

多変数 Padé 近似理論と補間について

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Summary. In [4], we studied multivariate Padé-type and Padé approximants by following similar ways to those of Brezinski[2] in univariate case. Brezinski[2] pointed out the fundamental fact that Padé-type approximants of $f(t)$ can be derived by operating the functional c on an interpolation polynomial of the generating function of $f(t)$. Sablonnière[5] and Arioka[1] extended this fact to the multivariate case by using their own functionals and generating functions. In this paper, we explain this fact from our viewpoint in [4] and study the relations to [1] and [5].

§1. Introduction. In [4], we introduced multivariate Padé-type approximants by the following ways. Let $f(t)=f(t_1, \dots, t_N)$ be a formal power series in N variables t_1, \dots, t_N with real coefficients,

$$(1.1) \quad f(t) = c_0 + c_1 + c_2 + \dots + c_l + \dots, \quad t = (t_1, \dots, t_N),$$

where c_i is a homogeneous polynomial of degree i in t_1, \dots, t_N with real coefficients. And let $P(X)$ be a "formal Laurent series" in X whose coefficients are polynomials in t_1, \dots, t_N .

$$P(X) = a_n X^n + a_{n+1} X^{n+1} + \dots, \quad a_i \in R[t_1, \dots, t_N], \quad i = n, n+1, \dots,$$

where $R[t_1, \dots, t_N]$ is the polynomial ring in t_1, \dots, t_N over the real number field R and n is an integer which may be negative. Let \mathcal{P} be the totality of the above "formal Laurent series". Then \mathcal{P} is an integral domain and contains the polynomial in X whose coefficients are polynomials in t_1, \dots, t_N . The inverse

element of a unit $P(X)$ of \mathcal{P} is denoted by $1/P(X)$. For example,

$$(1.2) \quad \frac{1}{1-X} = 1+X+X^2+X^3+\dots, \quad \frac{1}{X^n(1-X)} = X^{-n}+X^{-n+1}+X^{-n+2}+\dots.$$

For (1.1), an operator c acting on \mathcal{P} is defined by

$$c\left(\sum_i a_i X^i\right) = \sum_i a_i c_i \quad (\text{with the convention that } c_i=0 \text{ for } i < 0).$$

This operator c has the following property:

For $P(X), Q(X) \in \mathcal{P}$ and $a, b \in R[t_1, \dots, t_N]$,

$$(1.3) \quad c(aP(X)+bQ(X)) = ac(P(X)) + bc(Q(X)).$$

We define the operator $c^{(n)}$ by $c^{(n)}(P(X)) = c(X^n P(X))$ for $P(X) \in \mathcal{P}$,

where n is an integer. Then $c^{(n)}$ also has the same property as (1.3).

Operating c or $c^{(n)}$ on the special element X^i of \mathcal{P} , we have

$$c(X^i) = c_i \text{ and } c^{(n)}(X^i) = c_{n+i},$$

where $c_i=0$ for $i < 0$ and $c_{n+i}=0$ for $n+i < 0$.

We have immediately the following lemma by (1.2).

$$\text{Lemma 1.1} \quad c^{(n)}\left(\frac{1}{1-X}\right) = f(t), \quad t=(t_1, \dots, t_N), \quad (n \leq 0).$$

Here $1/(1-X)$ is called a generating function of $f(t)$.

The polynomial of \mathcal{P} is called a g -polynomial if it is a homogeneous polynomial with respect to $N+1$ variables t_1, \dots, t_N, X .

A g -polynomial $V(X)$ is expressed as follows,

$$(1.4) \quad V(X) = b_m X^q + b_{m+1} X^{q-1} + \dots + b_{m+i} X^{q-i} + \dots + b_{m+q}, \quad b_m \neq 0,$$

where b_{m+i} is a homogeneous polynomial of degree $m+i$ in t_1, \dots, t_N .

Then, we call $V(X)$ a g -polynomial of degree q with shift m .

$V(1) = b_m + b_{m+1} + \dots + b_{m+q}$ ($\in R[t_1, \dots, t_N]$) is called the reverse polynomial of $V(X)$ and denoted by $v(t)$ for $t=(t_1, \dots, t_N)$.

Multivariate Padé-type approximants "with shift m " are defined

as follows.

Definition 1.1. Let $V(X)$ be a g -polynomial of degree q with shift m and

$$(1.5) \quad w(t) = c \left(\frac{v(t) - X^{p+q+1} V(X)}{1 - X} \right), \quad t = (t_1, \dots, t_N),$$

where $v(t)$ is the reverse polynomial of $V(X)$. Then the rational function $w(t)/v(t)$ is called the (p/q) Padé-type approximant with shift m and denoted by $(p/q)_f^{\#}(t)$. We call the g -polynomial $V(X)$ a generating polynomial of the Padé-type approximant $(p/q)_f^{\#}(t)$.

Theorem 1.1 (Th.2.1 in [4]) In Definition 1.1, $v(t)$ and $w(t)$ are polynomials of degree $m+q$ and $m+p$ respectively. Moreover,

$$(1.6) \quad f(t)v(t) - w(t) = c^{(p+q+1)} \left(\frac{V(X)}{1-X} \right) = O(m+p+1), \quad t = (t_1, \dots, t_N).$$

Theorem 1.2 (Th.2.4 in [4]) Let $\bar{w}(t)$ be a function in t_1, \dots, t_N .

$$\bar{w}(t) = c^{(p+q+1)} \left(\frac{v(t) - V(X)}{1-X} \right), \quad t = (t_1, \dots, t_N).$$

(a) If $p < q$, then $(p/q)_f^{\#}(t) = \frac{\bar{w}(t)}{v(t)}$.

(b) If $p \geq q$, then $(p/q)_f^{\#}(t) = c_0 + \dots + c_{p-q} + \frac{\bar{w}(t)}{v(t)}$.

§2. Relation between Padé-type approximation and polynomial interpolation. Let Ω be a function field which contains all polynomials in t_1, \dots, t_N (i.e. the rational function field $R(t_1, \dots, t_N)$ or its extension field). $\Omega[X]$ and $\Omega(X)$ are the polynomial ring and the rational function field in X over the field Ω respectively. In considering the interpolation problem, since it is algebraically meaningless to substitute an element of Ω into the variable X of the formal infinite series $g(X) = 1 + X + X^2 + \dots$, we need regard the generating function $g(X)$ as an element $1/(1-X)$ of $\Omega(X)$.

Let us now define the Hermite interpolation polynomial of the generating function $g(X)=1/(1-X)$.

Definition 2.1 Let $\alpha_1, \dots, \alpha_s$ be s given distinct 'points' of Ω . If the polynomial $P_n(X)$ ($\in \Omega[X]$) of degree n in X satisfies the following condition,

$$(2.1) \quad P_n^{(j)}(\alpha_i) = g^{(j)}(\alpha_i), \quad 0 \leq j \leq k_i - 1, \quad i=1, \dots, s, \quad \sum_{i=1}^s k_i = n+1, \quad k_i \geq 1,$$

where $P_n^{(j)}(X)$ and $g^{(j)}(X)$ denote the j -th formal algebraic derivatives $\frac{d^j}{dX^j} P_n(X)$ and $\frac{d^j}{dX^j} g(X)$ respectively, then the polynomial $P_n(X)$ is called the Hermite interpolation polynomial of $g(X)$ at the nodes $\alpha_1, \dots, \alpha_s$.

We can prove the uniqueness of such interpolating polynomial in the same way as in the ordinary interpolation problem for a real valued function. If there exists an element α (a function in t_1, \dots, t_N) of Ω such that $P(\alpha) = 0 \in \Omega$, then the function α is called the zero of $P(X)$. We denote $\frac{1}{u(t)} c^{(n)}(P(X))$ by $c^{(n)}\left(\frac{P(X)}{u(t)}\right)$ for the sake of convenience.

The following theorem gives the relation between polynomial interpolation and Padé-type approximation.

Theorem 2.1 Let $V(X) = b_m X^q + b_{m+1} X^{q-1} + \dots + b_{m+q}$ be a g -polynomial of degree q with shift m . Suppose that s distinct functions (in t_1, \dots, t_N) $\alpha_1, \dots, \alpha_s$ of Ω are the zeros of multiplicity k_i of $V(X)$, that is, $V(X) = b_m \prod_{i=1}^s (X - \alpha_i)^{k_i}$, $k_1 + \dots + k_s = q$, $k_i \geq 1$.

(a) The case of $p > q - 1$. Let $P(X)$ be the Hermite interpolation polynomial of degree p of the generating function $1/(1-X)$ at

the nodes $\alpha_1, \dots, \alpha_s$ and 0 (with multiplicity $p-q+1$). Then

$$c(P(X)) = (p/q)_f^n(t), \quad t = (t_1, \dots, t_N).$$

(b) The case of $p \leq q-1$. Let $P(X)$ be the Hermite interpolation polynomial of degree $q-1$ of $1/(1-X)$ at the nodes $\alpha_1, \dots, \alpha_s$. Then

$$c^{(p-q+1)}(P(X)) = (p/q)_f^n(t), \quad t = (t_1, \dots, t_N).$$

In both cases, the denominator of the approximant is the reverse polynomial of $V(X)$ i.e. $V(1) = v(t) = b_m \prod_{i=1}^s (1-\alpha_i)^{k_i}$, $t = (t_1, \dots, t_N)$.

Proof. (a) Put $\bar{P}(X) = \frac{v(t) - X^{p-q+1}V(X)}{v(t)(1-X)}$. Then, from the expression

$$\begin{aligned} \bar{P}(X) &= \frac{b_m + \dots + b_{m+q} - b_m X^{p+1} - \dots - b_{m+q} X^{p-q+1}}{v(t)(1-X)} \\ &= \frac{1}{v(t)} \left\{ b_m(1+X+\dots+X^p) + \dots + b_{m+q}(1+X+\dots+X^{p-q}) \right\}, \end{aligned}$$

it follows that $\bar{P}(X)$ is a polynomial of degree p with respect to X . We are going to show that the polynomial $\bar{P}(X)$ satisfies the condition (2.1) for $n=p$. $\bar{P}(X)$ can be written as follows,

$$\bar{P}(X) = \frac{1}{1-X} - \frac{X^{p-q+1}V(X)}{v(t)(1-X)} = \frac{1}{1-X} - \frac{b_m \prod_{i=1}^{s+1} (X-\alpha_i)^{k_i}}{v(t)(1-X)},$$

where $\alpha_{s+1} = 0$ and $k_{s+1} = p-q+1$. Differentiating j times with respect to X and substituting α_i into X ,

$$\left(\bar{P}(X) \right)_{X=\alpha_i}^{(j)} = \left(\frac{1}{1-X} \right)_{X=\alpha_i}^{(j)} - \frac{b_m}{v(t)} \left(\frac{\prod_{i=1}^{s+1} (X-\alpha_i)^{k_i}}{1-X} \right)_{X=\alpha_i}^{(j)},$$

where $0 \leq j \leq k_i - 1$, $i=1, \dots, s+1$ and $\sum_{i=1}^{s+1} k_i = q + (p-q+1) = p+1$. As the last terms equal zero, the condition (2.1) holds. By the uniqueness of the interpolation polynomial, the polynomial $\bar{P}(X)$ coincides with $P(X)$. On the other hand,

$$c(\bar{P}(X)) = \frac{1}{v(t)} c \left(\frac{v(t) - X^{p-q+1}V(X)}{1-X} \right) = (p/q)_f^n(t),$$

which implies the result of the case (a).

(b) Put

$$\bar{P}(X) = \frac{v(t) - V(X)}{v(t)(1-X)} = \frac{1}{1-X} - \frac{b_m \prod_{i=1}^s (X - \alpha_i)^{k_i}}{v(t)(1-X)}.$$

Then, from the similar consideration to that of (a), it follows that $\bar{P}(X)$ is a polynomial of degree $q-1$ with respect to X and coincides with the Hermite interpolation polynomial of $1/(1-X)$ at the nodes $\alpha_1, \dots, \alpha_s$. On the other hand, by Theorem 1.2 (a),

$$c^{(p-q+1)}(\bar{P}(X)) = \frac{1}{v(t)} c^{(p-q+1)}\left(\frac{v(t) - V(X)}{1-X}\right) = (p/q)_f(t),$$

which proves the result of the case (b).

Example 2.1 The functions $\alpha_1 = \sqrt{t^2 + s^2}$ and $\alpha_2 = -\sqrt{t^2 + s^2}$ are the zeros of a g -polynomial $V(X) = X^2 - t^2 - s^2$, $t, s \in \mathbb{R}$. Let $P_1(X)$ be the interpolation polynomial of first degree of $1/(1-X)$ at the nodes α_1, α_2 and $P_2(X)$ the Hermite interpolation polynomial of third degree of $1/(1-X)$ at the nodes $\alpha_1, \alpha_2, 0, 0$. Then

$$P_1(X) = \frac{X+1}{1-t^2-s^2}, \quad P_2(X) = \frac{X^3 + X^2 + (1-t^2-s^2)X + 1 - t^2 - s^2}{1-t^2-s^2}$$

and

$$c(P_1(X)) = \frac{c_1 + 1}{1-t^2-s^2} = (1/2)_f(t, s), \quad t, s \in \mathbb{R},$$

$$c(P_2(X)) = \frac{c_3 + c_2 + (1-t^2-s^2)(c_1 + c_0)}{1-t^2-s^2} = (3/2)_f(t, s), \quad t, s \in \mathbb{R},$$

where

$$f(t, s) = \sum_{i=0}^{\infty} \left(\sum_{j+k=i} c_{jk} t^j s^k \right) = \sum_{i=0}^{\infty} c_i, \quad c_i = \sum_{j+k=i} c_{jk} t^j s^k, \quad c_{jk}, t, s \in \mathbb{R}.$$

Now let us consider the particular case in Theorem 2.1 such that $m=0$, $b_0=1$, $\alpha_i = \alpha^{(i)} \cdot t$, $i=1, \dots, s$, where $\alpha^{(i)} = (\alpha_1^{(i)}, \dots, \alpha_N^{(i)})$, $t = (t_1, \dots, t_N)$, $\alpha_j^{(i)}, t_i \in \mathbb{R}$ and \cdot denotes the scalar product in \mathbb{R}^N . Then, since the polynomial,

$$V(X) = \prod_{i=1}^s (X - \alpha_i)^{k_i} = \prod_{i=1}^s (X - \alpha(i) \cdot t)^{k_i}, \quad k_1 + \dots + k_s = q,$$

is a homogeneous polynomial of degree q in t_1, \dots, t_N, X , that is, a g -polynomial of degree q with shift 0 , we have the following corollary.

Corollary 2.1 (a) The case of $p > q - 1$. Let $P(X)$ be the Hermite interpolation polynomial of degree p of the generating function $1/(1-X)$ at the $p+1$ nodes, $\alpha(1) \cdot t, \dots, \alpha(q) \cdot t, 0, \dots, 0$. Then

$$c(P(X)) = (p/q)_f^n(t), \quad t = (t_1, \dots, t_N).$$

(b) The case of $p \leq q - 1$. Let $P(X)$ be the Hermite interpolation polynomial of degree $q-1$ of $1/(1-X)$ at the q nodes $\alpha(1) \cdot t, \dots, \alpha(q) \cdot t$. Then

$$c^{(p-q+1)}(P(X)) = (p/q)_f^n(t), \quad t = (t_1, \dots, t_N).$$

In both cases, the denominator of the approximant is a polynomial of degree q , $\prod_{i=1}^q (1 - \alpha(i) \cdot t)$, where $\{\alpha(i)\}$ are not always distinct.

In this corollary, the cases of $p = q - N$ and $p = q - 1$ correspond to [5] and [1] respectively (See §3 in detail).

Remark 2.1 In one variable case in [2], the polynomial $v(x) = \prod_i (x - \alpha_i)$ always becomes a generating polynomial of a Padé-type approximant for any given finite points $\{\alpha_i\}$. But in our case, this fact does not hold. In order to be applied Theorem 2.1 to the given functions $\{\alpha_i\}$, it is necessary that the polynomial $V(X) = b_n \prod_i (X - \alpha_i)$ is a g -polynomial. Let us give a simple counter example. The polynomial in X , $V(X) = (X - t^2)(X - s^2)$, $t, s \in \mathbb{R}$, is not a g -polynomial. Let $P(X)$ be the Hermite interpolation polynomial of first degree of $1/(1-X)$ at the nodes t^2, s^2 . Then

$$c(P(X)) = c\left(\frac{X+1-t^2-s^2}{(1-t^2)(1-s^2)}\right) = \frac{c_1+(1-t^2-s^2)c_0}{(1-t^2)(1-s^2)}, \quad t, s \in \mathbb{R}.$$

On the other hand, as the denominator $(1-t^2)(1-s^2)$ is the reverse polynomial of the g -polynomial $X^4-(t^2+s^2)X^2+t^2s^2$ and the numerator has second degree, we have, by the definition,

$$\begin{aligned} (2/4)_f(t, s) &= \frac{1}{(1-t^2)(1-s^2)} c\left(\frac{1-t^2-s^2+t^2s^2 - X^{-1}(X^4-(t^2+s^2)X^2+t^2s^2)}{1-X}\right) \\ &= \frac{1}{(1-t^2)(1-s^2)} c(1+X+X^2-(t^2+s^2)) = \frac{c_2+c_1+(1-t^2-s^2)c_0}{(1-t^2)(1-s^2)}. \end{aligned}$$

They are not coincident.

Remark 2.2 Let $P(X)$ be the polynomial such that $c(P(X)) = (p/q)_f^{\#}(t)$, $t = (t_1, \dots, t_n)$ for any formal power series $f(t)$, provided that the denominator is fixed. We note that the polynomial $P(X)$ is uniquely determined for $p \geq q-1$, but for $p < q-1$, such polynomial is not unique.

In fact, putting $P_1(X) = \frac{v(t)-V(X)}{v(t)(1-X)}$, then $c^{(p-q+1)}(P_1(X)) = (p/q)_f^{\#}(t)$ by Theorem 1.2(a). On the other hand, putting $P_2(X) = \frac{X^{q-p-1}v(t)-V(X)}{v(t)(1-X)}$, then $c^{(p-q+1)}(P_2(X)) = c\left(\frac{v(t)-X^{q-p-1}V(X)}{v(t)(1-X)}\right) = (p/q)_f^{\#}(t)$ by the definition.

Here, $P_1(X)$ and $P_2(X)$ are different polynomials of degree $q-1$ in X . We derived Theorem 2.1(b) by using the polynomial $P_1(X)$. By taking $P_2(X)$, we can also get the different result from Theorem 2.1(b):

"In the case of $p \leq q-1$, let $P(X)$ be the Hermite interpolation polynomial of degree $q-1$ of $X^{q-p-1}/(1-X)$ at the nodes $\alpha_1, \dots, \alpha_q$. Then $c^{(p-q+1)}(P(X)) = (p/q)_f^{\#}(t)$."

By operating c or $c^{(p-q+1)}$ on the determinantal expression of the Hermite interpolation polynomial $P(X)$ in Theorem 2.1, we can obtain the determinantal expression of Padé-type approximants

by the zeros of the generating polynomial.

Theorem 2.2 Let $\alpha_1, \dots, \alpha_q$ be the distinct zeros of a g -polynomial $V(X)$ of degree q with shift m , i.e $V(X) = b_m \prod_{i=1}^q (X - \alpha_i)$, $b_m \neq 0$. Then

$$(p/q)_f^m(t) = \frac{w(t)}{v(t)} = \frac{\sum_{i=0}^{p-q} c_i c_{p-q+1} c_{p-q+2} \dots c_p}{\begin{vmatrix} -\frac{1}{1-\alpha_1} & 1 & & \alpha_1 & \dots & \alpha_1^{q-1} \\ \vdots & \vdots & & \vdots & & \vdots \\ -\frac{1}{1-\alpha_q} & 1 & & \alpha_q & \dots & \alpha_q^{q-1} \end{vmatrix}}{\begin{vmatrix} 1 & \alpha_1 & \dots & \alpha_1^{q-1} \\ \vdots & \vdots & & \vdots \\ 1 & \alpha_q & \dots & \alpha_q^{q-1} \end{vmatrix}}$$

where $v(t) = V(1) = b_m \prod_{i=1}^s (1 - \alpha_i)^{k_i}$, $t = (t_1, \dots, t_N)$ and $\sum_{i=0}^{p-q} c_i = 0$ for $p - q < 0$.

Proof. Let $g(X)$ be the generating function $1/(1-X)$.

(a) For $p < q$, the interpolation polynomial in Theorem 2.1(b) is expressed by the determinant as follows,

$$P(X) = \frac{\begin{vmatrix} 0 & 1 & X & \dots & X^{q-1} \\ -g(\alpha_1) & 1 & \alpha_1 & \dots & \alpha_1^{q-1} \\ \vdots & \vdots & \vdots & & \vdots \\ -g(\alpha_q) & 1 & \alpha_q & \dots & \alpha_q^{q-1} \end{vmatrix}}{\begin{vmatrix} 1 & \alpha_1 & \dots & \alpha_1^{q-1} \\ \vdots & \vdots & & \vdots \\ 1 & \alpha_q & \dots & \alpha_q^{q-1} \end{vmatrix}}$$

Operating $c^{(p-q+1)}$ on $P(X)$, the result immediately follows.

(b) For $p \geq q$, $P(X)$ in Theorem 2.1(a) is written by

$$P(X) = \frac{\begin{vmatrix} 0 & 1 & X & X^2 & \dots & X^{p-q} & \dots & X^p \\ -1 & 1 & 0 & 0 & \dots & 0 & \dots & 0 \\ -1! & 0 & 1! & 0 & \dots & 0 & \dots & 0 \\ -2! & 0 & 0 & 2! & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ -(p-q)! & 0 & 0 & 0 & \dots & (p-q)! & \dots & 0 \\ -g(\alpha_1) & 1 & \alpha_1 & \dots & \alpha_1^{p-q} & \dots & \alpha_1^p \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ -g(\alpha_q) & 1 & \alpha_q & \dots & \alpha_q^{p-q} & \dots & \alpha_q^p \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1! & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & 2! & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & (p-q)! & \dots & 0 \\ 1 & \alpha_1 & \dots & \alpha_1^{p-q} & \dots & \alpha_1^p \\ \vdots & \vdots & & \vdots & & \vdots \\ 1 & \alpha_q & \dots & \alpha_q^{p-q} & \dots & \alpha_q^p \end{vmatrix}}$$

Taking account of $1 - g(\alpha_i) = -\alpha_i g(\alpha_i)$, we get

$$P(X) = \begin{vmatrix} \sum_{i=0}^{p-q} X_i & X^{p-q+1} & X^{p-q+2} & \dots & X^p \\ -g(\alpha_1) & 1 & \alpha_1 & \dots & \alpha_1^{q-1} \\ \vdots & \vdots & \vdots & & \vdots \\ -g(\alpha_q) & 1 & \alpha_q & \dots & \alpha_q^{q-1} \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & \alpha_1 & \dots & \alpha_1^{q-1} \\ \vdots & \vdots & & \vdots \\ 1 & \alpha_q & \dots & \alpha_q^{q-1} \end{vmatrix}.$$

Operating c on $P(X)$, we obtain the result.

§3. Relation to [1] and [5].

As an interpolating polynomial in many variables, Sablonnière[5] and Arioka[1] take up the Hakopian interpolation polynomial and the Kergin one respectively. We are going to study the relation between these polynomials and the polynomial $P(X)$ in §2.

(A) The relation to [1]. Let $f(t)$ be a formal power series,

$$(3.1) \quad f(t) = \sum_{n \geq 0} \sum_{|i|=n} \bar{c}_i t^i, \quad t = (t_1, \dots, t_N) \in \mathbb{R}^N, \quad i = (i_1, \dots, i_N) \in \mathbb{N}^N,$$

where $|i| = i_1 + \dots + i_N$. He defines the functional \bar{c} by $\bar{c}(x^i) = \bar{c}_i / \binom{n}{i}$, $x = (x_1, \dots, x_N)$, $n = |i|$ and shows that $f(t) = \bar{c}\left(\frac{1}{1-x \cdot t}\right)$, where $x \cdot t = x_1 t_1 + \dots + x_N t_N$, that is, the generating function in [1] is $1/(1-x \cdot t)$. Then it holds that

$$c(X^n) = c_n = \sum_{|i|=n} \bar{c}_i t^i = \bar{c}\left((x \cdot t)^n\right), \quad x = (x_1, \dots, x_N), \quad t = (t_1, \dots, t_N),$$

which implies that to obtain the expression in [1], it is sufficient to change c into \bar{c} after putting $X = x \cdot t$. For example, $c\left(\frac{1}{1-X}\right) = \bar{c}\left(\frac{1}{1-x \cdot t}\right)$.

Now, applying Corollary 2.1 for $p = q - 1$ and q distinct points $\alpha(1), \dots, \alpha(q)$ of \mathbb{R}^N , and putting $X = x \cdot t$, we have

$$(q-1/q)_f(t) = c(P(X)) = \bar{c}(P(x \cdot t)), \quad x = (x_1, \dots, x_N), \quad t = (t_1, \dots, t_N),$$

where

$$P(X) = \frac{v(t) - V(X)}{v(t)(1-X)}, \quad V(X) = \prod_{i=1}^q (X - \alpha_i), \quad v(t) = \prod_{i=1}^q (1 - \alpha_i), \quad \alpha_i = \alpha(i) \cdot t.$$

Here, $P(x \cdot t)$ is a polynomial of degree $q - 1$ in x because $P(X)$ is one of degree $q - 1$ in X . Moreover the condition of the interpolation,

$$P(\alpha_i) = \frac{1}{1-\alpha_i}, \quad i=1, \dots, q,$$

means the condition with respect to x ,

$$P(\alpha(i) \cdot t) = \frac{1}{1-\alpha(i) \cdot t}, \quad i=1, \dots, q,$$

that is, the polynomial $P(x \cdot t)$ is the interpolating polynomial of degree $q-1$ in x of $1/(1-x \cdot t)$ at the nodes $\alpha(1), \dots, \alpha(q)$ of \mathbb{R}^N and it is nothing else but the Kergin interpolation polynomial $K(x)$ of $1/(1-x \cdot t)$. In fact, from the expression of $K(x)$ (in the proof of Theorem 4.3 in [1]), we have

$$K(x) = \frac{1}{1-x \cdot t} \left(1 - \frac{\prod_{i=1}^q (x \cdot t - \alpha(i) \cdot t)}{\prod_{i=1}^q (1 - \alpha(i) \cdot t)} \right) = \frac{v(t) - v(X)}{v(t)(1-X)} \Big|_{x=x \cdot t} = P(x \cdot t).$$

(B) The relation to [5]. For a formal power series (3.1), Sablonnière[5] defines the functional \bar{c} by $\bar{c}_i = \binom{n+N-1}{N-1} \binom{n}{i} \bar{c}(x^i)$, $|i|=n$, $x=(x_1, \dots, x_N)$ and shows that $f(t) = \bar{c} \left(\frac{1}{(1-x \cdot t)^N} \right)$, that is, the function $1/(1-x \cdot t)^N$ is the generating function in [5]. Thus there is the following relation between our operation c and \bar{c} .

$$c(X^n) = c_n = \sum_{|i|=n} \bar{c}_i t^i = \bar{c} \left(\binom{n+N-1}{N-1} (x \cdot t)^n \right).$$

Taking account of the fact that $\frac{d^{N-1}}{dX^{N-1}} \left(\frac{X^{N-1}}{(N-1)!} \cdot X^n \right) = \binom{n+N-1}{N-1} X^n$, we have

$$(3.2) \quad c(X^n) = \bar{c} \left(\left(\frac{X^{N-1}}{(N-1)!} X^n \right) \Big|_{x=x \cdot t}^{(N-1)} \right),$$

where $(\dots)^{(N-1)}$ denotes the $(N-1)$ th derivative with respect to X .

By (3.2) we can obtain the expression in [5]. For examples,

$$(3.3) \quad c \left(\frac{1}{1-X} \right) = \sum_{n=0}^{\infty} c(X^n) = \sum_{n=0}^{\infty} \bar{c} \left(\left(\frac{X^{N-1}}{(N-1)!} X^n \right) \Big|_{x=x \cdot t}^{(N-1)} \right) \\ = \bar{c} \left(\left(\frac{X^{N-1}}{(N-1)!} \frac{1}{1-X} \right) \Big|_{x=x \cdot t}^{(N-1)} \right) = \bar{c} \left(\frac{1}{(1-X)^N} \Big|_{x=x \cdot t} \right) = \bar{c} \left(\frac{1}{(1-x \cdot t)^N} \right),$$

Moreover, (3.2) holds also for n such that $1-N \leq n < 0$ since the both sides equal zero. Thus we have

$$(3.4) \quad c\left(\frac{P(X)}{X^{N-1}}\right) = \bar{c}\left(\left(\frac{X^{N-1}}{(N-1)!} \frac{P(X)}{X^{N-1}}\right)\Big|_{X=x \cdot t}\right)^{(N-1)} = \bar{c}\left(\left(\frac{P(X)}{(N-1)!}\right)\Big|_{X=x \cdot t}\right)^{(N-1)},$$

where $P(X)$ is a polynomial in X .

Now let us apply Corollary 2.1 for $q=r+1$, $p=r-N+1$ and $\alpha(i+1)=x(i)$ ($i=0,1,\dots,r$). Then,

$$\begin{aligned} (r-N+1/r+1)_f(t) &= c^{(-N+1)}(P(X)) = c\left(\frac{P(X)}{X^{N-1}}\right), \quad t=(t_1, \dots, t_N), \\ &= \bar{c}\left(\left(\frac{P(X)}{(N-1)!}\right)\Big|_{X=x \cdot t}\right)^{(N-1)} \quad (\text{by (3.4)}), \end{aligned}$$

where

$$P(X) = \frac{v(t) - v(X)}{v(t)(1-X)}, \quad v(X) = \prod_{i=0}^r (X - x(i) \cdot t), \quad v(t) = \prod_{i=0}^r (1 - x(i) \cdot t).$$

Put $p(x,t) = \left(\frac{P(X)}{(N-1)!}\right)\Big|_{X=x \cdot t}^{(N-1)}$. Then it is a polynomial of degree $r-N+1$

in x because $P(X)$ is one of degree r in X . We are going to show that this polynomial $p(x,t)$ is nothing else but the Hakopian interpolation polynomial in [5]. From the expression

$$P(X) = \frac{1}{1-X} - \frac{1}{v(t)} \cdot \frac{\prod_{i=0}^r (X - x(i) \cdot t)}{1-X},$$

we have

$$\begin{aligned} (3.5) \quad p(x,t) &= \frac{1}{(N-1)!} \left(\frac{1}{1-X}\right)\Big|_{X=x \cdot t}^{(N-1)} - \frac{1}{(N-1)!v(t)} \left(\frac{\prod_{i=0}^r (X - x(i) \cdot t)}{1-X}\right)\Big|_{X=x \cdot t}^{(N-1)} \\ &= g(x,t) - \frac{1}{(N-1)!v(t)} U^{(N-1)}(x \cdot t), \end{aligned}$$

where

$$g(x,t) = \frac{1}{(1-x \cdot t)^N} \quad \text{and} \quad U(X) = \frac{\prod_{i=0}^r (X - x(i) \cdot t)}{1-X}.$$

We prepare some notations. Let $i=(i_0, i_1, \dots, i_{N-1})$ be a subset of $\{0, 1, \dots, r\}$ and $X_i = \{x(i_0), x(i_1), \dots, x(i_{N-1})\}$ a subset of points $\{x(0), x(1), \dots, x(r)\}$ in R^N . For a function $h(x)$, $h\{X_i\}$ is defined by

$$(3.6) \quad h\{X_i\} = (N-1)! \int_{Q^{N-1}} h(\lambda_0 x(i_0) + \dots + \lambda_{N-1} x(i_{N-1})) d\lambda,$$

where $Q^{N-1} = \{(\lambda_1, \dots, \lambda_{N-1}) \in \mathbb{R}^{N-1}; \lambda_1 + \dots + \lambda_{N-1} \leq 1, \lambda_i \geq 0\}$, $\lambda_0 = 1 - \sum_{i=1}^{N-1} \lambda_i$ and $d\lambda = d\lambda_1 \cdots d\lambda_{N-1}$. Now let us prove that $p(x, t)$ satisfies the condition of the Hakopian interpolation, i.e.

$$(3.7) \quad p(\{X_i\}, t) = g(\{X_i\}, t) \quad \text{for every multi-index } i = (i_0, i_1, \dots, i_{N-1}).$$

From the expression (3.5) and the definition (3.6), we obtain that

$$\begin{aligned} p(\{X_i\}, t) &= g(\{X_i\}, t) - \frac{1}{v(t)} \int_{Q^{N-1}} U^{(N-1)}(\{\lambda_0 x(i_0) + \dots + \lambda_{N-1} x(i_{N-1})\}, t) d\lambda \\ &= g(\{X_i\}, t) - \frac{1}{v(t)} \int_{Q^{N-1}} U^{(N-1)}(\lambda_0 \{x(i_0) \cdot t\} + \dots + \lambda_{N-1} \{x(i_{N-1}) \cdot t\}) d\lambda, \end{aligned}$$

by the Hermite-Genocchi formula,

$$= g(\{X_i\}, t) - \frac{(-1)^{N-1}}{v(t)} U[x(i_0) \cdot t, \dots, x(i_{N-1}) \cdot t],$$

where $U[x(i_0) \cdot t, \dots, x(i_{N-1}) \cdot t]$ denotes the divided difference of U at $x(i_0) \cdot t, \dots, x(i_{N-1}) \cdot t$. In the last expression, the second term in the right hand side is vanished by the fact $U(x(i) \cdot t) = 0$ ($i=0, 1, \dots, r$), which implies the result.

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