algebra of $X(T)$ over $\mathbb{Z}$ with a natural basis $e(\lambda)$, $\lambda \in X(T)$. For a finite dimensional T-module M, we put

(2) $$\mathrm{ch}\, M = \sum_{\lambda \in X(T)} \dim M_\lambda \; e(\lambda) \in \mathbb{Z}[X(T)]$$

and call it the (formal) character of M.

For each $\alpha \in R^+$ let $\alpha^\vee$ be its coroot and put $X(T)^+ = \{ \lambda \in X(T) \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \;\; \forall \alpha \in \Delta \}$. The simple G-modules are parametrized by $X(T)^+$:

(3) $$X(T)^+ \ni \lambda \longrightarrow L(\lambda) \text{ the simple } G\text{-module of highest weight } \lambda.$$

Thus we are after $\mathrm{ch}\, L(\lambda)$ $\forall \lambda \in X(T)^+$.

(1.2) Let $X_1(T) = \{ \nu \in X(T)^+ \mid \langle \nu, \alpha^\vee \rangle < p \;\; \forall \alpha \in \Delta \}$. For each $\lambda \in X(T)$ write

(1) $$\lambda = \sum_{i \geq 0} p^i \lambda^i, \quad \lambda^i \in X_1(T).$$

Then

<u>Steinberg's tensor product theorem</u> (cf. [11],(II.3.17)).

$$L(\lambda) = \bigotimes_{i\geq 0} L(\lambda^i)^{[i]},$$

where $M^{[i]}$ for a G-module M is the i-th Frobenius twist of M obtained from M by composing the i-th power of F : $G \xrightarrow{F^i} G \longrightarrow GL(M)$, says we have only to find ch $L(\lambda)$ $\forall\lambda \in X_1(T)$.

(1.3) For a B-module M define a sheaf $\mathcal{L}_{G/B}(M)$ on G/B by

(1) $\quad \forall V \in \mathrm{Top}(G/B), \quad \Gamma(V, \mathcal{L}_{G/B}(M)) =$

$$\{f \in \mathrm{Mor}(\pi^{-1}V, M) \mid f(bx) = b^{-1}f(x)\ \forall b \in B,\ x \in \pi^{-1}V\},$$

where $\pi : G \longrightarrow G/B$ is the natural projection. It is a quasi-coherent G-linearized sheaf, so each i-th cohomology $H^i(G/B, \mathcal{L}_{G/B}(M))$ comes equipped with the structure of a G-module. Let U be the unipotent radical of B. For each $\lambda \in X(T)$ we may regard the 1-dimensional T-module K_λ with weight λ as a B-module through the natural projection $B = T \ltimes U \longrightarrow T$. We often abbreviate $H^i(G/B, \mathcal{L}_{G/B}(K_\lambda))$ as $H^i(\lambda)$. Put

(2) $$\chi(\lambda) = \sum_{i\geq 0} (-1)^i \mathrm{ch}\, H^i(\lambda).$$

As usual, the alternating sum of ch $H^i(\lambda)$ is easy to find. Let $W = N_G(T)/T$ the Weyl group of G. With the set S of simple reflections, (W, S) forms a Coxeter system. Let $\ell : W \longrightarrow \mathbb{N}$ be the length function relative to S. We regard W as acting on $E = X(T) \otimes \mathbb{R}$ from the right.

Besides the usual action we introduce the dot action of W on E :

(3) $$v\cdot w = (v+\rho)w - \rho, \quad v \in E,\ w \in W,$$

where $\rho \in X(T)$ with $\langle \rho, \alpha^{\vee}\rangle = 1$ $\forall \alpha \in \Delta$.

<u>Weyl's character formula</u> (cf. [11],(II.5.10)). $\forall \lambda \in X(T)$,

$$\chi(\lambda) = \frac{\sum_{w\in W} (-1)^{\ell(w)} e(\lambda\cdot w)}{\sum_{w\in W} (-1)^{\ell(w)} e(0\cdot w)} .$$

Moreover, we have

<u>Kempf's vanishing theorem</u> (cf. [11],(II.4.5)). $\forall \lambda \in X(T)^{+}-\rho$ and $i \geq 0$, $H^{i}(\lambda) = 0$. In particular,

$$\text{ch } H^{0}(\lambda) = \frac{\sum_{w\in W} (-1)^{\ell(w)} e(\lambda\cdot w)}{\sum_{w\in W} (-1)^{\ell(w)} e(0\cdot w)} .$$

We also know (cf. [11],(II.2.4)) that $\forall \lambda \in X(T)^{+}$,

(4) $$\text{soc } H^{0}(\lambda) = L(\lambda)$$

(5) $$[H^{0}(\lambda) : L(\lambda)] = 1,$$

where [:] denotes the multiplicity of the second term in a composition series of the first.

(1.4) It has long been recognized that not the Weyl group W but the affine Weyl group $W_p = W \ltimes p\mathbb{Z}R$ plays a more important role in the representation theory of G, where $p\mathbb{Z}R$ consists of the translations t_γ by $\gamma \in p\mathbb{Z}R$. Under the dot action W_p is generated by the reflexions $s_{\alpha,n}$, $\alpha \in R$, $n \in \mathbb{Z}$, in the hyperplanes $H_{\alpha,n} = \{v \in E \mid \langle v+\rho, \alpha^\vee\rangle = np\}$. We will abbreviate $s_{\alpha,0}$ as s_α. Put $S_p = S \cup \{s_{\alpha_0,-1}\}$, where α_0 is the highest short root of R^+. Then (W_p, S_p) forms a Coxeter system with a subsystem $(W, S=\{s_\alpha|\alpha\in\Delta\})$. We extend the length function on (W, S) to one on (W_p, S_p), still denoted by ℓ.

We say λ is strongly linked to μ and write $\lambda \uparrow\uparrow \mu$, $\lambda, \mu \in X(T)$, iff there is a sequence of reflections $s_{\alpha_1,n_1}, \ldots, s_{\alpha_r,n_r}$ in W_p such that $\lambda \leq \lambda\cdot s_{\alpha_1,n_1} \leq \ldots \leq \lambda\cdot s_{\alpha_1,n_1}\cdots s_{\alpha_r,n_r} = \mu$.

Andersen's strong linkage principle (cf. [11],(II.6.13)). Let $\lambda \in X(T)^+-\rho$ and $\eta \in X(T)^+$. If $[H^i(\lambda\cdot w) : L(\eta)] \neq 0$ for some $i \geq 0$ and $w \in W$, then $\eta \uparrow\uparrow \lambda$.

(1.5) In analogy to the Kazhdan-Lusztig conjecture for the irreducible character formula of the complex simple Lie algebra (cf. [23] for a survey) G.Lusztig proposed a conjecture expressing ch $L(\lambda)$ in terms of various ch $H^0(\mu)$'s.

His strategy exploits another reduction of the problem. A connected component of $E \setminus \bigcup_{\alpha\in R, n\in\mathbb{Z}} H_{\alpha,n}$ is called an alcove. Let $\mathcal{A}$ be the set of alcoves on E. The affine Weyl group W_p permutes $\mathcal{A}$ simply and transitively. We will abbreviate its action $A\cdot w$ as Aw for $A \in \mathcal{A}$, $w \in W_p$. Note also that each translation t_γ by $\gamma \in pX(T)$ preserves $\mathcal{A}$.

Let $H^{\pm}_{\alpha,n} = \{v \in E \mid \langle v+\rho, \alpha^{\vee}\rangle > np\}$ and define a "distance" function $d : \mathscr{A} \times \mathscr{A} \longrightarrow \mathbb{Z}$ by

(1) $$d(A, B) = \#\{H_{\alpha,n} \text{ separating } A \text{ and } B \mid H^{-}_{\alpha,n} \supset A\} - \#\{H_{\alpha,n} \text{ separating } A \text{ and } B \mid H^{+}_{\alpha,n} \supset A\}.$$

From now on assume $p \geq h = \langle \rho, \alpha_0\rangle + 1$ the Coxeter number of G so that each alcove may contain an element of $X(T)$. Let A^{+} (resp. A^{-}) be the alcove containing 0 (resp. $0 \cdot w_0 = -2\rho$, where w_0 is the longest element of W). For each $A \in \mathscr{A}$ let 0_A be the image of 0 in A under W_p and let $\mathscr{A}^{+} = \{A \in \mathscr{A} \mid 0_A \in X(T)^{+}\}$, $\mathscr{A}^{-} = \mathscr{A}^{+}w_0$.

It is known (Jantzen's translation principle, cf. [11],(II.7)) that each ch $L(\lambda)$ can be obtained from ch $L(0_A)$ for a suitable $A \in \mathscr{A}$, and we are now ready to state

<u>Lusztig's conjecture</u> ([20], Problem IV). $\forall C \in \mathscr{A}$ with 0_C satisfying the Jantzen condition

(2) $$\langle 0_C+\rho, \alpha_0\rangle < p(p-h+2),$$

one shoud have

$$\text{ch } L(0_C) = \sum_{A\in\mathscr{A}} (-1)^{d(A,C)} P_{A,C}(1) \text{ch } H^0(0_A).$$

Here $P_{A,C} = P_{y,w}$ with $y, w \in W_p$ such that $A = A^{-}y$ and $C = A^{-}w$ are Kazhdan-Lusztig polynomials for the Coxeter system (W_p, S_p). It is known that the coefficients of $P_{y,w}$, $y, w \in W_p$, account for the dimensions of the hypercohomology of Deligne's complex of ℓ-adic sheaves on a certain variety (Kazhdan-Lusztig [19]), so they are nonnegative. Also (Kazhdan-Lusztig [18],(2.6))

(3) $$P_{y,w}(0) = 1 \quad \forall y \leq w .$$

(1.6) In this subsection we let (W, S) denote an arbitrary Coxeter system. The Kazhdan-Lusztig polynomials for (W, S) were introduced in the study of the representations of the Hecke-Iwahori algebra $\mathcal{H}$ associated to (W, S).

Let q be an indeterminate. The algebra $\mathcal{H}$ is a free $\mathbb{Z}[q,q^{-1}]$-module with a basis T_w, $w \in W$, and the multiplication given by

$$(T_s+1)(T_s-1) = 0 \qquad \forall s \in S,$$
$$T_w T_{w'} = T_{ww'} \quad \text{if} \quad \ell(w) + \ell(w') = \ell(ww').$$

There is a ring involution $\overline{}$ on $\mathcal{H}$ such that

(1) $$q \longmapsto q^{-1} \quad \text{and} \quad T_w \longmapsto T_{w^{-1}}{}^{-1} \quad \forall w \in W.$$

For y, $w \in W$ define $R_{y,w} \in \mathbb{Z}[q]$ by

(2) $$T_{w^{-1}}{}^{-1} = \sum_{y \in W} q^{-\ell(w)} \overline{R_{y,w}}\, T_y .$$

Then the Kazhdan-Lusztig polynomials $P_{y,w}$ are determined uniquely also as the polynomials that are 0 unless $y \le w$, of degree $\le \frac{1}{2}(\ell(w)-\ell(y)-1)$ if $y < w$, and 1 for $y = w$, satisfying

(3) $$q^{-\ell(w)}P_{y,w} = \sum_{z\in W} q^{-\ell(w)}\overline{R_{y,z}}\ \overline{P_{z,w}} \ .$$

In short, we have

Theorem ([18],(1.1.c)). $\forall w \in W$, $\exists!\ C_w^* \in \mathcal{H}$:

(i) $\overline{C_w^*} = q^{-\ell(w)}C_w^*$,

(ii) $C_w^* = \sum_{y\in W} P_{y,w}T_y$, where $P_{y,w} \in \mathbb{Z}[q]$ is 0 unless $y \le w$ in the Bruhat order, has degree $\le \frac{1}{2}(\ell(w)-\ell(y)-1)$, and $P_{w,w} = 1$.

There is also an inductive formula to define the polynomials. For y, $w \in W$ let $\mu(y, w)$ be the coefficient of $q^{\frac{1}{2}(\ell(w)-\ell(y)-1)}$ in $P_{y,w}$. We have for $w \in W$ and $s \in S$ with $sw > w$

(4) $$C_{sw}^* = (T_s+1)C_w^* + \sum_{y\in W,\, sy<y} \mu(y,w)(-1)^{\ell(w)-\ell(y)}q^{\frac{1}{2}(\ell(w)-\ell(y)+1)}C_y^* ,$$

from which we get $\forall y \in W$,

(5) $$P_{y,sw} = q^{1-c}P_{sy,w} + q^{c}P_{y,w} - \sum_{z\in W,\, sz<z} \mu(z,w)q^{\frac{1}{2}(\ell(w)-\ell(z)+1)}P_{y,z} ,$$

where $c = \begin{cases} 1 & \text{if } sy < y \\ 0 & \text{otherwise.} \end{cases}$

For the properties of the Kazhdan-Lusztig polynomials one can also check a concise survey in [22]. We only add a handy remark that

(6) $$P_{y^{-1},w^{-1}} = P_{y,w} \qquad \forall y,\ w \in W.$$

2. Q-polynomials. The study of Kazhdan-Lusztig polynomials in the representation theory of G started, however, really with Lusztig's [21], where he considered the inverse problem of his conjecture.

(2.1) The present representation theory has benefitted much from regarding G as a group scheme. It allows us to look at the representations of the Frobenius kernel $G_1 = \ker F$ of G. They are, equivalently, the right comodules over the Hopf algebra $K[G_1] = K[G]/\sum_{f\in I} K[G]f^p$ the coordinate algebra of G_1 , where I is the augmentation ideal of $K[G]$.

Let $G_1T = F^{-1}(T)$. J.C.Jantzen [10] has exhibitted us a tight relationship between the representations of G and G_1T. The simple G_1T-modules are parametrized by the entire $X(T)$:

(1) $X(T) \ni \lambda \longrightarrow \hat{L}_1(\lambda)$ the simple G_1T-module of highest weight λ.

For $\lambda \in X_1(T)$ the simple G-module $L(\lambda)$ remains G_1T-simple :

(2) $$L(\lambda) = \hat{L}_1(\lambda) \qquad \forall \lambda \in X(T),$$

so we may look for ch $\hat{L}_1(\lambda)$ instead of ch $L(\lambda)$.

Let $B_1T = F|_B^{-1}T$. For a B_1T-module M, define a sheaf $\mathcal{L}_{G_1T/B_1T}(M)$ on G_1T/B_1T just as for G/B, and take its cohomology $H^i(G_1T/B_1T, \mathcal{L}_{G_1T/B_1T}(M))$. Unlike the cohomology on G/B, all the higher cohomologies vanish on G_1T/B_1T by Serre's theorem as G_1T/B_1T is affine, so we put

(3) $$\hat{Z}_1(M) = H^0(G_1T/B_1T, \mathcal{L}_{G_1T/B_1T}(M)).$$

Its character is given by (cf. [11],(II.9.2))

(4) $$\text{ch } \hat{Z}_1(M) = \text{ch } M \frac{\prod_{\alpha\in R^+}(1-e(-p\alpha))}{\prod_{\alpha\in R^+}(1-e(-\alpha))} .$$

Also $\forall\lambda, \eta \in X(T)$, we have

(5) $$\hat{Z}_1(\lambda + p\eta) = \hat{Z}_1(\lambda) \otimes p\eta ,$$

(6) $$\text{soc } \hat{Z}_1(\lambda) = \hat{L}_1(\lambda), \text{ so } \hat{L}_1(\lambda + p\eta) = \hat{L}_1(\lambda) \otimes p\eta ,$$

(7) $$[\hat{Z}_1(\lambda) : \hat{L}_1(\lambda)] = 1,$$

(8) if $[\hat{Z}_1(\lambda) : \hat{L}_1(\eta)] \neq 0$, then $\eta \uparrow\uparrow \lambda$.

The Lusztig conjecture for G_1T-modules may be formulated as

$$(9)\qquad \mathrm{ch}\ \hat{L}_1(0_C) = \sum_{A\in\mathscr{A}} (-1)^{d(A,C)}\hat{P}_{A,C}(1)\mathrm{ch}\ \hat{Z}_1(0_A) \quad \forall A,\ C \in \mathscr{A}\ ,$$

where the $\hat{P}_{A,C}$ are generic Kazhdan-Lusztig polynomials introduced by Kato [17]. We will turn to those later in § 4. Note that by (2) the formula (9) will be enough (for $p \geq h$) to determine all the irreducible characters of G.

(2.2) Back to Lusztig's work, we call a connected component of $E \setminus \cup_{\alpha\in\Delta,n\in\mathbb{Z}} H_{\alpha,n}$ a box. For $\nu \in pX(T)$ let $A_\nu^{\pm} = A^{\pm}t_\nu$, and we denote by Π_ν (resp. Π_ν^-) the box containing A_ν^+ (resp. A_ν^-). In particular, we will abbreviate $\Pi_{-\rho}$ (resp. $\Pi_{-\rho}^-$) as Π (resp. Π^-). Put $W_\nu = t_{-\nu}Wt_\nu$ and $w_\nu = t_{-\nu}w_0t_\nu$. In the category of G_1T-modules a little bit of maneuvering is possible (cf. [11],(II.9.13)) : $\forall A,\ B \in \mathscr{A}$ with $B \subset \Pi_\nu^-$ and $w \in W_\nu$,

$$(1)\qquad [\hat{Z}_1(0_{Aw}) : \hat{L}_1(0_B)] = [\hat{Z}_1(0_A) : \hat{L}_1(0_B)].$$

Also from (2.1.5, 6) $\forall A,\ B \in \mathscr{A}$ and $\nu \in pX(T)$,

$$(2)\qquad [\hat{Z}_1(0_{At_\nu}) : \hat{L}_1(0_{Bt_\nu})] = [\hat{Z}_1(0_A) : \hat{L}_1(0_B)].$$

Consequently, the formal $\mathbb{Z}[q,q^{-1}]$-linear combination of alcoves

$$(3)\qquad \Sigma_A\ c_{BA}\ A \quad \text{with} \quad c_{BA} = [\hat{Z}_1(0_A) : \hat{L}_1(0_B)] \quad \text{for} \quad B \supset \Pi_\nu^-$$

is invariant under the action of W_ν :

(4) $$\Sigma_A c_{BA}Aw = \Sigma_A c_{BA}A \quad \forall w \in W_\nu .$$

Also $\forall \nu \in pX(T)$,

(5) $$c_{BA} = c_{Bt_\nu, At_\nu} .$$

Lusztig's objective was to construct a q-analogue D^B of the element (3) by replacing the coefficient c_{BA} by certain polynomials in q^{-1}. He poses some simple conditions on this element :

(i) it should satisfy a q-analogue of Weyl group invariance property (4),

(ii) each coefficient must have a certain explicit bound for its degree,

(iii) it must enjoy a simple symmetry property with respect to w_ν ,

and proceeds to show that these properties determine the element D^B uniquely. He does that by defining on the free $\mathbb{Z}[q,q^{-1}]$-module

(6) $$\mathcal{M} = \coprod_{A\in\mathcal{A}} \mathbb{Z}[q,q^{-1}]A$$

with basis corresponding to the alcoves a module structure over the Hecke-Iwahori algebra $\mathcal{H}$ for the affine Weyl group W_p (cf. (1.6)) via

(7) $$\forall s \in S_p \text{ and } A \in \mathcal{A},\ T_sA = \begin{cases} sA & \text{if } s \notin \mathcal{L}(A) \\ qsA + (q-1)A & \text{if } s \in \mathcal{L}(A). \end{cases}$$

Here we define the left action of W_p on $\mathscr{A}$ by

(8) $$w(A^- y) = A^- wy \quad \forall w,\ y \in W_p .$$

Also for each $A \in \mathscr{A}$ we set

(9) $$\mathscr{L}(A) = \{s \in S_p \mid sA < A\}.$$

In order to state Lusztig's result we introduce a partial order $\leq$ on $\mathscr{A}$ as follows :

(10) $A \leq B$ iff $\exists$ a sequence $A = A_0$, A_1 ,..., $A_n = B$: $\forall i \in [1,n]$, $\exists \alpha_i \in R$ and $n_i \in \mathbb{Z}$: $A_i = A_{i-1} s_{\alpha_i, n_i}$ and $d(A_{i-1}, A_i) = 1$

It is easy to show that

(11) $$A \leq B \quad \text{iff} \quad 0_A \uparrow\uparrow 0_B .$$

For $\nu \in pX(T)$ put

(12) $$e_\nu = \sum_{\bar{A} \ni \nu} A \in \mathscr{M},$$

and let $\mathscr{M}_\nu$ the $\mathscr{H}$-submodule of $\mathscr{M}$ generated by e_ν .

<u>Theorem</u> (Lusztig [21],(1.8)). Let $\nu \in pX(T)$ and $B \subset \Pi_\nu^-$. Then $\exists !\ D^B \in \mathscr{M}_\nu$:

(i) $D^B = \sum_A Q^{B,A}(q^{-1})A$, where $Q^{B,A} \in \mathbb{Z}[q]$ is 0 unless $B \leq A$, has degree $\leq \frac{1}{2}(d(B,A)-1)$ if $B < A$, and $Q^{A,A} = 1$.

(ii) $q^{d(B,A_\nu^+)}Q^{B,A}(q^{-1}) = Q^{B,Aw_\nu}(q)$.

The fact $D^B \in \mathcal{K}_\nu$ implies that $D^B(1)$ is invariant under W_ν :

(13) $$D^B(1)w = D^B(1) \quad \forall w \in W_\nu,$$

thus

(14) $$Q^{B,A}(1) = Q^{B,Aw_\nu}(1) \quad \forall w \in W_\nu .$$

(2.3) We have

(1) $$\sum_B(-1)^{d(A,B)}\hat{P}_{A,B}Q^{B,C} = \delta_{A,C} \quad \forall A,\ C \in \mathcal{A},$$

so the G_1T-Lusztig conjecture (2.1.9) is equivalent to

(2) $$[\hat{Z}_1(0_A) : \hat{L}_1(0_B)] = Q^{B,A}(1) \quad \forall A,\ B \in \mathcal{A}.$$

It is called the generic decomposition pattern conjecture by the following reason : in [10], Jantzen showed $\forall \lambda,\ \xi \in X(T)^+$,

(3) $$[H^0(\lambda) : L(\xi)] = \sum_{\eta\in X(T)} [\hat{Z}_1(\lambda) : \hat{L}_1(\eta)][L(\eta^0)\otimes\chi(\eta^1)^{[1]}: L(\xi)].$$

In particular, if $[\hat{Z}_1(\lambda) : \hat{L}_1(\eta)] = 0$ $\forall \eta^1 \notin \overline{A^+}$ (eg. if $4(h-1) \leq$

$\langle\lambda^1, \alpha^\vee\rangle \le p-4(h-1)$ $\forall\alpha \in R^+$), then we get via the strong linkage principle (1.4) and Steinberg's tensor product theorem (1.2)

(4) $$[H^0(\lambda) : L(\mu)] = [\hat{Z}_1(\lambda) : \hat{L}_1(\mu)],$$

thus $H^0(0_A)$ for 0_A in such a region exihibit a decomposition pattern depending only on the position of A in the box containing it (cf. (2.2.2)) and we expect "generically"

(5) $$[H^0(0_A) : L(0_B)] = Q^{B,A}(1) .$$

(2.4) Let $\nu \in pX(T)$ and define a map $\varphi_\nu : \mathcal{M} \longrightarrow \mathcal{M}$ via

(1) $$\Sigma_A\, c_A\, A \longrightarrow \Sigma_A\, \overline{c_A}\, Aw_\nu\ , \quad c_A \in \mathbb{Z}[q,q^{-1}].$$

Then φ_ν is an $\mathcal{H}$-antilinear, i.e., $\varphi_\nu(hm) = \overline{h}\varphi_\nu(m)$ $\forall h \in \mathcal{H}$ and $m \in \mathcal{M}$, involution leaving $\mathcal{M}_\nu$ invariant. For $B \subset \Pi_\nu^-$ put $C = Bw_\nu$, $Q_{A,C} = Q^{B,Aw_\nu}$ $\forall A \in \mathcal{A}$ and let $D_C = \varphi_\nu(D^B)$. Then $D_C = \Sigma_A\, Q_{A,C}A$, thus we can restate

Theorem ([21],(2.15)). Let $\nu \in pX(T)$ and $C \in \Pi_\nu$. Then $\exists!D_C \in \mathcal{M}_\nu$:

(i) $D_C = \Sigma_A\, Q_{A,C}A$, where $Q_{A,C} \in \mathbb{Z}[q]$ is 0 unless $A \le C$, has degree $\le \frac{1}{2}(d(A,C)-1)$ if $A < C$, and $Q_{C,C} = 1$,

(ii) $$q^{d(A_\nu^+,C)}\, Q_{A,C}(q^{-1}) = Q_{Aw_\nu,C}(q) .$$

Note, in particular,

(2) $$D_{A_\nu^+} = e_\nu \qquad \forall \nu \in pX(T) .$$

For a psychological reason we prefer to work with D_C whose coefficients are polynomials in q rather than in q^{-1}.

We call a function $\delta : \mathscr{A} \longrightarrow \mathbb{Z}$ a length function iff

(3) $$d(A, B) = \delta(B) - \delta(A) \quad \forall A, B \in \mathscr{A}.$$

By δ we will always mean such a function. Let $\mathscr{M}^0$ be the $\mathscr{H}$-submodule of $\mathscr{H}$ generated by all e_ν , $\nu \in pX(T)$:

(4) $$\mathscr{M}^0 = \sum_{\nu \in pX(T)} \mathscr{H} e_\nu .$$

We have ([21],(2.12)) an $\mathscr{H}$-antilinear involution Φ_δ of $\mathscr{M}^0$ such that

(5) $$\Phi_\delta\, e_\nu = q^{-\delta(A_\nu^+)} e_\nu \qquad \forall \nu \in pX(T).$$

Then the condition (ii) in the above theorem is equivalent (cf. [21], (2.13)) to

(6) $$\Phi_\delta\, D_C = q^{-\delta(C)} D_C .$$

(2.5) Let $C \subset \Pi_\nu$ and $w \in W_p$ with $wA_\nu^+ = C$. Using (2.4.6) Lusztig

[21], Theorem 5.2 shows

(1) $$D_C = \sum_{\substack{y \\ \ell(yw_\nu)=\ell(y)+\ell(w_\nu)}} P_{yw_\nu, ww_\nu} T_y e_\nu ,$$

consequently,

(2) $$Q_{A,C}(1) = P_{z,ww_\nu}(1) \quad \text{if} \quad A = zA_\nu^- .$$

Meanwhile, according to [21], Jantzen conjectured

(3) $$\operatorname{ch} L(0_C) = \Sigma_A (-1)^{d(A,C)} Q_{A,C}(1) \operatorname{ch} H^0(0_A) \quad \forall C \subset \Pi.$$

We see that it is compatible with Lusztig's conjecture (1.5) as

(4) $$Q_{A,C}(1) = P_{A,C}(1) \quad \forall C \subset \Pi \text{ and } A \in \mathcal{A}^+$$

by (2).

Kato [17] shows, conversely, that

(5) Jantzen's conjecture (3) implies the Lusztigs conjecture.

Again the formula (3) would be enough to determine all the irreducible characters of G while in Lusztig's conjecture not all 0_C, $C \subset \Pi$, may satisfy the Jantzen condition (1.5.2) for small p.

(2.6) For each A let $E_A = T_w e_\nu$, where $\nu \in pX(T)$ with $A \subset \Pi_\nu$ and $w \in W_p$ with $A = wA_\nu^+$. Then the E_A 's form a basis of $\mathcal{M}^0$ ([21],(6.1)) :

(1) $$\mathcal{M}^0 = \coprod_{A\in\mathcal{A}} \mathbb{Z}[q,q^{-1}]E_A \ .$$

Let $\hat{\mathcal{M}}$ be the set of formal $\mathbb{Z}[q,q^{-1}]$-linear combinations $\sum_{A\in\mathcal{A}} c_A A$ of alcoves such that $\{A \mid c_A \neq 0\}$ is bounded above. It forms an $\mathcal{H}$-module in a natural way, containing $\mathcal{M}$ as a subnodule. Moreover, each element of $\hat{\mathcal{M}}$ can be written uniquely in the form $\sum_{B\leq A_0} c_B E_B$, $c_B \in \mathbb{Z}[q,q^{-1}]$. We extend the $\mathcal{H}$-antilinear involution Φ_δ on $\mathcal{M}^0$ to a map $\hat{\Phi}_\delta : \hat{\mathcal{M}} \longrightarrow \hat{\mathcal{M}}$ via

(2) $$\sum_{B\leq A_0} c_B E_B \longmapsto \sum_{B\leq A_0} \overline{c_B}\, \Phi_\delta(E_B),$$

and write

(3) $$\hat{\Phi}_\delta(A) = q^{-\delta(A)} \textstyle\sum_B (-1)^{d(A,B)} \mathcal{R}_{B,A} B \ , \qquad \mathcal{R}_{B,A} \in \mathbb{Z}[q,q^{-1}].$$

Then the $Q_{A,C}$ are uniquely determined also as the polynomials that are 0 unless $A \leq C$, of degree $\leq \frac{1}{2}(d(A,C)-1)$ if $A < C$, and $Q_{C,C} = 1$, satisfying

(4) $$Q_{A,C} = \textstyle\sum_B (-1)^{d(A,B)} \mathcal{R}_{A,B}\, \overline{Q_{B,C}}\, q^{d(B,C)} \qquad \forall A,\ C \in \mathcal{A}.$$

In short,

<u>Theorem</u> ([21],(7.3)). $\forall C \in \mathcal{A}$, $\exists ! D_C \in \hat{\mathcal{H}}$:

(i) $\hat{\Phi}_\delta \, D_C = q^{-\delta(C)} D_C$,

(ii) $D_C = \sum_A Q_{A,C} A$, where $Q_{A,C} \in \mathbb{Z}[q]$ is 0 unless $A \leq C$, has degree $\leq \frac{1}{2}(d(A,C)-1)$ if $A < C$, and $Q_{C,C} = 1$.

It follows that

(5) $$D_C \, t_\nu = D_{Ct_\nu} \qquad \forall C \in \mathcal{A} \text{ and } \nu \in pX(T) .$$

(2.7) We have noted in (1.5) that the coefficients of $P_{y,w}$ are all nonnegative, from which one can also show that

(1) the coefficients of $Q_{A,C}$ are all nonnegative $\forall A, C \in \mathcal{A}$.

Define $\mu : \mathcal{A} \times \mathcal{A} \longrightarrow \mathbb{N}$ by

(2) $$\mu(A, C) = \text{the coefficient of } q^{\frac{1}{2}(d(A,C)-1)} \text{ in } Q_{A,C} ,$$

so $\mu(A,C) = 0$ unless $A \leq C$ and $d(A,C)$ is odd. Lusztig [21],Theorem 8.2 shows $\forall C \in \mathcal{A}$ and $s \in S_p$,

(3) $$T_s \, D_C = \begin{cases} qD_C & \text{if } s \in \mathcal{L}(C) \\ -D_C + D_{sC} + \sum_{A,\, s \in \mathcal{L}(A)} \mu(A,C) q^{\frac{1}{2}(d(A,C)+1)} D_A & \text{otherwise.} \end{cases}$$

It follows that

(4) $$\mathcal{M}^0 = \coprod_{C\in\mathcal{A}} \mathbb{Z}[q,q^{-1}]D_C \ .$$

Also $\forall A,\ C \in A$ and $s \in \mathcal{L}(C)$,

(5) $$Q_{A,C} = Q_{sA,C} \quad \forall A \in \mathcal{A},$$

(6) $$\mu(A,C) = 0 \quad \text{if } s \notin \mathcal{L}(A) \text{ and } A \neq sC \ .$$

For $\nu \in pX(T)$ define a new right action of W_p on $\mathcal{A}$ by

(7) $$A \longmapsto AI_{\nu,w} = At_{(\eta-\nu)w-(\eta-\nu)} \quad \forall w \in W_p \quad \text{if } A \subset \Pi_\eta^- \ .$$

There is also an $\mathcal{H}$-linear right action of W_p on $\mathcal{M}^0$ defined by

(8) $$e_\nu \longmapsto e_\nu\theta_w = q^{\frac{1}{2}d(A_{\nu w}^+,\ A^+)} e_{\nu w} \ .$$

We have ([21],(8.7))

(9) $$D_C\,\theta_w = q^{\frac{1}{2}d(CI_{-p\rho,w},\ C)} D_{CI_{-p\rho,w}} \quad \forall C \in \mathcal{A} \text{ and } w \in W_p \ ,$$

consequently,

(10) $$\mu(A,\ C) = \mu(AI_{-p\rho,w},\ CI_{-p\rho,w}) \quad \forall A,\ C \in \mathcal{A} \text{ and } w \in W_p \ .$$

(2.8) We will now describe an inductive algorithm to compute D_C. For $C \subset \Pi_\nu$ write $C = wA_\nu^+$, $w \in W_p$, and put $n_C = d(A_\nu^+, C)$. The induction will be on n_C. If $n_C = 0$, then $D_C = e_\nu$, so assume $n_C > 0$ and that the elements $D_{C'}$ with $n_{C'} < n_C$ have already been constructed. In particular, $\mu(A, C')$ are known for such C' and all $A \in \mathcal{A}$. Choose $s \in \mathcal{L}(C)$ with $sC \subset \Pi_\nu$. Then $n_{sC} = n_C - 1$ and we have from (2.7.3)

$$(1) \quad D_C = (T_s + 1)D_{sC} - \sum_{s \in \mathcal{L}(A)} \mu(A, sC) q^{\frac{1}{2}d(A,C)} D_A .$$

Here Lusztig [21],Corollary 10.6 shows

$$(2) \quad n_A < n_{sC} \quad \forall A \in \mathcal{A} \text{ with } s \in \mathcal{L}(A) \text{ and } \mu(A, sC) \neq 0 ,$$

consequently, the D_A 's appearing on the right hand side of (1) are already known. Thus (1) provides a desired inductive formula, from which we also get

$$(3) \quad Q_{A,C} = q^c Q_{sA,sC} + q^{1-c} Q_{A,sC} - \sum_{s \in \mathcal{L}(B)} \mu(B, sC) q^{\frac{1}{2}d(B,C)} Q_{A,B} ,$$

where $c = \begin{cases} 1 & \text{if } s \notin \mathcal{L}(A) \\ 0 & \text{if } s \in \mathcal{L}(A) . \end{cases}$

(2.9) Basic properties of $\mathcal{R}$-polynomials introduced in (2.6.3) can be found in [ ],§11 (see also Andersen-Kaneda [4],(4.2)). Using those Lusztig [21], Corollary 11.14 shows that the function

$(-1)^{d(B,C)}Q_{B,C}(0)$ is the Möbius function of the partially ordered set $(\mathcal{A}, \leq)$:

(1) $$\sum_{\substack{B \\ A\leq B\leq C}} (-1)^{d(B,C)}Q_{B,C}(0) = \delta_{A,C} \qquad \forall A,\ C \in \mathcal{A}\ .$$

Also for $\nu \in pX(T)$ we have ([21],(11.15))

(2) $$Q_{yA_\nu^-,wA_\nu^-} = P_{y,w} \qquad \forall y,\ w \in W_\nu\ .$$

(2.10) One finds in [21], §12 beautiful pictures of D_C for the groups of type A_1 , A_2 , B_2 , and G_2 .

For $C \subset \Pi_\nu$ define

(1) $$\text{supp } D_C = \{A \in \mathcal{A} \mid Q_{A,C} \neq 0\}\ .$$

We have noted in (2.7.1) that

(2) $$\text{supp } D_C = \{A \in A \mid Q_{A,C}(1) \neq 0\}\ ,$$

so it is invariant under the action of W_p by (2.2.14), consequently

(3) $$\text{supp } D_C \subset \{A \in \mathcal{A} \mid Cw_\nu \leq A \leq C\}\ .$$

One observes, moreover, that the pictures of D_C in [21],§12 have no holes, that is indeed a general fact (Kaneda [12]) :

(4) $\quad \operatorname{supp} D_C = \{A \in \pi_\nu \mid A \le C\}\, W_\nu \qquad \forall C \subset \pi_\nu.$

This was proved in response to

Ye's theorem [25]. Let $\nu \in pX(T)$ and $C \subset \pi_\nu^-$. If $p \ge 2(h-1)$, then $\{A \in \mathcal{A} \mid [\hat{Z}_1(0_A) : \hat{L}_1(0_C)] \ne 0\} = \{A \subset \pi_\nu^- \mid A \ge C\}\, W_\nu$.

There is yet another symmetry in the pattern D_C. It was discovered (Andersen-Kaneda [4]) in the process of studying the structure of the injective hull of $\hat{L}_1(C)$. Let $\nu, \eta \in pX(T)$ and $A \subset \pi_\nu$, $C \subset \pi_\eta$. Then $\forall w \in W$,

(5) $$\Sigma_B\, q^{\delta(B)} Q_{B,A}\, Q_{Bt_\zeta, C} = q^{n_w(\nu-\eta)} \Sigma_B\, q^{\delta(B)} Q_{B,A}\, Q_{B,C},$$

where $\zeta = (\nu-\eta)w-(\nu-\eta)$ and $n_w(\nu-\eta) = \frac{1}{2}d(A^-, A^- t_{(\nu-\eta)-(\nu-\eta)w})$. In particular,

(6) $$\sum_{\substack{B \\ \bar{B}\ni\nu}} q^{\delta(B)} Q_{Bt_\zeta, C} = q^{n_w(\nu-\eta)} \sum_{\substack{B \\ \bar{B}\ni\nu}} q^{\delta(B)} Q_{B,C}.$$

3. Inverse Kazhdan-Lusztig polynomials $Q'_{A,C}$. By the equation

$$\sum_{B\in\mathcal{A}^-} (-1)^{d(A,B)} P_{Aw_0, Bw_0} Q'_{C,B} = \delta_{A,C} \qquad \forall A, C \in \mathcal{A}^-$$

we can define polynomials $Q'_{A,C} \in \mathbb{Z}[q]$, A, $C \in \mathcal{A}^-$, called the inverse Kazhdan-Lusztig polynomials for the affine Weyl group (W_p, S_p). Much alike characterization of the Q'-polynomials as for Lusztig's Q-polynomials are available by Andersen [1].

(3.1) Lusztig [21],Corollary 11.9 showed

(1) $$Q'_{A,C} = Q_{A,C} \quad \text{if } A, C \in \mathcal{A}^- \text{ are sufficiently far from the hyperplanes } H_{\alpha,0} \ \forall\alpha \in \Delta,$$

thus Lusztig's Q-polynomials are sometimes called the generic inverse Kazhdan-Luztig polynomials. More precisely, we have (Kaneda [13],(2.2))

(2) $$Q'_{A,C} = \sum_{w\in W} (-1)^{\ell(w)} q^{\frac{1}{2}d(CI_{p\rho,w},\ C)} Q_{A,CI_{p\rho,w}} \qquad \forall A,\ C \in \mathcal{A}^- .$$

In characteristic 0 the Borel-Weil-Bott theorem (cf. [11], (II.5.5)) brings complete information about all $H^i(\lambda)$: $\forall\lambda \in X(T)^+-\rho$, $w \in W$, and $i \geq 0$,

(3) $$H^i(\lambda\cdot w) = \begin{cases} H^0(\lambda) & \text{if } \lambda \in X(T)^+ \text{ and } i = \ell(w) \\ 0 & \text{otherwise.} \end{cases}$$

A similar result holds in our situation generically (cf. [11],(II.9.14)), but fails badly when λ is close to an $H_{\alpha,0}$, $\alpha \in \Delta$. Andersen [1] asks how the cancellation on the right hand side of (3) is related to the failure of the Borel-Weil-Bott theorem in positive characteristic.

(3.3) With the Q'-polynomials we can invert the Lusztig conjecture : $\forall A, C \in \mathcal{A}^+$ with 0_C satisfying the Jantzen condition (1.5.2),

(1) $$[H^0(0_C) : L(0_A)] = Q'_{Cw_0, Aw_0}(1) .$$

On the other hand, we have (Humphreys [8], Jantzen, Doty-Sullivan [6]) $\forall A, C \in \mathcal{A}^+$ with 0_C satisfying the Jantzen condition,

(2) $$[H^0(0_C) : L(0_A)] = \sum_{w \in W} (-1)^{\ell(w)} [\hat{Z}_1(0_C) : \hat{L}_1(0_{AI_{0,w}})],$$

so the inversion formula (1) for the G-module would follow from the inversion formula for the G_1T-modules via (3.1.3), i.e.,

(3) the G_1T-Lusztig cinjecture (2.1.9) implies the Lusztig conjecture (1.5).

For $p >> 0$ this was known before (Kato [17]). The converse is also known to hold if p is large enough that $0_{\bar{A}_{p\rho}}$ should satisfy the Jantzen condition (Kaneda [14]).

Can we show Jantzen's conjecture (2.5.3) is equivalent to the G_1T-Lusztig conjecture : $\forall C \in \mathcal{A}^+$ with $0_A \in X_1(T)$,

$$(4)\qquad \sum_{A\in\mathcal{A}^+}(-1)^{d(A,C)}P_{A,C}(1)\,\frac{\sum_{w\in W}(-1)^{\ell(w)}e(0_A\cdot w)}{\sum_{w\in W}(-1)^{\ell(w)}e(0\cdot w)} = \sum_{A\in\mathcal{A}}(-1)^{d(A,C)}\hat{P}_{A,C}(1)e(0_A)\,\frac{\prod_{\alpha\in R^+}(1-e(-p\alpha))}{\prod_{\alpha\in R^+}(1-e(-\alpha))} \quad ?$$

4. <u>Generic Kazhdan-Lusztig polynomials</u> $\hat{P}_{A,C}$. There are several ways to define the generic Kazhdan-Lusztig polynomials for (W_p, S_p), due to Kato [17], one of which is already given at (2.3.1).

(4.1) For $\gamma \in p\mathbb{Z}R$ choose $\xi \in p\mathbb{Z}R \cap X(T)^+$ such that $\gamma + \xi \in X(T)^+$ and set

$$(1)\qquad \tilde{T}_\gamma = T_{\gamma+\xi}T_\xi^{-1} ,$$

which can be shown to be well-defined. For $w \in W_p$ write $w = xt_\gamma$ with $x \in W$ and $\gamma \in p\mathbb{Z}R$, and set

$$(2)\qquad \tilde{T}_w = T_x\,\tilde{T}_\gamma .$$

Kato [17], Proposition 1.10 shows

$$(3)\qquad \mathcal{H} = \coprod_{w\in W_p} \mathbb{Z}[q,q^{-1}]\tilde{T}_w ,$$

(4) $\mathcal{M} \simeq \mathcal{H}$ as $\mathcal{H}$-modules via $A^{-}w \longmapsto \tilde{T}_w$.

Using the isomorphism he transfers the map $\hat{\Phi}_\delta$ of (2.6) on $\hat{\mathcal{M}}$ to define an $\mathcal{H}$-antilinear involution Ψ on $\hat{\mathcal{H}}$ via

$$(5) \qquad \Psi(\tilde{T}_w) = q^{d(A^{-}w,\ A)} \sum_{\substack{y \\ A^{-}y \leq A^{-}w}} (-1)^{d(A^{-}y, A^{-}w)} \mathcal{R}_{A^{-}y, A^{-}w} \tilde{T}_y .$$

Then the generic Kazhdan-Lusztig polynomials $\hat{P}_{A,C}$ are uniquely determined as the polynomials that are 0 unless $A \leq C$, of degree $\leq \frac{1}{2}(d(A,C)-1)$ if $A < C$, and $\hat{P}_{C,C} = 1$, satisfying

$$(6) \qquad q^{-\delta(A)} \hat{P}_{A,C} = \Sigma_B\, q^{-\delta(A)} \overline{\mathcal{R}_{Bw_0, Aw_0}}\, \hat{P}_{B,C} .$$

In short, if we define an $\mathcal{H}$-antilinear involution $\tilde{\Phi}_\delta : \hat{\mathcal{M}} \longrightarrow \hat{\mathcal{M}}$ via

$$(7) \qquad A \longmapsto \Sigma_B\, q^{-\delta(B)} \mathcal{R}_{Aw_0, Bw_0} B ,$$

then $\forall C \in \mathcal{A}$, $\exists!\ E_C = \Sigma_A \hat{P}_{A,C} A \in \hat{\mathcal{M}}$:

$$(8) \qquad \tilde{\Phi}_\delta E_C = q^{-\delta(C)} E_C ,$$

where $\hat{P}_{A,C} \in \mathbb{Z}[q]$ is 0 unless $A \leq C$, has degree $\leq \frac{1}{2}(d(A,C)-1)$ if $A < C$, and $\hat{P}_{C,C} = 1$.

It follows that

(9) $\hat{P}_{A,C} = P_{A,C}$ if $A,\ C \in \mathscr{A}^+$ are sufficiently far from $H_{\alpha,0}$ $\forall \alpha \in \Delta$,

suggesting the name "generic" Kazhdan-Lusztig polynomial for $\hat{P}_{A,C}$. More precisely, Kato [17],Corollary 4.3 shows

$$(10) \qquad P_{A,C} = \sum_{w\in W} (-1)^{\ell(w)} q^{\frac{1}{2}d(CI_{0,w},\ C)} \hat{P}_{A,CI_{0,w}} \qquad \forall A,\ C \in \mathscr{A}^+.$$

(4.2) We now turn to the extension problem in the G_1T-module category following Vogan [24] and Andersen [1].

The automorphism φ of G corresponding to the root system automorphism $\alpha \longmapsto -\alpha$ $\forall \alpha \in R$ leaves G_1T invariant, so we may define the contravariant dual $D\mathscr{M}$ of each G_1T-module $\mathscr{M}$ by the composition $G_1T \xrightarrow{\varphi} G_1T \longrightarrow GL(\mathscr{M}^*)$. We have (cf. [11],(II.11.1)) $\forall \lambda,\ \eta \in X(T)$ and $i \geq 0$,

$$(1) \qquad \mathrm{Ext}^i_{G_1T}(\hat{Z}_1(\lambda),\ D\hat{Z}_1(\eta)) \simeq \mathrm{Ext}^i_{G_1T}(D\hat{Z}_1(\lambda),\ \hat{Z}_1(\eta)) \simeq \begin{cases} K & \text{if } \lambda = \eta \text{ and } i = 0 \\ 0 & \text{otherwise,} \end{cases}$$

from which we get $\forall \lambda \in X(T)$,

$$(2) \qquad \mathrm{ch}\ \hat{L}_1(\lambda) = \sum_{\eta\in X(T)} \sum_{i\geq 0} (-1)^i \dim \mathrm{Ext}^i_{G_1T}(\hat{L}_1(\lambda),\ \hat{Z}_1(\eta))\mathrm{ch}\ \hat{Z}_1(\eta),$$

so we can reformulate the G_1T-Lusztig conjecture (2.1.9) as

$$(3) \quad (-1)^{d(A,C)}\hat{P}_{A,C}(1) = \sum_{i\in\mathbb{N}} (-1)^i \dim \mathrm{Ext}^i_{G_1T}(\hat{L}_1(0_C),\ \hat{Z}_1(0_A)) \quad \forall A,\ C \in \mathscr{A}.$$

It is even equivalent (cf. Kaneda [14], (4.12)) for $p > h$ to

(4) $\hat{P}_{A,C} = \sum_{i\geq 0} q^i \dim \mathrm{Ext}_{G_1 T}^{d(A,C)-2i}(\hat{L}_1(0_C), \hat{Z}_1(0_A)) \quad \forall A, C \in \mathcal{A}.$

The conjecture (3) has been verified for $C = A^+$ by Andersen-Jantzen (cf. Kaneda [14],(4.6)) :

(5) $\hat{P}_{A,A^+} = \sum_{i\geq 0} q^i \dim H^{d(A,A^+)-2i}(B_1, 0_A)^T \quad \forall A \in \mathcal{A},$

putting together Kato [16],(1.8) with the determination of the B_1-cohomology by Andersen-Jantzen [3],(2.3) and (2.9) : for $p > h$

(6) $H^\bullet(B_1, K) \simeq S^\cdot(\mathfrak{u}^*)^{[1]}$ as graded B-algebras,

(7) $\forall \lambda \in X(T)$ and $i \in \mathbb{N}$, as B-modules

$$H^i(B_1, \lambda) \simeq \begin{cases} S^{\frac{i-\ell(w)}{2}}(\mathfrak{u}^*)^{[1]} \otimes p\gamma & \text{if } \lambda = 0\cdot w + p\gamma \text{ for some } w \in W \text{ and } \gamma \in X(T) \text{ with } i-\ell(w) \text{ even} \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathfrak{u}$ is the Lie algebra of U and $S'(\mathfrak{u}^*)$ is the symmetric algebra on $\mathfrak{u}^*$ with each $S^i(\mathfrak{u}^*)$ given the degree $2i$.

The cohomology of higher Frobenius kernel $B_r = \ker(F|_B)^r$, $r > 1$, is unknown. As usual, their alternating sum is easy to find, however (Kaneda-Shimada-Tezuka-Yagita [15],(2.5)) : for $p > h$

(8) $\forall\eta \in X(T)$ and $r > 0$, $\sum_{i\geq 0}(-1)^i \dim H^i(B_{r+1}, K)_{p^{r+1}\eta} =$

$$\sum_{\substack{w\in W \\ \lambda\in X(T)}} \sum_{i\geq 0}(-1)^i \dim H^i(B_r, K)_{p^r(0\cdot w+p(\eta-\lambda))} \sum_{j\geq 0}(-1)^j \dim H^{j-\ell(w)}(B_1, K)_{p\lambda} .$$

One suspects if $H^{\cdot}(B_r, K)$ for $r > 1$ may also be described using the generic Kazhdan-Lusztig polynomials. If $r = 2$, ch $H^{\cdot}(B_2, K)$ is available for SL_2 (Andersen-Jantzen [3],(2.4.2)) and for SL_3 (Kaneda-Shimada-Tezuka-Yagita [15],(5.11) for $p > 3$).

5. Some consequences of the Lusztig-conjecture. In this section assume the G_1T-Lusztig conjecture. We will state some consequences.

(5.1) As already suggested in (4.2.4), the $\hat{P}$-polynomials seem to carry information on the structure of $\hat{Z}_1(\lambda)$, $\lambda \in X(T)$. Indeed, following Andersen [1], Gaber-Joseph [7] and Irving [9], it was proved (Andersen-Kaneda [4],(6.3)) that the socle series and the radical series of each $\hat{Z}_1(0_C)$ coincide and that $\forall C \subset \Pi_\nu$,

(1) $$Q_{A,C} = \sum_j q^{\frac{1}{2}(d(A,C)-j)} [\mathrm{rad}_j\, \hat{Z}_1(0_A) : \hat{L}_1(0_{Cw_\nu})],$$

where $\mathrm{rad}_j\, \hat{Z}_1(0_A) = \mathrm{rad}^j\, \hat{Z}_1(0_A)/\, \mathrm{rad}^{j+1}\hat{Z}_1(0_A)$ is the j-th level in the radical series of $\hat{Z}_1(0_A)$.

(5.2) From (5.1.1) it follows ([4],(6.5)) that

(1) $\quad \mathrm{Ext}^1_{G_1T}(\hat{L}_1(0_C), \hat{Z}_1(0_A)) = \mathrm{Ext}^1_{G_1T}(\hat{L}_1(0_C), \hat{L}_1(0_A)) \quad \forall A \le C.$

On the other hand, from (4.2.4) one expects

(2) $\quad \mu(A, C) = \dim \mathrm{Ext}^1_{G_1T}(\hat{L}_1(0_C), \hat{Z}_1(0_A)),$

consequently,

(3) $\quad \mu(A, C) = \dim \mathrm{Ext}^1_{G_1T}(\hat{L}_1(0_C), \hat{L}_1(0_A)) \quad \forall A \le C.$

For $A, C \in \mathscr{A}$ set

(4) $$\tilde{\mu}(A, C) = \begin{cases} \mu(A, C) & \text{if } A \le C \\ \mu(C, A) & \text{otherwise,} \end{cases}$$

and put $\tilde{\mu}(A) = \{B \in \mathscr{A} \mid \tilde{\mu}(A, B) \neq 0\}$. Doty-Sullivan [5] conjectures

(5) for $A \subset \Pi^-_\nu$, $\tilde{\mu}(A)$ should be the union of I_{ν,W_ν}-orbits of

$\{A^\alpha \mid \alpha \in \Delta\}$, $\{B \in \mathscr{A} \mid B$ is adjacent to $A\}$,

$\{B \in \mathscr{A}^+ t_{\nu-p\rho} \mid B < A, \mathscr{L}(A) \subset \mathscr{L}(B), d(B,A)$ odd $\}$, and

$\{B \in \mathscr{A}^- t_\nu \mid A < B, \mathscr{L}(B) \subset \mathscr{L}(A), d(A,B)$ odd $\}$,

where $A^\alpha = As_{\alpha,n}$ if $pn < \langle 0_A, \alpha^\vee \rangle < p(n+1)$. It has been verified in [5] (cf. also Kaneda [14]) that $\tilde{\mu}(A)$ is I_{ν,W_ν}-invariant and is contained in the union of the prescribed orbits. Conversely, it is easy to see that the first two sets in the list are contained in $\mu(A)$

For G of rank ≤ 2 one observes also

$$(6) \qquad \tilde{\mu}(A, B) = \tilde{\mu}(Aw_0, Bw_0) \quad \forall A, B \in \mathcal{A}.$$

Does it hold in general ?

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