数理解析研究所講究録
679 巻 1989 年 250-260

SOLUTIONS OF SOME SEMILINEAR ELLIPTIC PROBLEMS

BY

Hwai-chiuan Wang

Abstract

In this article we study some semilinear elliptic problems on an infinite strip, and prove their existences of various classical solutions, which are spherically symmetric and decreasing in the $|x|$-direction and decay exponentially at infinite.

0. INTRODUCTION

In the part III of his lecture notes [5], Ni gave systematic studies of semilinear elliptic equaitons on unbounded domains in the Euclidean space $\mathbb{R}^n$, and gave extensive references. A typical equation in [5] is as follows:

AMS(MOS) subject classification (1980), primary 35J20, 35J25.
This research was partially supported by the National Science Council, Republic of China

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where Ω is an unbounded domain in $\mathbb{R}^n$. This type of equaitons in the case $\Omega = \mathbb{R}^N$ have been studied in great detail in[3,5,7]. The treatments in which use variational arguments to solve the problems. Those techniques, espectially from [3] involving the radial and the compactness theorems of strauss, form one of our basic methods. This type of equations in the case $\Omega = \mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$ have been studied in [1,2,4,7]. In [2,4], they use the finite domain approximations to treat the existence results. The bifurcation and asymptotic bifurcation of these equations have been studied in great detail in [2]. In [7] the double Steiner symmetrizations have been used, and in [1], finite domain approximaitons have been used to study the bifurcation problem of some more general equations. We treat here in the case $\Omega = \mathbb{R}^N \times (0,1)$, $N = 2, 3$, and develop some new techniques of uniform analysis to obtain our results. Throughout this article we use the same notation C for different constants in various inequalities.

1. EXISTENCES

Let $\Omega = \mathbb{R}^2 \times (0,1)$ or $\Omega = \mathbb{R}^3 \times (0,1)$. Denote by a point (x,y) in Ω with $x \in \mathbb{R}^N$, $N = 2$ or 3, $y \in (0,1)$. Consider the semilinear elliptic eigenvalue equation

$$(A) \qquad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $f: \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, odd, $f(0) = 0$, and satisfies the following conditions:

(1.1) $\quad -\infty < \underline{\lim}_{s\to 0+} \frac{f(s)}{s} \leq \overline{\lim}_{s\to 0+} \frac{f(s)}{s} = -m \leq 0$

$(1.2)_2$ $\quad -\infty < \overline{\lim}_{s\to\infty} \frac{f(s)}{s^\ell} \leq 0$ for any $\ell > 1$, if $\Omega = \mathbb{R}^2 \times (0,1)$

$(1.2)_3$ $\quad -\infty < \overline{\lim}_{s\to\infty} \frac{f(s)}{s^3} \leq 0$ if $\Omega = \mathbb{R}^3 \times (0,1)$

(1.3) There is $\alpha > 0$ with $F(\alpha) = \int_0^\alpha f(s)ds > 0$.

Define a new function $\tilde{f}: \mathbb{R} \longrightarrow \mathbb{R}$ as follows:

(i) if $f(s) \geq 0$ for all $s \geq \alpha$, put $\tilde{f} = f$

(i i) if there is $s_0 \geq \alpha$ with $f(s_0) = 0$ put

$$\tilde{f}(s) = \begin{cases} f(s) & \text{on } [0,s_0] \\ 0 & \text{for } s \geq s_0 \end{cases}$$

(iii) for $s \leq 0$, $\tilde{f}(s) = -\tilde{f}(-s)$.

Observe that $\tilde{f}$ satisfies the same condition as f. Furthermore, by the maximum principle, solutions of problem (A) with $\tilde{f}$ are also solutions of (A) with f. We henceforth adopt that f has been replaced by $\tilde{f}$. In this case, $(1.2)_2$ and $(1.2)_3$ can be replaced by the followings respectively

$$\lim_{|s|\to\infty} \frac{|f(s)|}{|s|^{\ell}} = 0 \quad \text{for any } \ell > 1, \text{ in case } \Omega = \mathbb{R}^2 \times (0,1)$$

and

$$\lim_{|s|\to\infty} \frac{|f(s)|}{|s|^3} = 0 \quad \text{in case } \Omega = \mathbb{R}^3 \times (0,1).$$

There are some typical examples of the equation (A)

1.4. <u>EXAMPLE</u>. Consider the equation

$$\begin{cases} -\Delta u + mu = \beta|u|^{p-1}u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where m, β are positive constants, and $p > 1$.

1.5. <u>EXAMPLE</u>. Consider the equation

$$\begin{cases} -\Delta u + mu = \beta |u|^{p-1}u - \gamma |u|^{q-1}u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where m, β, γ are positive constants, and $1 < q < p < \infty$ for the case $\Omega = \mathbb{R}^2 \times (0,1)$ and $1 < q < p < 3$ for the case $\Omega = \mathbb{R}^3 \times (0,1)$.

1.6. <u>THEOREM</u>. Suppose f satisfies the conditions (1.1) - (1.3). There is a solution (λ,u) of the equation (A), where u is of $C^2(\Omega)$, and is spherically symmetric and decreasing in the $|x|$-direction.

1.7. <u>REMARK</u>. In Theorem 1.6, we obtained a solution (λ,u) of equation (A)

$$\text{(A)} \qquad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In general, the Lagrange multiplier λ can not be absorbed.

Note that λ can be absorbed implies that u is a solution of the equation

$$(A1)\quad \begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

However, the equation (A1) has a solution in the following particular cases.

(1) In Theorem 1.8 below we modify our proof of Theorem 1.6 to obtain a solution of the equation

$$(B)\quad \begin{cases} -\Delta u + mu = u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $m > 0$ a constant.

(2) In Theorem 1.13 below, we use Nehari's method to construct a solution of the equation

$$(C)\quad \begin{cases} -\Delta u = g(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

1.8. **THEOREM**. For either $\Omega = \mathbb{R}^2 \times (0,1)$, $2 < p < \infty$ or $\Omega = \mathbb{R}^3 \times (0,1)$, $2 < p < 3$, there is a C^2 solution $u(x,y)$ of the equation

$$(B)\qquad \begin{cases} -\Delta u + mu = u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where m is a positive constant. Moreover $u(x,y)$ is spherically symmetric and decreasing in the $|x|$-direction for each y in $(0,1)$.

Let the function $g: \mathbb{R} \longrightarrow \mathbb{R}$ be continuous, odd, $g(0) = 0$, satisfies (1.1)-(1.3), and

(1. 9) g is increasing on $[0,\infty)$

(1.10) $tg(t) - 2G(t) \geq \theta G(t)$ for large t, where θ a positive constant, and $G(t) = \int_0^t g(s)ds$

(1.11) Consider the equation $g \in C^1(0,\infty)$ with $g'(t) > \frac{g(t)}{t}$ for all $t > 0$

1.12. **EXAMPLE**. $g(u) = u^p$, $2 < p < \infty$ in case $\Omega = \mathbb{R}^2 \times (0,1)$ or $2 < p < 3$ in case $\Omega = \mathbb{R}^3 \times (0,1)$.

Consider the equation

$$
\text{(C)} \qquad \begin{cases} -\Delta u = g(u) & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega . \end{cases}
$$

1.13. **THEOREM**. There is a C^2 solution $u(x,y)$ of the equation (C). The solution $u(x,y)$ is spherically symmetric and decreasing in the $|x|$-direciton for each y in $(0,1)$.

Follow from the proof of Theorem 1.6, 1.8 and Berestycki-Lions [3], we obtain

1.14. **THEOREM**. Let w be the solution of the equaiton (B) obtained as Theroem 1.8, and u any other solution of (B), then

$$
0 < s(w) \leq s(u)
$$

where $s(v) = A(v) - B(v)$, $A(v) = \frac{1}{2}\int_{\Omega} [|Dv|^2 + m|v|^2]$, $B(v) = \frac{1}{p+1} \int_{\Omega} |v|^{p+1}$.

Such a solution w is called a ground state for the equaiton (B). Any solutions u of (B) with

$$
s(w) < s(u) < \infty
$$

are called bound states. We'll prove that the equation (B) possesses infinite many solutions of buond states, through a dual variational method: For $n = 1,2,\cdots$

$$\text{maximize } \{B(u) \mid u \in H_0^1(\Omega),\ A(u) = n^2\}.$$

1.15. **THEOREM**. For either $\Omega = \mathbb{R}^2 \times (0,1)$, $2 < p < \infty$, or $\Omega = \mathbb{R}^3 \times (0,1)$, $2 < p < 3$, and for $n = 1,2,\cdots$, there is a C^2 solution $w_n(x,y)$ of the equation (B), which is spherically symmetric and decreasing in the $|x|$-direction with $A(w_n) = n^2$.

We study the decay property of the solutions of the equaiton (B).

1.16. **THEOREM**. If $u(x,y)$ is a C^2 solution of the equation (B) which is spherically symmetric and decreasing in the $|x|$-direciton, then

$$|D^\alpha u(x,y)| \le ce^{-\delta|x|} \qquad \text{for large } x$$

where $C, \delta > 0$ are constants independent of y in $(0,1)$ and $|\alpha| \le 1$.

1.17. **REMARK**. In an article in preparation, Nirenberg-Berestycki asserts that if $\Omega = \mathbb{R}^N \times (0,1)$, and u is a solution of the equation

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \\ \lim\limits_{|x|\to\infty} u(x) = 0 & \end{cases}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is locally Lipschtz continuous function, $f'(0) < \pi^2$, then u is symmetric in x about some x_0 and $u_{|x|} > 0$ for $|x| < |x_0|$. After shifting, u can be considered symmetric in $x = 0$. If we apply this result, in the assumptions of Theorem 2.1, we may only assume that u is a C^2 solution of the equation (B).

ACKNOWLEDGEMENTS. I would like to thank Professor W.M. Ni. for his lectures in Tsing Hua University which attract my attention to this problem. I also wish to express my gratitude to Professor H. Berestycki for his kindly suggestions on various possible extension of our previous article [8]. As a matter of fact, this article contains several extensions of [8] subjecting to his suggestions. I also would like to thank Professor J.F. Toland for his kindly reminding of several related articles.

REFERENCES

[1] Amick C.J., Semilinear elliptic eigenvalue problems on an infinite strip with an applicaiton to stratified fluids, Annal. Scu. Norm. Sup. Pisa. 1984, 441-499.

[2] Amick, C.J. and Toland J.F., Nonlinear elliptic eigenvalue problems on an infinite strip, global theory of bifurcation and asymptotic bifurcation and asymptotic bifurcation, Math. Ann. 262 (1983), 313-342.

[3] Berestycki, H. and Lions, P.-L., Nonlinear scalar field equations I, Arch. Rational Mech. Anal. 82(1983), 313-345.

[4] Bona, J.L., Bose, D.K. and Turner, R.E.L., Finite-amptitude steady waves in stratified fluids, J. Math. Pures et Appl. 62(1983), 389-439.

[5] Ni, W.M., Some Aspects of Semilinear elliptic Equations, Tsing Hua University, Taiwan, May 1987.

[6] Stakgold, I., Boundary Value Problems of Mathematical Physics, Vol. II, New York, MacMillan, 1968.

[7] Stuart, C.A., A variaitonal approach to bifurcation in L^p on an unbounded symmetrical domain, Math. Ann. 263(1983), 51-59.

[8] Wang, H.C., Some semilinear elliptic equaitons on a strip, Preprint.

Department of Mathematics
National Tsing Hua University
Hsinchu, Taiwan