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Entropy of automorphisms on the hyperfinite II_1 factor

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1. Introduction.

Connes and Stormer ([5]) extended the notion of the entropy from the classical ergodic theory to the frame of II_1 von Neumann algebras and showed that the n-shift of the hyperfinite II_1 factor is not conjugate to the m-shift for $n \neq m$. For each n, the n-shift is the automorphism corresponding to the translate of 1 in the infinite tensor product $R = \bigotimes_{i\in Z} (M_i, tr_i)$ of the algebra M_i of $n \times n$ matrices with the normalized trace tr_i on M_i, for each i.

In the index theory([7]) for II_1 factors, Jones gives the sequence $\{e_i; i=1,2,...\}$ of projections related on the number $\lambda \in (0,1/4]\cup\{(1/4)\sec^2(\pi/n); n=3,4,...\}$, which generates the hyperfinite II_1 factor R. By his method, from this sequence $\{e_i; i\in N\}$, we get the two sided sequence $\{e_i; i\in Z\}$ with the same property as $\{e_i; i\in N\}$. Pimsner-Popa([8]) computed the entropy of automorphism θ_λ of R generated by $\{e_i; i\in Z\}$ translating e_i to e_{i+1}. However, the value of it has not been obtained in the case $\lambda=1/4$. Powers [9] developed those results to another direction, considering the conjugacy problem of *-endomorphisms of R generated by the sequence $\{u_i; i\in N\}$ of self adjoint unitaries which translate u_i to u_{i+1}.

Those *-endomorphisms ,which are not automorphisms, are called binary shifts. After then, in [2],[4],[1] and [10], the conjugacy problem of *-endomorphisms corresponding to sequences more general unitaries are discassed. We call those *-endomorphisms unitary shifts. On the other hand, the author([2]) treated *-endomorphisms of R genetated by sequence $(p_i; i \in N)$ of projections translating p_i to p_{i+1}, which we call projection shifts. Those unitary shifts and projection shifts turn out ergodic automorphisms of the hyperfinite II_1 factor by the natural method.

In this paper, we shall study the entropy of those automorphisms. First, we shall get simple formulas of the entropy of ergodic automorphisms of the hyperfinite II_1 factor, which have applications to compute the entropies of the above automorphisms. Under some good conditions, the entropy $H(\theta)$ of an automorphism θ of a hyper finite II_1 factor R is determined by the entropy $H(A_n)$'s for an increasing sequence $(A_n)_{n \in N}$ of finite dimensional subalgebras which generating R:

$$H(\theta) = \lim_{n\to\infty} H(A_n)/n.$$

The above all automorphisms satisfy the conditions. As an application of this result to the automorphism θ_λ treated by Pimsner-Popa, we show that $H(\theta_\lambda) = \log 2$, in the case $\lambda = 1/4$. As another application, we shall show that if the inclusion data for the above sequence $(A_n)_{n \in N}$ is periodic in the sense below, then

$$H(\theta) = (1/p) \log \beta,$$

where p is the period of the data and β is the Perron-Frobenius eigen value of the inclusion matrix. This gives another proof of the results on the n-shift given by Connes-Stormer and also the results on θ_λ's by Pimsner=Popa for λ strictly larger than 1/4.

2. General results.

In this section, we shall get two formulas for the entropy of $*$-automorphisms of the hyperfinite II_1 factor.

Theorem 1. Let N be an approximately finite dimensional finite von Neumann algebra with a faithful normal normalized trace tr. Let θ be a $*$-automorphism of N with $tr\theta = tr$. Let $\{N_j; j=1,2,...\}$ be an increasing sequence of finite dimensional subalgebras of N such that N is the weak closure of $\cup_j N_j$. Assum that the following two conditions are satisfied:

(1) For j and m, there is an $*$-automorphism σ such that $\sigma(N_{j+m})$ contains the weak closure of $N_j \cup \theta(N_j) \cup ... \cup \theta^m(N_j)$.

(2) There exists an sequence $(n(j))_{j\in\mathbb{N}}$ such that

$$x\theta^m(y) = \theta^m(y)x \qquad (x,y \in N_j,\ \ m \geq n(j)),$$

$$\lim_{j\to\infty} (n(j) - j)/j = 0$$

and

$$tr(x\theta^m(y)) = tr(x)tr(y) \qquad (x,y \in N_j,\ \ m \geq n(j)).$$

Then

$$H(\theta) = \lim_{j\to\infty} H(N_j)/j.$$

Proof. BY the properties of functions H(...) ([5]),

$$H(\theta) = \lim_j H(N_j,\theta)$$
$$= \lim_j \lim_m (1/m)H(N_j,\theta(N_j), \quad \theta^m(N_j))$$
$$\leq \lim_{j,m}(1/m)H(\{N_j,...\theta^{m-j}(N_j)\}'',\{\theta^{m-j+1}(N_j),...,\theta^m(N_j)\}'')$$

$$\le \lim_j \liminf_m H(\sigma(N_m), \theta^{m-j+1}\sigma(N_{2j-1}))$$
$$\le \lim_j \liminf_m (1/m) \{H(N_m) + H(N_{2j-1})\}$$
$$= \liminf_{m\to\infty} (1/m) H(N_m).$$

On the other hand,

$$n(j)H(\theta) = H(\theta^{n(j)}) \ge \lim_m (1/m) H(N_j, \theta^{n(j)}(N_j), \ldots, \theta^{mn(j)}(N_j))$$
$$= H(N_j),$$

because $\{N_j, \theta^{n(j)}(N_j), \ldots, \theta^{mn(j)}(N_j)\}$ are pairwise commuting , θ is tr-preserving and $tr(x\theta^m(y)) = tr(x)tr(y)$ for x, y in N_j an $m \ge n(j)$. Hence

$$H(\theta) \ge (1/j)H(N_j) - ((n(j) - j)/j)H(\theta) \quad \text{for all } j.$$

Since $\lim_{j\to\infty}(n(j) - j)/j = 0$, we have $H(\theta) \ge \limsup_j (1/j)H(N_j)$. Therefore $H(\theta) = \lim_{j\to\infty} (1/j)H(N_j)$.

Theorem 1 is a generalization of a similar result in the proof of the computation of θ_λ of $\lambda > 1/4$.

Next, we shall give a formula for the entropy of automorphisms of a factor generated by a periodic sequence of finite dimensional algebras.

Let N be a finite von Neumann algebra generated by an ncreasing sequence $(N_j)_{j\in\mathbb{N}}$ of finite dimensional von Neumann algebras. Let tr be a faithful normal normalized trace on N. Then following after [7], we get the *dimension vector* d_j of N_j, the *trace vector* t_j of the restriction of tr on N_j and the *inclusion matrix* $[N_j \to N_{j+m}]$ for $N_j \subset N_{j+m}$;

$$d_j[N_j \to N_{j+m}] = d_{j+m}, \quad [N_j \to N_{j+m}]t_{j+m} = t_j \quad \text{and} \quad d_j t_j = 1,$$

where the dimension vector $d = (d(i))$ of A means that A is decom posed into the direct sum $\Sigma\oplus A_i$ of the $d(i) \times d(i)$ matrix algebra A_i and the trace vector $t = (t(i))$ (which is given as a column vector) means that $tr(x) = \Sigma\, t(i)\, Tr(x(i))$ for an $x = \Sigma\oplus x(i)$ in A. If the inclusion data for the sequence $(N_j)_{j\in\mathbb{N}}$ satisfies the following two conditions, then the sequence $(N_j)_{j\in\mathbb{N}}$ is said to be *periodic*([11]):

There is a $n_0 \geq 0$ and $p \geq 1$ such that for all $j \geq n_0$;

(i) $[N_j \to N_{j+1}] = [N_{j+p} \to N_{j+p+1}]$

(ii) The matrix $[N_j \to N_{j+p}]$ is primitive.

Remark. If N is generated by a periodic sequence $(N_j)_{j\in\mathbb{N}}$, then N is a factor([6], [11]).

Lemma 2. Let N be a hyperfinite II_1 factor generated by a periodec sequence $(N_j)_{j\in\mathbb{N}}$. Then for all $j \geq n_0$,

$$t_j = \|\, [N_j \to N_{j+p}] \,\|\, t_{j+p}$$

and

$$\lim_{j\to\infty} H(N_j)/\, j = (1/p) \log \|[N_j \to N_{j+p}]\|.$$

Proof. For $j \geq n_0$, we denote by T_j the matrix $[N_j \to N_{j+p}]$ and by β_j a Perron-Frobenius eigen value of T_j. Since $t_{j+p} = T_j^{\,k}\, t_{j+kp}$ for all $k \geq 1$ and T_j is primitive for all $j \geq n_0$, there is an $\alpha > 0$ such that $t_{j+p} = \alpha v$ for a unique Perron-Frobenius eigen vector v of T_j by the well known theorem. Hence

$$t_j = T_j t_{j+p} = \beta_j t_{j+p} = ||[N_j \to N_{j+p}]|| t_{j+p}.$$

By the property of the entropy of finite dimensional algebras,

$$H(N_j) = -\Sigma_k d_j(k) t_j(k) \log t_j(k)$$

$$= -\Sigma_k d_j(k) t_j(k) \log \beta_j^{-n} t_{j-np}(k)$$

$$= n \log \beta_j - \Sigma_k d_j(k) t_j(k) \log t_{j-np}(k).$$

Let m be the largest natural number such that $j - np \geq n_0$, that is $m = [\frac{j-n_0}{p}]$. Since $\Sigma_k d_j(k) t_j(k) = 1$ and $\{t_{j-mp}(k);k\}$ is a finite set, we have that

$$\lim_{j\to\infty} H(n_j)/j = \lim_{j\to\infty} (m/j) \log \beta_j = (1/p) \log||[N_j \to N_{j+p}]||.$$

Theorem 3. Let θ be an automorphism of a hyperfinite II_1 factor N which is generated by a periodic sequence $(N_j)_{j\in\mathbb{N}}$ of finite dimensional subalgebras of N. Let P be the period of $(N_j)_{j\in\mathbb{N}}$. If θ and $(N_j)_{j\in\mathbb{N}}$ satisfy the conditions (1) and (2) in Theorem1, then

$$H(\theta) = (1/p) \log ||[N_j \to N_{j+p}]|| \quad \text{for large } j.$$

As an application of this, we have the following result by Connes-Stormer.

Corollary 4. Let S_n be the n-shift of the hyperfinite II_1 factor, then $H(S_n) = \log n$.

Proof. Let M be the algebra of $n \times n$ -matrices. For an integer j, let

$$x_j = \ldots \otimes 1 \otimes \ldots \otimes 1 \otimes \underset{j}{\breve{x}} \otimes 1 \otimes \ldots \quad (x \in M).$$

Then $S_n(x_j) = x_{j+1}$ for all $x \in M$ and $j \in \mathbb{Z}$. For an integer $j \geq 0$, let

$$N_{2j+1} = (\bigcup M_k ; |k| \leq j)'' \quad \text{and} \quad N_{2j} = (M_j \bigcup N_{2j-1})''.$$

Then the sequence $(N_j)_{j \in \mathbb{N}}$ is a periodic sequence which generates the hyperfinite II_1 factor N. The period of $(N_j)_{j \in \mathbb{N}}$ is 1 andthe inclusion matrix $[N_j \to N_{j+1}]$ is the number n for all $j \geq 0$. Hence we have $H(S_n) = \log n$.

For another application of Corollary 3, we shall discuss in the next section.

3. Some Applications.

In this section, we shall give some applications of the preceding results. First, as an application of Theorem 1, we compute the entropy $H(\theta_\lambda)$ for the automorphism θ_λ due to Pimsner-Popa in the case $\lambda = 1/4$ (only which they did not compute). The automorphism θ_λ is defined as follows.

Let $(e_i)_{i \in \mathbb{Z}}$ be the two sided sequence of projections satisfying the axioms;

a) $e_i e_{i\pm 1} e_i = \lambda\, e_i$ for a $\lambda \in (0,1/4] \cup \{1/4 \sec^2 \pi/m; m\geq 3\}$,

b) $e_i e_j = e_j e_i$ for $|i-j| \geq 2$

c) the von Neumann algebra P generated by $(e_i)_{i\in\mathbb{Z}}$ is a hyperfinite II_1 factor

d) $tr(we_i) = \lambda\, tr(w)$ for the trace tr of P if w is a word on 1 and $\{e_j; j<i\}$.

The automorphism θ_λ of P is defined by $\theta_\lambda(e_i) = e_{i+1}$. We define the sequence $(A_j)_{j\in\mathbb{N}}$ of finite dimensional subalgebras of P which generates P by

$$A_{2j} = \{e_j; |j|\leq j-1\}'', \qquad A_{2j+1} = \{M_{2j}, e_j\}''.$$

Then we proved in [3] that the inclusion data for $(A_j)_{j\in\mathbb{N}}$ is same as the data in [7]. In case $\lambda = 1/4$, the sequence of trace vectors has beautiful values as follows.

Lemma 4. Let t_j be the trace vector for the restriction of t on A_j. If $\lambda = 1/4$, then for all $k\in\mathbb{N}$,

$$t_{2k} = (1/4^k, 3/4^k, 5/4^k, \ldots, (2k+1)/4^k)$$

and

$$t_{2k+1} = (1/4^k, 2/4^k, 3/4^k, \ldots, (k+1)/4^k).$$

Proof. We shall prove it by the induction on k. It is obvious for $k = 1$. Assume that Lemma holds for $k = m$. It is known that $t_{2(m+1)}(j) = (1/4)t_{2m}(j)$ for $j = 1,2,\ldots,m+1$. Hence we just have to show $t_{2(m+1)}(m+2) = (2m+3)/4^{2m+1}$. The dimension vector d_{2k} satisfies $d_{2k}(i) = \binom{2k}{k-i+1} - \binom{2k}{k-i}$, where $\binom{n}{i}$ is the binomial symbols with the convention $\binom{n}{-1} = 0$. Since $\Sigma_i d_j(i) t_j(i) = 1$, we have $\alpha = t_{2p+1}(p+1) = (2p+3)/4^{2p+1}$ by the equality:

$$4^{p+1} = \binom{2(p+1)}{p+1} - (2p+1) + 2\Sigma_{j=0}^{p-1} \binom{2(p+1)}{p-j} + 4^{p+1}\alpha.$$

Similarly, we have the values for t_{2k+1}.

Theorem 5. Let $\lambda = 1/4$, then we have;

$$H(\theta_\lambda) = \log 2.$$

Proof. We denote θ_λ by θ. It is easy to check that the sequence $(A_j)_{j\in\mathbb{N}}$ and θ satisfy the condition (1) and (2) in Theorem 1. Hence $H(\theta) = \lim_{j\to\infty} H(A_j)/j$. On the other hand,

$$H(A_{2k}) = -\Sigma_{j=1}^{k+1} d_{2k}(j)t_{2k}(j)\log t_{2k}(j)$$

$$= -\Sigma_j d_{2k}(j)t_{2k}(j)(\log(2j+1) - \log 4^k)$$

$$= \log 4^k - \Sigma_j d_{2k}(j)t_{2k}(j)\log(2j+1), \quad \text{for } k=1,2,\ldots .$$

Similarly, $H(A_{2k+1}) = \log 4^k - \Sigma_j d_{2k+1}(j)t_{2k+1}(j)\log j$ for all k. On the other hand,

$$0 \le \lim_{k\to\infty}(1/2k)\Sigma_j d_{2k}(j)t_{2k}(j)\log(2j+1) \le \lim_{k\to\infty}(1/2k)\log(2k+1) = 0$$

and

$$\lim_{k\to\infty}(1/2k+1)\Sigma_j d_{2k+1}(j)t_{2k+1}(j)\log j = 0.$$

Hence we have

$$H(\theta) = \lim_{j\to\infty}(1/j)H(A_j) = \lim_j (1/2j)\log 4^j = \log 2.$$

By applying Corollary 3 to θ_λ ,we have the following results by Pimsner-Popa:

Theorem 6. Let $\lambda \geq 1/4$. Then $H(\theta_\lambda) = -(1/2)\log\lambda$.

Proof. Since the inclusion data for the sequence $(A_j)_{j\in\mathbb{N}}$ is same as one obtained by Jones [7] ([3]), the sequence $(A_j)_{j\in\mathbb{N}}$ is periodic and the period of it is 2. For a sufficiently large j, the Perron-Frobenius eigen value of the inclusion matrix $[A_j \to A_{j+2}]$ is $1/\lambda$ if $\lambda > 1/4$. Hence $H(\theta_\lambda) = -(1/2)\log\lambda$ by Corollary 3.

In [2], we discussed on projection shifts and unitaryi shifts on the hyperfinite II_1 factor which are defined as follows.

Let take an $\lambda \in (0,1/4] \cup \{(1/4)\sec^2(\pi/m); m \geq 3\}$. Let k be a positive integer. Then there exists a sequence $(p_j)_{j\in\mathbb{N}}$ of projection which generates a hyperfinite II_1 factor R and satisfies the followings; for all positive integers i and j,

(i) $p_i p_j = p_j p_i$ if $|i-j| \neq k$

(ii) $p_i p_j p_i = \lambda p_i$ if $|i-j| = k$

(iii) $tr(wp_i) = \lambda\, tr(w)$ if w is an associative word on $\{1, p_1, p_2, \ldots, p_{i-1}\}$.

The λ-projection shift σ is an $*$-endomorphism on R defined by $\sigma(p_i) = p_{i+1}$.

Let S be a subset of $\mathbb{N}$. Let $\gamma = \exp(2\pi i/n)$ for some positive integer n. Then there exists a sequence $(u_i)_{i\in\mathbb{N}}$ of unitaries which generates the hyperfinite II_1 factor R and satisfy the following conditions;

(i') $u_i^n = 1$

(ii') $u_i u_j = \gamma u_j u_i$ if $|i-j| \varepsilon S$

(iii') $u_i u_j = u_j u_i$ if $|i-j|$ is not contained in S.

The n-unitary shift ρ on R is a $*$-endomorphism defined by $\sigma(u_i) = u_{i+1}$.

Those sequences $(p_i)_{i \varepsilon \mathbb{N}}$ and $(u_i)_{i \varepsilon \mathbb{N}}$ are extended to two sided sequences $(p_i)_{i \varepsilon \mathbb{Z}}$ and $(u_i)_{i \varepsilon \mathbb{Z}}$ with a similar property as (i),(ii),(iii) and (i'),(ii'),(iii') by methods discussed in [2] and [4]. Then the endomorphisms σ and ρ on R are extended to automorphisms on the hyperfinite II_1 factor R generated by $(p_i)_{i \varepsilon \mathbb{Z}}$ or $(u_i)_{i \varepsilon \mathbb{Z}}$. The automrphisms σ and ρ are ergodic because the weak closure of linear span of reduced words (with respect to the properties (j) and (j') for j = i, ii and iii) on the products of powers of $(p_i)_{i \varepsilon \mathbb{Z}}$ or $(u_i)_{i \varepsilon \mathbb{Z}}$ is R and the trace of R has the multiplicative property; $tr(x\sigma(y)) = tr(x)tr(y)$ for the reduced words x and y.

Applying the results in section 2, we can compute the entropy of those automorphisms σ and ρ. As projection shifts, all σ for (λ, k) are not conjugate to σ for (λ', k') if $(\lambda,k) \neq (\lambda',k')$. However, the entropies of σ for (λ,k) are all $-(1/2)\log\lambda$. As are all $-(1/2)\log\lambda$. As unitary shifts, ρ for (γ,S) are not conjugate to ρ for (γ',S') if (γ,S) is not equal to (γ', S'). However, for an example, the entropies of ρ for (γ,k) are all $(1/2)\log n$, where $\gamma = \exp(2\pi i/n)$ and k is a positive integer.

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