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Eigenvalue Problems for Some Qusilinear Equations

By

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Let Ω be a bounded domaim in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and consider the following well-known Poincaré's inequality :

$$(P) \qquad |u|_{L^p} \leqq C\,|\nabla u|_{L^p} \qquad u \in W_0^{1,p}(\Omega),\ 1<p<\infty.$$

Since the injection from $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact, it is easy to find an element $u \neq 0$ in $W_0^{1,p}(\Omega)$ which attains the best possible constant for (P), that is to say

$$R(u) = \sup\{R(v)\ ;\ v \in W_0^{1,p}(\Omega),\ v \neq 0\} =: \frac{1}{\lambda_1}, \quad R(v) = |v|_{L^p}/|\nabla v|_{L^p}$$

Then it can be shown that u must satisfy the equation :

$$(E)_\lambda \quad \begin{cases} -\Delta_p u = \lambda\,|u|^{p-2}u & \text{in } \mathcal{D}'(\Omega), \\ u \in W_0^{1,p}(\Omega), \end{cases}$$

with $\lambda = \lambda_1$, where $\Delta_p u = \operatorname{div}(\,|\nabla u|^{p-2}\nabla u\,)$.

For the case $p = 2$, it is well known that λ_1 is <u>the first eigenvalue</u> of $-\Delta$ with zero Dirichlet boundary condition and that its eigenfunctions form a one dimensional linear subspace of $H_0^1(\Omega)$, i.e., λ_1 is <u>simple</u>. As for the case $p \neq 2$, this kind of result was first obtained by [4] for the case $N = 1$, where it is shown that eigenvalues form a countable set and are all simple.

For higher dimension $N \geqq 2$, Sakaguchi [6] showed that the first eigenvalue λ_1 is simple, provided that $\partial\Omega$ is connected. His method of proof relies on a maximum principle for $-\Delta_p$.

The main purpose of this note is to introduce a method based on " variational principle " to show the following properties without assuming that $\partial\Omega$ is connected.

(I) The first eigenvalue λ_1 is simple ;

(II) $(E)_\lambda$ has a positive solution if and only if $\lambda = \lambda_1$,

(i.e., other eigenfunctions must change their sign.)

Quite recently, Anane [1] also proved these results. However, her method of proof essentially depends on the peculiarity of the operator Δ_p and it seems that this does not work for other similar operators. Our method of proof is quite different from those of [1] and [6] and can cover other similar operators. In order to emphasize this advantage, we here formulate our results in an abstract form.

Let A and B be Fréchet derivatives of functionals f^1 and f^2 defined on a Banach space V , (we denote $A = \partial f^1$ and $B = \partial f^2$). Consider the following abstract eigenvalue problem :

$$(AE)_\lambda \qquad Au = \lambda Bu .$$

We impose the following conditions on f^i.

(A1) (i) $R(|v|) \geqq R(v) := f^2(v)/f^1(v) \quad \forall v \in V$,

(ii) $f^1(v) \geqq 0 \quad \forall v \in V$ and $f^1(v) = 0$ if and only if $v = 0$,

(iii) $\exists\, u \in V$ s.t. $u \neq 0$ and $R(u) = \sup\{R(v) ; v \in V, v \neq 0\} > 0$.

(A2) $\exists\, \alpha > 1$ s.t. $f^i(tv) = t^\alpha f^i(v)$ $\forall v \in V, \forall t > 0$, $i=1,2$.

(A.3) (i) $f^1(u \vee w) + f^1(u \wedge w) \leqq f^1(u) + f^1(w)$, $\forall u, \forall w \in V$,

(ii) $f^2(u \vee w) + f^2(u \wedge w) \geqq f^2(u) + f^2(w)$, $\forall u, \forall w \in V$,

where $(u \vee w)(x) = \max(u(x), w(x))$ and $(u \wedge w)(x) = \min(u(x), w(x))$.

Furthermore we assume

(A0) Every non-negative nontrivial solution u of $(AE)_\lambda$ belongs to $C^1(\Omega)$ and satisfies $u(x) > 0$ for all $x \in \Omega$.

Then our first result can be stated as follows.

<u>THEOREM I</u> Assume (A0)-(A3). Put $\lambda_1 = 1/\sup\{R(v)\,;\, v \in V, v \neq 0\}$. Then we have

(i) $(AE)_\lambda$ has no nontrivial solution for $\lambda \in (0, \lambda_1)$,

(ii) λ_1 is simple, i.e., $(AE)_{\lambda_1}$ has a positive solution and the set of all positive solutions of $(AE)_{\lambda_1}$ is a one dimensional subspace of V.

<u>Remark 1</u>. (i) Take $V = W_0^{1,p}(\Omega)$, $f^1(v) = |\nabla v|^p_{L^p}/p$ and $f^2(v) = |v|^p_{L^p}/p$, then $(E)_\lambda$ can be reduced to $(AE)_\lambda$, and all above conditions (A0)-(A3) and (A4),(A5),(A0)' in Theorem II are satisfied.

(ii) If $Au = \operatorname{div} \vec{a}(x,u, u) + a_0(x,u, u)$ and $Bu = b(x,u)$, then some sufficient conditions for (A0) can be given in terms of $\vec{a}$, a_0 and b. Since these are somewhat complicated, but general enough, we do not go into the details here. (See Ladyzhenskaya and Ural'tseva [3], Trudinger [8] and Tolksdorf [7].)

(iii) Conditions (A1)-(A3) are not enough to assure the simplicity of λ_1. In fact, it is easy to give a trivial counterexample by taking $f^1 = f^2 = f$ for a suitable f. In this case, $\lambda_1 = 1$ and the set of all eigenvectors becomes V.

Proof of Theorem I. (i) First of all, we note that (A2) implies

$${}_{V^*}\langle f^i(v), v\rangle_V = \alpha f^i(v), \quad \forall v \in V,\ i=1,2.$$

Hence, if u is a solution of $(AE)_\lambda$, then multiplication of $(AE)_\lambda$ by u gives

(1) $\alpha f^1(u) = \lambda\alpha f^2(u)$, i.e., $R(u) = 1/\lambda$.

This is a contradiction, since $1/\lambda = R(u) > 1/\lambda_1 = \sup R(v)$.

(ii) Set $J_\lambda(v) = f^1(v) - \lambda f^2(v)$, then

(2) $J_\lambda(v) \underset{(=)}{\leq} 0$ if and only if $R(v) \underset{(=)}{\geq} 1/\lambda$.

Therefore it follows from (iii) of (A1) that $\exists u \in V$ s.t. $u \neq 0$ and

(3) $\min\{ J_{\lambda_1}(v)\,;\, v \in V, v \neq 0\} = 0 = J_{\lambda_1}(u)$.

Hence, Fréchet derivative of J_{λ_1} at u must vanish, i.e., u becomes a nontrivial solution of $(AE)_{\lambda_1}$. Conversely, if u is a solution of $(AE)_{\lambda_1}$, then, by (1), $J_{\lambda_1}(u) = 0$. Thus we find that

(4) u is a solution of $(AE)_{\lambda_1}$ if and only if $J_{\lambda_1}(u) = 0$.

Furthermore, by (2),(3) and (i) of (A1), $J_{\lambda_1}(u) = 0$ implies $J_{\lambda_1}(|u|) = 0$. Hence

(5) If u is a solution of $(AE)_{\lambda_1}$, then $|u|$ is also a solution of $(AE)_{\lambda_1}$.

Then, by (A0), $|u|$ has no zero in Ω. Consequently, every solution u of $(AE)_{\lambda_1}$ is positive or negative in Ω.

Let u, v be two positive solutions of $(AE)_{\lambda_1}$ and put

$$M(t,x) = \max(u(x), t\,v(x)) \quad \text{and} \quad m(t,x) = \min(u(x), t\,v(x)).$$

Then, by (A3), we get

$$0 \leqq J_{\lambda_1}(M) + J_{\lambda_1}(m) \leqq J_{\lambda_1}(u) + J_{\lambda_1}(t\,v) = J_{\lambda_1}(u) + t\,J_{\lambda_1}(v) = 0,$$

whence follows $J_{\lambda_1}(M) = J_{\lambda_1}(m) = 0$. Hence, by (4), M and m turn out to be solutions of $(AE)_{\lambda_1}$ for all $t \geqq 0$. For any $x_0 \in \Omega$, set $t_0 = u(x_0) / v(x_0) > 0$. Then, for any unit vector e, we have

$$u(x_0 + h\,e) - u(x_0) \leqq M(t_0, x_0 + h\,e) - M(t_0, x_0),$$

$$t_0 v(x_0 + h\,e) - t_0 v(x_0) \leqq M(t_0, x_0 + h\,e) - M(t_0, x_0).$$

Dividing these inequalities by $h > 0$ and $h < 0$ and letting t tend to ± 0, we get

$$\nabla_x u\,(x_0) = \nabla_x M(t_0, x_0) = t_0\,\nabla_x v(x_0). \quad \text{Hence}$$

$$\nabla \frac{u}{v}(x_0) = (\nabla u(x_0)\,v(x_0) - u(x_0)\nabla v(x_0)) / v(x_0)^2 = 0.$$

Thus we see that $u(x) / v(x) = \text{Const.}$ in Ω. (QED)

Now we state our second result :

THEOREM II Assume (A0)-(A3) and

(A4) f^1 is strictly convex.

(A5) $B(u) \leqq B(v)$ if $0 \leqq u \leqq v$.

Furthermore, we assume

(A0)' Every positive solution u of $(AE)_\lambda$ satisfies $u \in C^1(\bar{\Omega})$ and $\partial u/\partial n(x) < 0$ on $\partial\Omega$.

Then $(AE)_\lambda$ has a positive solution if and only if $\lambda = \lambda_1$.

Remark 2. To assure condition (A0)', one must prove a Hopf-type maximum principle, and generally this is not so easy. In this sense, this is rather restrictive. However, in most cases, we can exclude this condition by applying some approximation procedure. (Say for the case $B(u) = |u|^{p-2}u$, set $B_\varepsilon(u) = b_\varepsilon(x)|u|^{p-2}u$ with $b_\varepsilon(x) = 1$ if $dis(x,\partial\Omega) \geq \varepsilon$, $b_\varepsilon(x) = 0$ if $dis(x,\partial\Omega) < \varepsilon$ and prove the corresponding first eigenvalue λ_1^ε converges to λ_1 as ε tends to zero. Then it suffices to repeat the same argument as in the proof of Theorem II (for a sufficiently small ε).) In order to present the basic idea of the proof, we here give a proof by assuming (A0)'.

Proof of Theorem II. By using convex analysis, we can prove the following lemma.

Lemma 3. Assume (i) of (A3) and (A4). Then $Au \leqq Av$ implies $u \leqq v$.

Suppose that $(AE)_\lambda$ with $\lambda > \lambda_1$ has a positive solution v, and let u be a positive solution of $(AE)_{\lambda_1}$. By virtue of (A0)' and the fact that tv is also a solution of $(AE)_\lambda$, we may assume without loss of generality that $u \leqq v$. Then, by (A5), we get

$$Au = \lambda_1 Bu \leqq \lambda_1 Bv = \lambda B(\eta v) = A(\eta v) \quad \text{with } \eta = (\lambda_1/\lambda)^{1/(\alpha-1)} < 1,$$

where we used the fact that B is a homogeneous operator of order $\alpha - 1$. Then it follows from Lemma 3 that $u \leqq \eta v$. Now, repeating this argument n times, we deduce $0 \leqq u \leqq \eta^n v$. Then, by letting n tend to ∞, we finally have $u \equiv 0$. This is a contradiction. (QED)

As is mentioned above, our abstract framework can cover some problems more complicated than $(E)_\lambda$. For example, let $V = W_0^{1,p}(\Omega)$ and put

$$f^1(u) = \int_\Omega \{(a_1(x)|u|^2 + a_2(x)|\nabla u|^2)^{p/2} + a_3(x)|u|^p\}\, dx / p,$$

$$f^2(u) = \int_\Omega b(x)|u|^p\, dx / p,$$

where a_i, $b \in L^\infty(\Omega)$; $a_1(x), a_3(x), b(x) \geqq 0$, $a_2(x) \geqq \rho > 0$ a.e. $x \in \Omega$.

Then
$$Au = \partial f^1(u) = \operatorname{div}\left((a_1|u|^2 + a_2|\nabla u|^2)^{(p-2)/2} a_2 \nabla u\right)$$
$$+ (a_1|u|^2 + a_2|\nabla u|^2)^{(p-2)/2} a_1 u + a_3|u|^{p-2}u,$$

$$Bu = b\,|u|^{p-2}u.$$

It can be shown that assertions of Theorems I and II hold good for these operators.

Theorems I and II have another type of application :

Theorem 4. Let $b \in L^{\infty}(\Omega)$, $b(x) \geqq 0$ a.e. $x \in \Omega$ and $1 < q < p$. Then

$$(6) \quad \begin{cases} -\Delta_p u = b(x)\,|u|^{q-2}u & \text{in } \mathcal{D}'(\Omega), \\ u \in W_0^{1,p}(\Omega) \setminus \{0\}, \ u(x) \geqq 0 & \text{a.e. } x \in \Omega \end{cases}$$

has a unique solution.

Proof. Existence part is easy. Let $V = W_0^{1,p}(\Omega)$, and put $f^1(v) = |\nabla v|^p_{L^p}/p$ and $f^2(v) = \frac{1}{p}(\int_\Omega b(x)|v|^q dx)^{p/q}$. Then $(AE)_\lambda$ becomes

$$(7) \qquad -\Delta_p u = \lambda\, |b^{1/q}u|^{p-q}_{L^q}\, b(x)\,|u|^{q-2}u,$$

and all assumptions in Theorems I and II are satisfied. (f^2 satisfies (ii) of (A3) if and only if $q \leqq p$.)

Let u and v be different solutions of (6). Then u and v satisfy (7) with $\lambda = |b^{1/q}u|^{q-p}_{L^q}$ and $\lambda = |b^{1/q}v|^{q-p}_{L^q}$ respectively. Then Theorems I and II say that

$$|b^{1/q}u|_{L^q} = |b^{1/q}v|_{L^q} \quad \text{and} \quad u = tv \quad \text{for some} \quad t > 0,$$

whence follows $u = v$. (QED)

Remark 5. A same type of result as Theorem 4 is already obtained by Diaz and Saa [2]. But their result does not cover the case where meas $\{x \in \Omega;\ b(x) = 0\} > 0$.

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