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Bivariate Thiele-type branched continued fraction and general order Padé approximation

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0. Introduction

The problems of interpolation and approximation by rational functions raise in various fields and it is very inportant in application to study them. In univariate case it is well-known that there is a close relation between continued fraction and univariate interpolation or univariate Padé approximation. The two latters can be obtained by truncating their corresponding interpolating continued fraction [1][2]. The generalization to the multivariate case has recently been considered [3]. Siemaszko,W. gives a bivariate Thiele-type branched continued fraction (TBCF) interpolating formula on a rectangular subset of grid [4]. We develop Siemaszko's formula to a more general case, and show that this interpolation will yield a general order Padé approximation. The interpolant of TBCF is one of the interpolant of TWO-STEP METHOD which we introduce in section 3 and we give the proof of the convergence theorem briefly.

1. TBCF interpolation

Let $I = \bigcup_{j=0}^{t}\{(x_i, y_j) \ : 0 \le i \le s_j\}$ be a given point set in x and y complex plants where $s_0 \ge s_1 \ge \ldots \ge s_t$, and $f(x,y)$ a function whose value is given on I. We consider the problem of constructing a rational function $R(x,y)$ such that $R(x,y)$ is a interpolant for $f(x,y)$ on I.

Let $R(x,y)$ be a rational function which has following form:

$$R(x,y) = B_0(x) + \frac{y-y_0}{B_1(x)} + \cdots + \frac{y-y_{t-1}}{B_t(x)}$$

where

$$B_j(x) = b_0^j + \frac{x-x_0}{b_1^j} + \cdots + \frac{x-x_{s_j-1}}{b_{s_j}^j} \qquad 0 \le j \le t\,.$$

Now we seek for b_i^j $(0 \le i \le s_j, 0 \le j \le t)$ so that $R(x,y)$ is a interpolant for $f(x,y)$ on i. It is clear that if such b_i^j $(0 \le i \le s_j, 0 \le j \le t)$ exist, they are unique.

In order to construct $R(x,y)$ we first introduce the **partial inverted difference** and **partial reciprocal difference** [4]:

$$\varphi_{0,0}[x_i;y_k] = f_{i,k}\,,$$

$$\varphi_{0,1}[x_i;y_k,y_l] = \frac{y_k - y_l}{\varphi_{0,0}[x_i;y_k] - \varphi_{0,0}[x_i;y_l]}\,,$$

$$\varphi_{0,r+1}[x_i;y_{q_0},\ldots,y_{q_{r-1}},y_k,y_l] = \frac{y_k - y_l}{\varphi_{0,r}[x_i;y_{q_0},\ldots,y_{q_{r-1}},y_k] - \varphi_{0,r}[x_i;y_{q_0},\ldots,y_{q_r-1},y_l]}\,, \quad \textit{for } r \ge 1$$

$$\varphi_{1,r}[x_i,x_j;y_{q_0},\ldots,y_{q_r}] = \frac{x_i - x_j}{\varphi_{0,r}[x_i;y_{q_0},\ldots,y_{q_r}] - \varphi_{0,r}[x_j;y_{q_0},\ldots,y_{q_r}]}\,,$$

$$\varphi_{s+1,r}[x_{p_0},\ldots,x_{p_{s-1}},x_i,x_j;y_{q_0},\ldots,y_{q_r}] = \frac{x_i - x_j}{\varphi_{s,r}[x_{p_0},\ldots,x_{p_{s-1}},x_i;y_{q_0},\ldots,y_{q_r}] - \varphi_{s,r}[x_{p_0},\ldots,x_{p_{s-1}},x_j;y_{q_0},\ldots,y_{q_r}]}\,, \quad \textit{for } s \ge 1$$

$$\rho_{0,0}[x_i;y_k] = f_{i,k}\,,$$

$$\rho_{0,1}[x_i;y_k,y_l] = \frac{y_k - y_l}{\rho_{0,0}[x_i;y_k] - \rho_{0,0}[x_i;y_l]}\,,$$

$$\rho_{0,r+1}[x_i;y_0,\ldots,y_{r+1}] = \rho_{0,r-1}[x_i;y_1,\ldots,y_r] + \frac{y_0 - y_{r+1}}{\rho_{0,r}[x_i;y_0,\ldots,y_r] - \rho_{0,r}[x_i;y_1,\ldots,y_{r+1}]}\,, \quad \textit{for } r \ge 1$$

$$\rho_{1,r}[x_i,x_j;y_0,\ldots,y_r] = \frac{x_i - x_j}{\rho_{0,r}[x_i;y_0,\ldots,y_r] - \rho_{0,r}[x_j;y_0,\ldots,y_r]}\,,$$

$$\rho_{s+1,r}[x_0,\ldots,x_{s+1};y_0,\ldots,y_r] = \rho_{s-1,0}[x_1,\ldots,x_s;y_0,\ldots,y_r] + \frac{x_0 - x_{s+1}}{\rho_{s,r}[x_0,\ldots,x_s;y_0,\ldots,y_r] - \rho_{s,r}[x_1,\ldots,x_{s+1};y_0,\ldots,y_r]}\,, \quad \textit{for } s \ge 1$$

If there exists a $\varphi_{s,r}[x_{p_0},\dots,x_{p_s};y_{q_0},\dots,y_{q_r}]$ or a $\rho_{s,r}[x_{p_0},\dots,x_{p_s};y_{q_0},\dots,y_{q_r}]$, then this is called the $(s,r)th$ partial inverted difference or the $(s,r)th$ partial reciprocal difference for function $f(x,y)$ respectively.

Like in the univariate case, the partial reciprocal difference is independent of the order of $x_{p_i},(i=0,\dots,s)$ or $y_{q_j},(j=0,\dots,r)$ and, for $i\geq 1$ and $l\geq 1$ there is the following relationship :

$$\varphi_{0,l}[x_j;y_0,\dots,y_l]=\rho_{0,l}[x_j;y_0,\dots,y_l]-\rho_{0,l-2}[x_j;y_0,\dots,y_{l-2}]\ ,$$

$$\varphi_{i,l}[x_0,\dots,x_i;y_0,\dots,y_l]=\rho_{i,l}[x_0,\dots,x_i;y_0,\dots,y_l]-\rho_{i-2,l}[x_0,\dots,x_{i-2};y_0,\dots,y_l]\ .$$

where $\rho_{0,-1}=\rho_{-1,l}=0$.

Theorem 1. *For $r=0,\dots,t$, if all the partial inverted differences $\varphi_{s,r}[x_0,\dots,x_s;y_0,\dots,y_r]$ for $f(x,y)$, $(0\leq s\leq s_r,)$, are well-defined, then*

$$f(x,y)=B_0(x)+\frac{y-y_0}{B_1(x)}+\dots+\frac{y-y_{t-1}}{B_t(x)}+\frac{y-y_t}{B_{t+1}(x,y)} \tag{1}$$

where

$$B_j(x)=\varphi_{0,j}+\frac{x-x_0}{\varphi_{1,j}}+\dots+\frac{x-x_{s_j-1}}{\varphi_{s_j,j}}+\frac{x-x_{s_j}}{\varphi_{s_j+1,j}},\qquad 0\leq j\leq t\ ,$$

$$B_{t+1}(x,y)=\varphi_{0,t+1}+\frac{x-x_0}{\varphi_{1,t+1}}+\dots\frac{x-x_{s_t-1}}{\varphi_{s_t,t+1}}+\frac{x-x_{s_t+1}}{\varphi_{s_t+1,t+1}}\ .$$

Here for $j=0,\dots,t$, $i=0,\dots,s_j$, $\varphi_{i,j}=\varphi_{i,j}[x_0,\dots,x_i;y_0,\dots,y_j]$, $\varphi_{s_j+1,j}=\varphi_{s_j+1,j}[x_0,\dots,x_{s_j},x;y_0,\dots,y_j]$ and for $i=0,\dots,s_t$, $\varphi_{i,t+1}=\varphi_{i,t+1}[x_0,\dots,x_i;y_0,\dots,y_t,y]$, $\varphi_{s_t+1,t+1}=\varphi_{s_t+1,t+1}[x_0,\dots,x_{s_t},x;y_0,\dots,y_t,y]$.

Moreover, if $R_{k,l}(x,y)$ denotes the $(k,l)th$ convergent of BCF (1), $(k=(k_0,\dots,k_l)$, $(k_0\geq k_1\geq\dots\geq k_l)$, $0\leq k_j\leq s_j$, $0\leq j\leq l\leq t)$, then

$$R_{k,l}(x_i,y_j)=f_{i,j}\ ,\qquad i=0,\dots,k_l,\qquad j=0,\dots,l\ . \tag{2}$$

Proof: Following the definition of inverted difference we can soon obtain the equation (1). For the proof of (2), since

$$R_{k,l}(x,y) = \left[\varphi_{0,0} + \frac{x-x_0}{\varphi_{1,0}} + \cdots + \frac{x-x_{k_0-1}}{\varphi_{k_0,0}}\right] + \left[\frac{y-y_0}{\varphi_{0,1} + \frac{x-x_0}{\varphi_{1,1}} + \cdots + \frac{x-x_{k_1-1}}{\varphi_{k_1,1}}}\right]$$
$$+ \cdots + \left[\frac{y-y_{l-1}}{\varphi_{0,l} + \frac{x-x_0}{\varphi_{1,l}} + \cdots + \frac{x-x_{k_l-1}}{\varphi_{k_l,l}}}\right],$$

thus we have for $i = 0, \ldots, k_l,\ j = 0, \ldots, l$,

$$R_{k,l}(x_i, y_j) = \varphi_{0,0}[x_i; y_0] + \frac{y_j - y_0}{\varphi_{0,1}[x_i; y_0, y_1]} + \cdots + \frac{y_j - y_{j-1}}{\varphi_{0,j}[x_i; y_0, \ldots, y_j]}$$
$$= \varphi_{0,0}[x_i; y_j] = f_{i,j}\ . \qquad \square$$

Thus $R_{s,t}(x,y)$ is the interpolant $R(x,y)$ which we are looking for. The degree of the numerator and the denominator of $R(x,y)$ in y are $[\frac{1}{2}(t+1)]$ and $[\frac{1}{2}t]$ respectively, and the degree of the numerator and the denominator of $B_j(x)$ are $[\frac{1}{2}(s_j+1)]$ and $[\frac{1}{2}s_j]$, respectively. We call $R(x,y)$ a $\left[[\frac{1}{2}(\mathbf{s}+1)]/[\frac{1}{2}\mathbf{s}]; [\frac{1}{2}(t+1)]/[\frac{1}{2}t]\right]$TBCF interpolant.

Here we add that if a subset of grid can be written as $G = \{\cup_\alpha^N (x_\alpha, y_\beta); 0 \le \beta \le M_\alpha\} \cup \{\cup_\beta^M (x_\alpha, y_\beta); 0 \le \alpha \le N_\beta\}$ where it is not necessary that $M_\alpha (\alpha = 0, \ldots, N)$, $N_\beta (\beta = 0, \ldots, M)$ are on the decrease. Then we can interpolate f on G, because by renumbering, G can be written as I .

Now to give the error formula, we define following functions:
for $j = 0, \ldots, t$ let

$$K_j(x) = \varphi_{0,j} + \frac{x-x_0}{\varphi_{1,j}} + \cdots + \frac{x-x_{s_j-1}}{\varphi_{s_j,j}} + \frac{x-x_{s_j}}{\varphi_{s_j+1,j}},$$

$$L_j(x) = \varphi_{0,j} + \frac{x-x_0}{\varphi_{1,j}} + \cdots + \frac{x-x_{s_j-1}}{\varphi_{s_j,j}},$$

$$\frac{P(x,y)}{Q(x,y)} = R(x,y)\ ,$$

$$T_j(x,y)=\frac{P_j(x,y)}{Q_j(x,y)}=K_0(x)+\frac{y-y_0}{K_1(x)}+\cdots+\frac{y-y_{j-1}}{K_j(x)}$$
$$+\frac{y-y_j}{L_{j+1}(x)}+\cdots+\frac{y-y_{t-1}}{L_t(x)},$$

$$\Phi_0(x,y)=Q_0(x,y)Q(x,y)(T_0(x,y)-R(x,y)),$$

$$\Phi_j(x,y)=Q_j(x,y)Q_{j-1}(x,y)(T_j(x,y)-T_{j-1}(x,y)),$$

$$\Phi_{t+1}(x,y)=Q_t(x,y)(f(x,y)-T_t(x,y)),$$

$$V_j(y)=\prod_{l=0}^{j-1}(y-y_l),$$

$$W_j(x)=\prod_{l=0}^{s_j}(x-x_l).$$

Theorem 2. *If for the function $f:D\to R$, $f\in C_D^{s_0,t}$, all the respective partial inverted differences are well-defined, then for an arbitrary chosen $(x,y)\in D$, $\exists\xi_j,\eta_j$ $(j=0,\ldots,t+1)$ such that*

$$f(x,y)-R(x,y)=\frac{W_0(x)}{(s_0+1)!Q_0(x,y)Q(x,y)}D_x^{s_0+1}\Phi_0(\xi_0,y)$$
$$+\sum_{j=1}^{t}\frac{V_j(y)W_j(x)}{j!(s_j+1)!Q_j(x,y)Q_{j-1}(x,y)}D_{x,y}^{s_j+1,j}\Phi_j(\xi_j,\eta_j)$$
$$+\frac{V_{t+1}(y)}{(t+1)!Q_n(x,y)}D_y^{t+1}\Phi_{t+1}(x,\eta_{t+1}). \qquad (3)$$

Proof: For $j=0,\ldots,t+1$, let

$$F_j^1(x,y,u)=\Phi_j(x,u)-\Phi_j(x,y)\frac{V_j(u)}{V_j(y)},$$

$$F_j^2(x,y,v)=\Phi_j(v,y)-\Phi_j(x,y)\frac{W_j(v)}{W_j(x)}.$$

Since for $j=0,\ldots,t$, $K_j(x_i)=L_j(x_i)$, $(i=0,\ldots,s_j)$, we have for $j=1,\ldots,t$,

$$T_j(x,y_l)-T_{j-1}(x,y_l)=0, \qquad (l=0,\ldots,j-1),$$

$$T_j(x_i,y)-T_{j-1}(x_i,y)=0, \qquad (i=0,\ldots,s_j).$$

Therefore $F_j^1(x,y,u)=0$ at $u=y_0,\ldots,y_{j-1},y$, and $F_j^2(x,y,v)=0$ at $v=x_0,\ldots,x_{s_j},x$. From the Rolle theorem we conclude that $\exists\xi_j,\eta_j$ such that

$$\Phi_j(x,y)=\frac{V_j(y)}{j!}D_y^j\Phi_j(x,\eta_j)=\frac{V_j(y)W_j(x)}{j!(s_j+1)!}D_{x,y}^{s_j+1,j}\Phi_j(\xi_j,\eta_j)\;.$$

Thus for $j=1,\ldots,t$,

$$T_j(x,y)-T_{j-1}(x,y)=\frac{V_j(y)W_j(x)}{j!(s_j+1)!Q_j(x,y)Q_{j-1}(x,y)}D_{x,y}^{s_j+1,j}\Phi_j(\xi_j,\eta_j)\;.$$

Similarly we obtain

$$f(x,y)-T_t(x,y)=\frac{V_{t+1}(y)}{(t+1)!Q_t(x,y)}D_y^{t+1}\Phi_{t+1}(x,\eta_{t+1})\;,$$

$$T_0(x,y)-R(x,y)=\frac{W_0(x)}{(s_0+1)!Q_0(x,y)Q(x,y)}D_x^{s_0+1}\Phi_0(\xi_0,y)\;.$$

Using these above three equations we can prove theorem 2. □

2. General order Padé approximation

Let all points of I coincide at $(\bar{x},\bar{y})$. If the limit

$$\lim_{y_0,\ldots,y_l\to\bar{y}}\;[\lim_{x_0,\ldots,x_k\to\bar{x}}\rho_{k,l}[x_0,\ldots,x_k;y_0,\ldots,y_l\;]\;]$$

exist and is finite then we denote this limit by $\rho_{k,l}(\bar{x},\bar{y})$. Due to the ralationship between $\varphi_{k,l}$ and $\rho_{k,l}$, we can deduce that the limit $\varphi_{k,l}(\bar{x},\bar{y})$ also exists. We can compute $\varphi_{k,l}(\bar{x},\bar{y})$ by using following scheme:

$$\rho_{-1,0}=\rho_{k,-1}=0\;,$$

$$\rho_{0,0}(x,y)=\varphi_{0,0}(x,y)=f(x,y)\;,$$

$$\varphi_{i,0}(x,y)=\frac{i}{D_x\rho_{i-1,0}(x,y)}\;,$$

$$\rho_{i,0}(x,y)=\rho_{i-2,0}(x,y)+\varphi_{i,0}(x,y)\;,$$

$$\varphi_{i,j}(x,y)=\frac{j}{D_yD_x\rho_{i,j-1}(x,y)}\;,$$

$$\rho_{i,j}(x,y)=\rho_{i,j-2}(x,y)+\varphi_{i,j}(x,y)\;.$$

Theorem 3. *Let $f(x,y): D \to R$ be analytic in the neighbourhood of the point $(x_0, y_0) \in \Delta$, and let all $\rho_{k,l}(x_0,y_0)$ exist, and $\varphi_{k,l}(x_0,y_0) \neq 0$, $(k = 0,\dots,n, l = 0,\dots,m_k+1)$. Let also $R(x,y)$ be the (s,t)th convergent of BCF (1) where all (x_i, y_j) coincide to (x_0, y_0), then*

$$f(x,y) - R(x,y) = \sum_{k,l}^{\infty} d_{k,l}(x-x_0)^k(y-y_0)^l$$

where $d_{k,l} = 0$ for $k = 0,\dots,s_l$, $l = 0,\dots,t$.

Proof: By reducing $T_j(x,y)$ we obtain that for $j = 0,\dots,t$,

$$Q_j(x_0,y_0) = (\prod_{l=0}^{j}(\prod_{k=1}^{s_l+1}\varphi_{k,l}(x_0,y_0)))(\prod_{l=j+1}^{t}(\prod_{k=1}^{s_l}\varphi_{k,l}(x_0,y_0))) \neq 0\ .$$

Therefore $1/Q_j(x,y)$ can formally be written as $\sum_{i,j\geq 0} c_{i,j}(x-x_0)^i(y-y_0)^j$ where $c_{0,0} \neq 0$. The same is true for $Q(x,y)$. Using the error formular (3), we obtain the conclusion. □

For any $\mathbf{m}$ and n where $0 \leq m_j \leq [\frac{1}{2}s_j]$ $(j = 0,\dots,t)$, $0 \leq n \leq [\frac{1}{2}t]$, let

$$R(x,y) = \sum_{j=0}^{t-2n}\Big(B_j(x)\prod_{k=0}^{j-1}(y-y_k)\Big) + \prod_{k=0}^{t-n-1}(y-y_k)\Big[\frac{y-y_t-2n}{B_{t-2n+1}(x)} + \dots + \frac{y-y_{t-1}}{B_t(x)}\Big],$$

where for $j = 0,\dots,t$,

$$B_j(x) = \sum_{i=0}^{s_j-2m_j}\Big(b_i^j\prod_{l=0}^{i-1}(x-x_l)\Big) + \prod_{l=0}^{s_j-2m_j-1}(x-x_l)\Big[\frac{x-x_{s_j-2m_j}}{b_{s_j-2m_j+1}} + \dots + \frac{x-x_{s_j-1}}{b_{s_j}}\Big].$$

If all the nessasary suitable combinations of partial devided difference and partial inverted diiference exist, then let b_i^j be these combinations of partial devided difference and partial inverted difference, $R(x,y)$ is the $[\frac{\mathbf{s}-\mathbf{m}}{\mathbf{m}}; \frac{t-n}{n}]$ TBCF interpolant for $f(x,y)$ on I.

In next section we introduce a two-step method and we can find that the interpolant of TBCF is one of the interpolant of two-step method.

3. Two-step method for bivariate Hermite rational interpolations

For two given families of complex numbers $\{X\}_s = \{x_0, x_1, \ldots, x_s\}$ and $\{Y\}_t = \{y_0, y_1, \ldots, y_t\}$, we consider the direct product of them $\{X\}_s \times \{Y\}_t = \{x_0, x_1, \ldots, x_s\} \times \{y_0, y_1, \ldots, y_t\} (\subset C^2)$, where we allow that some of the x_i coincide. Assume that the values of a function $f(x, y)$ are known on the points set $\{X\}_s \times \{Y\}_t$ and if $x_{i_1} = x_{i_2} = \cdots = x_{i_\mu}$, then

$$\frac{\partial^k}{\partial x^k} f(x_{i_1}, y_j) \,, \qquad 0 \le k \le \mu - 1$$

are given.

Now we are going to seek for a rational function $r(x,y) = p(x,y)/q(x,y)$, such that $r(x, y)$ is a Hermite interpolant for $f(x, y)$ on $\{X\}_s \times \{Y\}_t$.

There might be many different results of this problem depending on the degrees of p and q. Here we introduce a two-step method to construct rational interplants, which is of practical use by using the computer algebra system "Reduce", and we remark that this interpolating rational function is an extended bivariate Thiele-type interpolant.

Step 1. Let us search for $t + 1$ univariate rational functions $f_j^{[s,t][m]}(x)$, $(j = 0, \ldots, t)$, such that $f_j^{[s,t][m]}(x)$ is a Hermite interpolant for $f(x, y_j)$ on $\{X\}_s$, where the index m indicates the degree of the denominator and the degree of the numerator is not larger than $s - m$.

Step 2. After obtaining every $f_j^{[s,t][m]}(x)$ in Step 1, regarding x as a parameter, we seek for a rational function $f^{[s,t][m,n]}(x, y)$ such that $f^{[s,t][m,n]}(x, y)$ is a interpolant for $f_0^{[s,t][m]}(x)$, $f_1^{[s,t][m]}(x)$, ..., $f_t^{[s,t][m]}(x)$ on $\{Y\}_t$ and has the denominator of degree n and the numerator of degree $\le t - n$ with respect to y.

Remark. If the interpolating function $f_j^{[s,t][m]}(x)$ exists, then it is unique because of the uniqueness of univariate rational interpolant. As for $f^{[s,t][m,n]}(x,y)$, the uniqueness of its existence is guaranteed also. Furthermore, if $f^{[s,t][m,n]}(x,y)$ exists and $x_{i_1} = x_{i_2} = \cdots = x_{i_\mu}$, it holds that

$$\frac{\partial^k}{\partial x^k} f^{[s,t][m,n]}(x_{i_1}, y_j) = \frac{\partial^k}{\partial x^k} f_j^{[s,t][m]}(x_{i_1}, y_j) = \frac{\partial^k}{\partial x^k} f(x_{i_1}, y_j)$$

for $0 \le k \le \mu - 1$, which means that $f^{[s,t][m,n]}(x,y)$ is surely a Hermite interpolant for $f(x,y)$ on $\{X\}_s \times \{Y\}_t$.

4. Convergence of interpolating rational functions

Let X and Y be the following two bounded triangular interpolation schemes in the x and y complex planes respectively

$$\begin{array}{lll} x_0^0 & & \\ x_0^1 & x_1^1 & \\ x_0^2 & x_1^2 & x_2^2 \\ \vdots & & \end{array} \qquad\qquad \begin{array}{lll} y_0^0 & & \\ y_0^1 & y_1^1 & \\ y_0^2 & y_1^2 & y_2^2 \\ \vdots & & \end{array}$$

and $\hat{X}$ and $\hat{Y}$ the smallest closed (hence compact) sets containing all points of X and Y respectively. We denote the sets of all nomalized positive Borel measures with their supports in $\hat{X}$ and $\hat{Y}$ by $M_1(\hat{X})$ and $M_2(\hat{Y})$ respectively and define the elementary measures $\{\mu_n\}$ and the logarithmic potential $u(z;\mu)$ as follows.

Definition 1. For a triangular interpolation scheme Z and for any Borel set B, the associated elementary measures $\{\mu_n\}$ are defined by

$$\mu_n(B) = \frac{1}{n+1}\sum_{i=0}^{n} \chi(z_i^n \in B) \ ,$$

where $\chi(P) = 1$ if P is true and 0 if P is false.

A sequence of elementary measures $\{\mu_n\}$ will be called regular if there exists a $\mu \in M(\hat{X})$ such that $\mu_n \to \mu$.

Definition 2. Let E be a compact subset of the complex plane and μ a measure of $M(E)$. The logarithmic potential of μ, $u(z;\mu)$, is the function

$$u(z;\mu) \equiv \int_E log \frac{1}{|z-\xi|} d\mu(\xi) .$$

Now let X and Y be two bounded triangular interpolation schemes, whose respective associated elementary measures $\{\mu_n\}$ and $\{\nu_n\}$ are regular, as $\mu_n \to \mu$ and $\nu_n \to \nu$. Let $X_\rho \equiv \{x \in C : e^{-u(x;\mu)} < \rho\}$ and $Y_\varrho \equiv \{y \in C : e^{-u(y;\nu)} < \varrho\}$ for positive real numbers ρ and ϱ. Then we have following three lemmas.

Lemma 1. [5] *Let Γ_1 and Γ_2 be cycles which contain the closure of $\hat{X} \cup X_\rho$ and $\hat{Y} \cup Y_\varrho$ respectively. For two compact subsets K_1 and K_2 of X_ρ and Y_ϱ respectively, we put $\gamma_1 = max\{e^{-u(x;\mu)} : x \in K_1\}$ and $\gamma_2 = max\{e^{-u(y;\nu)} : y \in K_2\}$. Then for any given ϵ_1 and $\epsilon_2 > 0$, there exist two positive numbers s_0 and t_0 such that for $s > s_0$ and $t > t_0$*

$$\left|\frac{(x-x_0^s)^{[s+1]}}{(\xi-x_0^s)^{[s+1]}}\right| \leq [\frac{\gamma_1}{\rho}e^{2\epsilon_1}]^{s+1} \qquad for\ (x,\xi) \in K_1 \times \Gamma_1$$

and

$$\left|\frac{(y-y_0^t)^{[t+1]}}{(\eta-y_0^t)^{[t+1]}}\right| \leq [\frac{\gamma_2}{\varrho}e^{2\epsilon_2}]^{t+1} \qquad for\ (y,\eta) \in K_2 \times \Gamma_2,$$

where

$$(z-z_0^n)^{[n+1]} \equiv (z-z_0^n)(z-z_1^n)\cdots(z-z_n^n) .$$

Lemma 2. [6] *Let $Q(x,y)$ be a bivariate polynomial,*

$$Q(x,y) = d_0(y) + d_1(y)x + \cdots + d_{M-1}(y)x^{M-1} + x^M \ ,$$

where the degrees of $d_i(y)$ $(i = 0, \dots, M-1) \leq P$ and there is a y_0 such that $Q(x, y_0)$ has M simple roots. Then for suitablely choosed two bounded triangular interpolation schemes X and Y where the points of each row of Y do not coincide, there exist two positive real numbers ρ, ϱ and M algebraic functions $\alpha_i(y)$ $(i = 1, \dots, M)$ $: Y_\varrho \to X_\rho$, such that

$$Q(x,y) = (x - \alpha_1(y))(x - \alpha_2(y)) \cdots (x - \alpha_M(y))$$

for all $(x, y) \in X_\rho \times Y_\varrho$, where X_ρ and Y_ϱ are domains defined just after definition 2 and $\alpha_i(y)$ $(i = 1, \dots, M)$ are holomorphic in Y_ϱ and any two of $\{\alpha_i(y)\}$ do not intersect in Y_ϱ.

Lemma 3. [6] *Let $Q(x, y)$ be a bivariate polynomial such that for given X_ρ, Y_ϱ and for a $x_0 \in X_\rho$, $Q(x_0, y)$ has N simple roots $y_1, \dots, y_N$ in Y_ϱ. Then for suitable neighbourhoods $U_\delta(x_0)$ and $U_\varepsilon(y_i)$ $(i = 1, \dots, N)$ there are exact N functions $\beta_i(x) : U_\delta(x_0) \to U_\varepsilon(y_i)$ $(i = 1, \dots, N)$ such that*

$$Q(x, \beta_i(x)) = 0 \qquad x \in U_\delta(x_0) \ .$$

Let $\{q^{(m)}(x, y)\}$ be a sequence of holomorphic functions in $X_\rho \times Y_\varrho$ which uniformly converges to $Q(x, y)$ on any compact subset of $X_\rho \times Y_\varrho$. Then there exist exact N holomorphic functions $\beta_i^{(m)}(x)$ $(i = 1, \dots, N)$ $: U_\delta(x_0) \to U_\varepsilon(y_i)$ such that

$$q^{(m)}(x, \beta_i^{(m)}(x)) = 0 \qquad x \in U_\delta(x_0)$$

and $\beta_i^{(m)}(x) \to \beta_i(x)$ uniformly on any compact set of $U_\delta(x_0)$.

Now we are going to prove the convergence theorem. In the sequel, for X and Y, we choose a Y with each row of distinct points and sufficient large positive real number ρ and ϱ such that $\hat{X} \subset X_\rho$ and $\hat{Y} \subset Y_\varrho$.

theorem 4. *Let X and Y be two bounded triangular interpolation schemes in the x and y complex planes respectively and, Ω_x and Ω_y two domains such as $\hat{X} \subset X_\rho$, $\overline{X_\rho} \subset \Omega_x$ and $\hat{Y} \subset Y_\varrho$, $\overline{Y_\varrho} \subset \Omega_y$. Let $f(x,y) = g(x,y)/Q(x,y)$ be holomorphic on $\hat{X} \times \hat{Y}$ and meromorphic on $X_\rho \times Y_\varrho$, where $Q(x,y)$ is a bivariate polynomial satisfying following conditions:*

1) $Q(x,y) = d_0(y) + d_1(y)x + \cdots + d_{M-1}(y)x^{M-1} + x^M$, where degrees of $d_i(y)$ $(i = 0, 1 \ldots, M-1) \leq P$,

2) $Q(x,y)$ can be factorized as $(x - \alpha_1(y)) \cdots (x - \alpha_M(y))$, where $\alpha_i(y)$ $(i = 1, \ldots, M)$ are holomorphic in Ω_y, any two of $\{\alpha_i\}$ do not intersect each other in $\overline{\Omega_y}$, and $\cup_{i=1}^{M}\{\alpha_i(y) : y \in \overline{\Omega_y}\} \subset X_\rho$,

and $g(x,y)$ is holomorphic on $\Omega_x \times \Omega_y$, such that $g(x,y) \neq 0$ for all $(x,y) \in \{(x,y) \in \Omega_x \times \Omega_y : Q(x,y) = 0\}$.

If $Q(x_0,y)$ has N simple zero points y_i $(i = 1, \ldots, N)$ in Y_ϱ for a $x_0 \in X_\rho$ and $f^{[s,t][M,N]}(x,y)$ is a Hermite interpolant for $f(x,y)$ on $X \times Y$ in Step 2 of section 3, then there are neighbourhoods $V(x_0)$ and $U_\varepsilon(y_i)$ $(i = 1, \ldots, N)$ of x_0 and y_i respectively such that

$$\lim_{s\to\infty} [\lim_{t\to\infty} f^{[s,t][M,N]}(x,y)] = f(x,y)$$

uniformly on $\overline{V(x_0)} \times K$ where K is any compact subset of $Y_\varrho \backslash \cup_{i=1}^{N} U_\varepsilon(y_i)$.

Proof. Let K_1 be a compact subset of $X_\rho \times Y_\varrho \backslash \{(x,y) : Q(x,y) = 0\}$ such that $\hat{X} \times \hat{Y} \subset K_1$. First we prove that there are constants ρ' and B_1 such that $0 < \rho' < 1$ and, for the $(t+1)$th row of Y $\{y_0^t, y_1^t, \ldots, y_t^t\}$,

$$\left| f_j^{[s,t][M]}(x) - f(x, y_j^t) \right| \leq B_1 (\rho')^{s+1} \qquad \textit{for } (x, y_j^t) \in K_1 \ ,$$

where $f_j^{[s,t][M]}(x)$ is a Hermite interpolant for $f(x, y_j^t)$ on the $(s+1)$th row of X $\{x_0^s, x_1^s, \ldots, x_s^s\}$.

Let K_2 be a compact subset of X_ρ such that $K_2 \supset \hat{X} \cup \cup_{i=1}^{M}\{\alpha_i(y) : y \in \overline{Y_\varrho}\} \cup \{x : (x, y_j^t) \in K_1, y_j^t \in Y\}$. We put

$$q^s(x,y) = Q(x,y) + \sum_{k=0}^{M-1} a_k^s(y)(x-\alpha_1(y))^{[k]} ,$$

where $a_k^s(y)$ $(k = 0, \ldots, M-1)$ are holomorphic in a domain Ω_y' which $\overline{Y_\varrho} \subset \Omega_y' \subset \Omega_y$ and $(x-\alpha_1(y))^{[k]}$ $(k = 0, \ldots, M-1)$ are the defined ones at the last part of section 3. Then the function $h^s(x,y) = q^s(x,y)g(x,y)$ is holomorphic in $\Omega_x \times \Omega_y'$. Considering that y is a parameter, we construct a polynomial interpolant $\pi^s(x,y)$ with degree at most s for $h^s(x,y)$ on the points set $\{x_0^s, \ldots, x_s^s\}$. Then by the integral representation of Hermite interpolating polynomal, we have

$$\pi^s(x,y) = \frac{1}{2\pi i}\int_{\Gamma_1}\left(1 - \frac{(x-x_0^s)^{[s+1]}}{(\xi - x_0^s)^{[s+1]}}\right)\frac{h^s(\xi,y)}{\xi - x}d\xi ,$$

where Γ_1 is a cycle homologous to 0 in Ω_x, with winding number $n(x,\Gamma_1) = 1$ for all $x \in \overline{X_\rho}$.

We show that it is possible to choose $a_k^s(y)$ $(0 \le k \le M-1)$ so that $(x-\alpha_1(y))^{[M]}$ is a factor of $\pi^s(x,y)$. Substituting $\alpha_i(y)$ $(i = 1, \ldots, M)$ into x in the above integral representation of $\pi^s(x,y)$, we have

$$\pi^s(\alpha_i(y), y) = \sum_{k=1}^{M}[c_{ik}^1(y) + c_{ik}^{2s}(y)]a_{k-1}^s(y) - c_i^{3s}(y)$$

where

$$c_{ik}^1(y) = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{(\xi-\alpha_1(y))^{[k-1]}g(\xi,y)}{\xi - \alpha_i(y)}d\xi ,$$

$$c_{ik}^{2s}(y) = -\frac{1}{2\pi i}\int_{\Gamma_1}\frac{(\alpha_i(y)-x_0^s)^{[s+1]}}{(\xi - x_0^s)^{[s+1]}}\frac{(\xi-\alpha_1(y))^{[k-1]}g(\xi,y)}{\xi - \alpha_i(y)}d\xi,$$

$$c_i^{3s}(y) = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{(\alpha_i(y)-x_0^s)^{[s+1]}}{(\xi - x_0^s)^{[s+1]}}\frac{(\xi-\alpha_1(y))^{[M]}g(\xi,y)}{\xi - \alpha_i(y)}d\xi .$$

It is clear that for all $y \in Y_\varrho$,

$$c_{ik}^1(y) = 0 , \qquad\qquad i < k$$

$$c^1_{ik}(y) = (\alpha_i(y) - \alpha_1(y))^{[k-1]} g(\alpha_i(y), y) \neq 0 \ , \quad i \geq k$$

We have $c_k = min\{|c^1_{kk}(y)| : y \in \overline{Y_\varrho}\} > 0$ $(k = 1, \ldots, M)$. Hence the matrix $C^1(y) = (c^1_{ik})$ is a regular triangular matrix for any $y \in \overline{Y_\varrho}$.

On the other hand, for any given $\epsilon > 0$, by Lemma 1, we can find a positive number s_0, such that if $s > s_0$, for all $y \in \overline{Y_\varrho}$,

$$|c^{2s}_{ik}(y)| \leq \frac{B}{2\pi\delta} |\Gamma_1| (\frac{\gamma_1}{\rho} e^{2\epsilon})^{s+1}$$

$$|c^{3s}_i(y)| \leq \frac{B}{2\pi\delta} |\Gamma_1| (\frac{\gamma_1}{\rho} e^{2\epsilon})^{s+1}$$

where $|\Gamma_1|$ is the length of Γ_1, $\delta = dis(\Gamma_1, K_2) > 0$, $B = max\{max\{|(\xi - \alpha_1(y))^{[k-1]} g(\xi, y)| : \xi \in \Gamma_1, y \in \overline{Y_\varrho}\}, 1 \leq k \leq M\}$ and $\gamma_1 = max\{e^{-u(x;\mu)} : x \in K_2\} < \rho$. If we choose ϵ such as $0 < \epsilon < \frac{1}{2} log_e(\frac{\rho}{\gamma_1})$, then there is a real number ρ' which satisfies the condition $0 < (\frac{\gamma_1}{\rho} e^{2\epsilon}) < \rho' < 1$.

Hence we can find a positive number s_1 such that for all $s > s_1$

$$min\{|det[C^1(y) + C^{2s}(y)]| : y \in \overline{Y_\varrho}\} = \delta_1 > 0 \ ,$$

where $C^{2s}(y)$ is a matrix (c^{2s}_{ik}). Thus, for sufficient large s, the linear system

$$\Big[C^1(y) + C^{2s}(y)\Big] a^s(y) - c^{3s}(y) = 0$$

where $a^s(y) = (a^s_0(y), \ldots, a^s_{M-1}(y))^T$ and $c^{3s}(y) = (c^{3s}_1(y), \ldots, c^{3s}_M(y))^T$, is solvable. Since the elements of $C^1(y), C^{2s}(y), c^{3s}$ are holomorphic in Ω_y and, for all $y \in \overline{Y_\varrho}$, $det[C^1(y) + C^{2s}(y)] \neq 0$, then there is a domain Ω'_y such that $\Omega_y \supset \Omega'_y \supset \overline{Y_\varrho}$ and all elements of $[C^1(y) + C^{2s}(y)]^{-1}$ are holomorphic in Ω'_y. Thus $a^s_i(y)$ $(i = 0, \ldots, M-1)$ are holomorphic in Ω'_y and therefore we can choose $a^s(y)$ so that $q^s(x, y)$ is holomorphic in $\Omega_x \times \Omega'_y$ and $(x - \alpha_1(y))^{[M]}$ is a factor of $\pi^s(x, y)$.

By using the estimate inequality of $c^{3s}_i(y)$, we get for $(x, y) \in K_2 \times \overline{Y_\varrho}$

$$|a^s_i(y)| = |([C^1(y) + C^{2s}(y)]^{-1} c^{3s}(y))_i| \leq \frac{MB_2}{\delta_1} \frac{B|\Gamma_1|}{2\pi\delta} (\rho')^{s+1}$$

and

$$|Q(x,y)-q^s(x,y)| \le \sum_{i=0}^{M-1} |a_i^s(y)|\cdot|(x-\alpha_1(y))^{[i]}| \le \frac{M^2B_2B_3}{\delta_1}\frac{B|\Gamma_1|}{2\pi\delta}(\rho')^{s+1} ,$$

where $\delta_1 = min\{|det[C^1(y)+C^{2s}(y)]| : y \in \overline{Y_\varrho}\}$, $B_2 = max\{|c_{i,k}^1(y) + c_{i,k}^{2s}(y)| : y \in \overline{Y_\varrho}\}$ and $B_3 = max\{|(x-\alpha_1(y))^{[k]}| : x \in \overline{X_\rho}, y \in \overline{Y_\varrho}, k = 0,\dots,M\}$.

Since K_1 is a compact subset of $X_\rho \times Y_\varrho \backslash \{(x,y) : Q(x,y) = 0\}$, there is a positive real number δ_2 such that $|Q(x,y)| \ge \delta_2$ for all $(x,y) \in K_1$ and then, for sufficient large number s, $q^s(x,y) \ge \delta_2/2$ for all $(x,y) \in K_1$. Put

$$p^s(x,y) = \pi^s(x,y)/(x-\alpha_1(y))^{[M]} .$$

Then the degree of $p^s(x,y)$ in x is not larger than $s-M$ by the above consideration. Since $\pi^s(x,y_j^t)$ is an Hermite interpolant for $q^s(x,y_j^t)g(x,y_j^t)$ on $x_0^s,\dots,x_s^s$, $q^s(x_i^s,y_j^t) \ne 0$ and $(x_i^s-\alpha_1(y_j^t))^{[M]} \ne 0$, $p^s(x,y_j^t)/q^s(x,y_j^t)$ is a Hermite interpolant for $f(x,y_j^t)$ on $x_0^s,\dots,x_s^s$. Hence, when s is sufficient large, there exists $f_j^{[s,t][M]}(x)$ which can be expressed by

$$f_j^{[s,t][M]}(x) = \frac{p^s(x,y_j^t)}{q^s(x,y_j^t)} .$$

Let $B_4 = max\{|\frac{1}{2\pi i}\int_{\Gamma_1} \frac{(\xi-\alpha_1(y))^{[k]}}{\xi - x} d\xi| : (x,y) \in K_1, k = 0,\dots,M\}$. Then

$$\left|\frac{p^s(x,y)}{q^s(x,y)} - f(x,y)\right| = \frac{|\pi^s(x,y)-q^s(x,y)g(x,y)|}{|q^s(x,y)Q(x,y)|} \le \frac{2}{\delta_2^2}[\sum_{k=0}^{M-1}|a_k^s(y)|+1]B_4(\rho')^{s+1}$$

and consequently, there exists a positive constant B_1 such that

$$|f_j^{[s,t][M]}(x) - f(x,y_j^t)| \le B_1(\rho')^{s+1} \qquad \textit{for all } (x,y_j^t) \in K_1$$

Now we prove the rest half part of the theorem. By Lemma 3, there are N holomorphic functions $\beta_i(x)$ $(i = 1,\dots,N)$ such that $Q(x,\beta_i(x)) =$

0 in a suitable neighbourhood $U_\delta(x_0)$. In the above argument, we have already proved that $q^s(x,y)$ uniformly convergs to $Q(x,y)$ in $K_2 \times \overline{Y_\varrho}$. Then by Lemma 3, $q^s(x,y) = 0$ also decides N holomorphic functions $\beta_i^s(x)$ $(i = 1,\ldots,N)$ and $\beta_i^s(x) \to \beta_i(x)$ uniformly in any compact set $\overline{V(x_0)} \subset U_\delta(x_0)$.

We put

$$q^{s,t}(x,y) = (y - \beta_1^s(x))^N + \sum_{k=0}^{N-1} b_k^{s,t}(x)(y - \beta_1^s(x))^{[k]} ,$$

where $b_k^{s,t}(x)$ is holomorphic in an open set $V'(x_0)$ which $\overline{V(x_0)} \subset V'(x_0) \subset U_\delta(x_0)$. Then $q^{s,t}(x,y)$ is a polynomial of degree N with respect to y. If we set $g^s(x,y) = p^s(x,y)(y - \beta_1^s(x))^{[N]}/q^s(x,y)$, then $g^s(x,y)$ is holomorphic in $V'(x_0) \times \Omega'_Y$.

Here regarding x as a parameter, we construct a polynomial interpolant $\pi^{s,t}(x,y)$ with degree $\leq t$ for $q^{s,t}(x,y)g^s(x,y)$ at $y_0^t,\ldots,y_t^t$.

Let K be a compact subset of $Y_\varrho \backslash \cup_{i=1}^N U_\varepsilon(y_i)$ where $U_\varepsilon(y_i)$ $(i = 1,\ldots,N)$ are neighbourhoods of y_i. Discussing in the same way as in previous, for sufficient large t, we can find constants ρ'', B_5, B_6 and N holomorphic functions $b_k^{s,t}(y)$ $(k = 1,\ldots,N)$ so that $0 < \rho'' < 1$, $(y - \beta_1^s(x))^{[N]}$ is a factor of $\pi^{s,t}(x,y)$ and, for $x \in \overline{V(x_0)}$ and $y \in K$

$$|b_k^{s,t}(y)| \leq B_5(\rho'')^{t+1}$$

and

$$|q^{s,t}(x,y) - q^s(x,y)| \leq B_6(\rho'')^{t+1} .$$

Now we put

$$p^{s,t}(x,y) = \pi^{s,t}(x,y)/(y - \beta_1^s(x))^{[N]} .$$

Then, finding a suitable constant B_7, we have

$$\left|\frac{p^{s,t}(x,y)}{q^{s,t}(x,y)} - \frac{g^s(x,y)}{q^s(x,y)}\right| \leq B_7(\rho'')^{t+1} \qquad (x,y) \in \overline{V(x_0)} \times K .$$

If we choose sufficient small $\varepsilon > 0$ such that $(V(x_0) \times \cup_{i=1}^{N} U_\varepsilon(y_i)) \cap (\hat{X} \times \hat{Y}) = \emptyset$, for sufficient large s and t, $f^{[s,t][M,N]}(x,y)$ surely exists, and

$$f^{[s,t][M,N]}(x,y) = p^{s,t}(x,y)/q^{s,t}(x,y) \ ,$$
$$|f^{[s,t][M,N]}(x,y) - f(x,y)| \leq B_7(\rho'')^{t+1} + B_1(\rho')^{s+1}$$

for $(x,y) \in \overline{V(x_0)} \times K$. Thus we complete the proof. □

References

[1] George A. Baker, Jr. and Peter Graves-Morris; *Padé Approximation*, Addison-Wesley Publishing Company, 1981.

[2] Cuty A. and Wuytack L.; *Nonlinear methods in numarical analysis*, North Holland, Amsterdam,1986.

[3] Cuty A. and Verdonk B.; *Multivariate rational interpolation*, Computing **34** (1985), 41-61.

[4] Siemaszko W.;*Branched continued fraction for two-variable functions*, J. Comp. Appl. Math **9** (1983), 137-153.

[5] D. D. Warner; *An extension of Saff's theorem on the convergence of interpolating rational functions*, J. Approximation Theory **18** (1976), 108-118.

[6] Zhang Guofeng and Shiro Ogawa; *Uniform convergence of multivariate interpolating rational functions*, Kobe Journal of Mathematics, to appear.

[7] Zhang Guofeng; *Bivariate Thiele-type branched continued fraction and general order Padé approximation*, submitted.