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On Higher Differentiability and Partial Regularity of the Minimizers in the Calculus of Variations

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1. Introduction

In this paper we shall treat with the following problem in the calculus of variations: Let $n$ and $N$ be positive integers and suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with the $C^2$-class boundary. Then we consider the functional,

\[ I[v] = \sum_{\alpha,\beta=1}^{n} \sum_{i,j=1}^{N} \int_{\Omega} a_{\iota,j^\beta}^\alpha(x, v) D_{\alpha}v^i D_{\beta}v^j dx \]

for $v : \Omega \rightarrow \mathbb{R}^N$,

where $D_{\alpha}v^i = \frac{\partial v^i}{\partial x_\alpha}$ ($\alpha = 1, \ldots, n$), $i = 1, \ldots, N$) and $a_{\iota,j^\beta}^\alpha$ ($\alpha, \beta = 1, \ldots, n$, $i, j = 1, \ldots, N$) are continuously differentiable functions in $\Omega \times \mathbb{R}^N$ satisfying the following: There exist positive numbers $\lambda$ and $\Lambda$ ($0 < \lambda \leq \Lambda < +\infty$) such that $a_{\iota,j^\beta}^\alpha$ ($\alpha, \beta = 1, \ldots, n$, $i, j = 1, \ldots, N$) satisfy for $\forall (x, v) \in \Omega \times \mathbb{R}^N$

\[ \lambda|\zeta|^2 \leq \sum_{i,j=1}^{N} \sum_{\alpha,\beta=1}^{n} a_{\iota,j^\beta}^\alpha(x, v) \zeta_{\alpha} \zeta_{\beta}^j \leq \Lambda|\zeta|^2 \]

for $\forall (x, v) \in \Omega \times \mathbb{R}^N$

\[ a_{\iota,j^\beta}^\alpha = a_{j,i}^\beta, \alpha. \]

In addition, since the coefficients $a_{\iota,j^\beta}^\alpha$ belong to $C^1(\Omega \times \mathbb{R}^N; \mathbb{R})$, for positive numbers $K_1$ and $K_2$ there exists a positive number $L(K_1, K_2)$ such that

\[ \max_{1 \leq i,j \leq N} \max_{|x| \leq K_1} |a_{\iota,j}^{\alpha}(x, z)| + \max_{1 \leq i,j \leq N} \max_{|x| \leq K_2} |\frac{\partial a_{\iota,j}^{\alpha}}{\partial e'}(x, z)| \]

\[ + \max_{1 \leq i,j,k \leq N} \max_{|x| \leq K_1} |a_{\iota,j,k}^{\alpha}(x, z)| \leq L(K_1, K_2) \]

where $a_{\iota,j,k}^{\alpha}(x, z) \equiv \frac{\partial a_{\iota,j}^{\alpha}}{\partial z_k}(x, z)$.
and
\[
\frac{\partial a_{i,j^\beta}^\alpha}{\partial e'}(x,z)\text{ denotes the derivative in a direction of a vector } e \text{ in } \mathbb{R}^n.
\]
This implies the existence of at least a minimizer of the functional \( I \) in the Sobolev space \( H^{1,2}(\Omega; \mathbb{R}^N) \) and \( I \) is lower semicontinuous with respect to the weak topology of \( H^{1,2}(\Omega; \mathbb{R}^N) \) (see [Mo]) under an appropriate boundary condition.

First, we show that the first-derivatives of minimizers satisfies a modulus of uniform continuity in the norm \( L^2_{\text{loc}}(\Omega; \mathbb{R}^N) \).

Secondly, we mention a convergence theorem and a partial regularity result of the weak differentials of minimizers. However, we remark that the former theorem was proved in [Gm], [HKL] and [Mm].

We use the summation convention that Latin indices run from 1 to \( N \) and Greek indices run from 1 to \( n \).

We conclude this introduction by recalling other notational conventions:

(1.5) \[ B_R(x_0) \equiv \{ x \in \mathbb{R}^N : |x-x_0| < R \}. \]

For a set \( A \subset \mathbb{R}^N \), we denote by \( \text{mes} A \) and \( |A| \) the \( n \)-dimensional Lebesgue measure of \( A \).

For \( u \in L^1(B_R(x_0); \mathbb{R}^N) \), we define

(1.6) \[ u_{x_0,R} = \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} u(x) dx. \]

For a sufficiently small number \( d \), we define an open set

(1.7) \[ \Omega_d = \Omega - \{ x \in \Omega : \text{dist}(x, \partial \Omega) \leq d \}, \]

where \( \text{dist}(x, \partial \Omega) \) means the Euclidean metric between \( x \) and \( \partial \Omega \).

For a set \( A \) in \( \mathbb{R}^n \), \( H^{(k)}(A) \) denotes the \( k \)-dimensional Hausdorff measure of \( A \) (for the definition, see [Gm]).

\( e_i (i = 1, \cdots, n) \) means the unit vector in \( \mathbb{R}^N \) parallel to the \( x_i \)-axis. We define a translate operator \( \Delta_m (m = 1, 2, \cdots, n) \) by

(1.8) \[ (\Delta_m f)(x) = f(x + h e_m) - f(x) \quad \text{for} \quad f \in L^p(\Omega; \mathbb{R}^N). \]

2. Main Result

Under the above preparations, we can describe

**Theorem 1.** Let \( u \) be a minimizer of the functional \( I \) in \( H^{1,2}(\Omega; \mathbb{R}^N) \) and let us suppose that \( u \) is a bounded, namely there exists some positive constant \( M \) such that \( \text{ess.sup}|u| \leq M \). Then, for any fixed domain \( \tilde{\Omega} \) compactly contained in \( \Omega \), there exists positive number \( \alpha = \alpha (n, N, \lambda, \Lambda, M, L) \) (\( 0 < \alpha \leq 1 \)) and \( C = C(n, N, \lambda, \Lambda, \Omega, M, L) \) such that for \( h > 0 \) with \( h \leq \text{dist}(\tilde{\Omega}; \partial \Omega) \) \( u \) satisfies

\[
\int_{\Omega} |\Delta_m(\nabla u(x))|^2 dx \leq C \cdot h^\alpha \quad \text{for} \quad \forall m (m = 1, 2, \cdots, n)
\]
THEOREM 2. Suppose that \( \{u_i\}_{i \geq 1} \) is a sequence of minimizers of \( I \) in the space \( H^{1,2}(\Omega; R^N) \) such that \( \{u_i\}_{i \geq 1} \) converges strongly to a function \( u_0 \) in \( L^2_{\text{loc}}(\Omega; R^N) \). Then the function \( u_0 \) belongs to \( H^{1,2}_{\text{loc}}(\Omega; R^N) \) and moreover a suitable subsequence of \( \{u_i\}_{i \geq 1} \) converges strongly to \( u_0 \) in \( H^{1,2}_{\text{loc}}(\Omega; R^N) \).

THEOREM 3. Let \( u \) be a minimizer of the functional \( I \) in \( H^{1,2}(\Omega; R^N) \). Then, for a singular set defined by

\[
S = \{ x \in \Omega : \lim_{\varepsilon \to +0} |(Du)_{x,\varepsilon}| = +\infty \} 
\]

the following

\[
\mathcal{H}^{(\beta)}(S) = 0
\]

holds for any positive number \( \beta \) satisfying \( n - 2\alpha < \beta < n \).

Remark. In the following proof, the letter \( C_i \) \((i = 1, \ldots, 14)\) means a various constant depending only on \( n, N, \lambda, \Lambda, \Omega, \tilde{\Omega}, M \) and \( L \).

PROOF OF THEOREM 1

First, a minimizer \( u \) is a weak solution of the Euler-Lagrange equations of the functional \( I \), \( u \) satisfies

\[
2 \int_{\Omega} a_{i,j}^{\alpha,\beta}(u(x), x)D_{\alpha}u^{i}(x)D_{\beta}\psi^{j}(x) dx + \int_{\Omega} a_{i}^{\alpha,j,k}(u(x), x)D_{\alpha}u^{i}(x)D_{\beta}u^{j}(x)\phi^{k}(x) dx = 0
\]

for \( \forall \phi(x) \in H^{1,2}(\Omega; R^N) \).

Next, let \( \delta \) be a positive number satisfying \( \delta < \frac{1}{8}\text{dist}(\tilde{\Omega}, \partial\Omega) \). For each number \( h \) \((0 < h < \delta)\), the parallel transition along with \( x_m \)-axis \((m = 1, \ldots, n)\) leads to

\[
2 \int_{\Omega} a_{i,j}^{\alpha,\beta}(u(x+he_{m}), x+he_{m})D_{\alpha}u^{i}(x+he_{m})D_{\beta}\psi^{j}(x) dx + \int_{\Omega} a_{i}^{\alpha,j,k}(u(x+he_{m}), x+he_{m})D_{\alpha}u^{i}(x+he_{m})D_{\beta}u^{j}(x+he_{m})\phi^{k}(x) dx = 0
\]

for \( \forall \phi(x) \in C^{\infty}(\Omega_{\delta}; R^N) \).

Then we have

\[
2 \int_{\Omega} a_{i,j}^{\alpha,\beta}(u(x), x)D_{\alpha}\Delta_{m}u^{i}(x)D_{\beta}\psi^{j}(x) dx = 2 \int_{\Omega} [a_{i,j}^{\alpha,\beta}(u(x), x) - a_{i,j}^{\alpha,\beta}(u(x+he_{m}), x+he_{m})]D_{\alpha}u^{i}(x+he_{m})D_{\beta}\psi^{j}(x) dx
\]

\[
- \int_{\Omega} a_{i}^{\alpha,j,k}(x+he_{m}, u(x+he_{m}))D_{\alpha}u^{i}(x+he_{m})D_{\beta}u^{j}(x+he_{m})\phi^{k}(x) dx + \int_{\Omega} a_{i,j}^{\alpha,\beta}(x, u(x))D_{\alpha}u^{i}(x)D_{\beta}u^{j}(x)\phi^{k}(x) dx
\]

\[
+ \int_{\Omega} a_{i}^{\alpha,j,k}(x, u(x))D_{\alpha}u^{i}(x)D_{\beta}u^{j}(x)\phi^{k}(x) dx
\]
after subtracting (2.4) from (2.3).

We now substitute \( \Delta_m u(x) \zeta^2(x) \) into \( \phi(x) \) in (2.5), where \( \zeta(x) \in \mathcal{C}^\infty(\Omega_\delta; R) \) is defined by

\[
(2.6) \quad \zeta(x) = \begin{cases} 1 & : \Omega_{4\delta} \\ 0 & : \Omega / \Omega_{3\delta} \end{cases} \quad \text{with} \quad |D\zeta(x)| \leq \frac{2}{\delta}.
\]

Then we have

\[
2 \int_\Omega a_{i,j}^{\alpha,\beta}(u(x), x)D_\alpha(\Delta_m u^i(x))D_\beta(\Delta_m u^j(x)) \zeta^2(x) dx \\
+ 4 \int_\Omega a_{i,j}^{\alpha,\beta}(u(x), x)D_\alpha(\Delta_m u^i(x))D_\beta \zeta(x) \Delta_m u^j(x) \zeta(x) dx \\
= 2 \int_\Omega [a_{i,j}^{\alpha,\beta}(x, u(x)) - a_{i,j}^{\alpha,\beta}(u(x+h\epsilon_m), x+h\epsilon_m)]D_\alpha u^i(x+h\epsilon_m) \\
[D_\beta(\Delta_m u^j(x))\zeta^2(x) + 2\Delta_m u^j(x)D_\beta \zeta(x)\zeta(x)] dx \\
- \int_\Omega a_{i,j,k}^{\alpha,\beta}(x+h\epsilon_m, u(x+h\epsilon_m))D_\alpha u^i(x+h\epsilon_m)D_\beta u^j(x+h\epsilon_m) \Delta_m u^k(x) \zeta^2(x) dx \\
+ \int_\Omega a_{i,j,k}^{\alpha,\beta}(x, u(x))D_\alpha u^i(x)D_\beta u^j(x) \Delta_m u^k(x) \zeta^2(x) dx
\]

(2.7)

Here, we estimate the left-hand side of (2.7), which we call \( (L) \), from below. First, by using (1.2), we have

\[
(2.8) \quad \frac{\lambda}{4nN\Lambda} \int_\Omega |D(\Delta_m u(x))|^2 \zeta^2(x) dx - 4nN\Lambda \int_\Omega |D(\Delta_m u(x))| |\zeta(x)| |\Delta_m u(x)||D\zeta(x)| dx.
\]

Second, applying the Schwarz inequality to the second term of (2.8) with \( \epsilon = \frac{\lambda}{2nN\Lambda} \), we have

\[
(2.9) \quad (L) \geq 2\lambda \int_\Omega |\Delta_m(Du(x))|^2 \zeta^2(x) dx \\
- 2\epsilon nN\Lambda \int_\Omega |\Delta_m(Du(x))|^2 \zeta^2(x) dx \\
- \frac{2nN\Lambda}{\epsilon} \int_\Omega |D\zeta(x)|^2 |\Delta_m u(x)|^2 dx \\
\geq \lambda \int_\Omega |\Delta_m(Du(x))|^2 \zeta^2(x) dx \\
- \frac{2(nN\Lambda)^2}{\lambda} \int_\Omega |D\zeta^2(x)||\Delta_m u(x)|^2 dx.
\]

On the other hand, we perform the estimates of the right-hand side of (2.7), which we call
\[ (R) = -2 \int_{\Omega} \int_{0}^{1} \frac{d\alpha_{i,j}^{\alpha}}{dt'}(x + t\Delta u(x))dt D_{\alpha}u^{i}(x + he_{m}) \]

\[ [D_{\beta}(\Delta u^{j}(x))(2(x) + 2\Delta u^{j}(x)D_{\beta}\zeta(x)\zeta(x)]dx \]

\[ - \int_{\Omega} a_{i,j,k}^{\alpha}(x + he_{m}, u(x + he_{m}))D_{\alpha}u^{i}(x + he_{m})D_{\beta}u^{j}(x + he_{m})\Delta_{m}u^{k}(x)\zeta^{2}(x)dx \]

\[ + \int_{\Omega} a_{i,j,k}^{\alpha}(x, u(x))D_{\alpha}u^{i}(x)D_{\beta}u^{j}(x)\Delta_{m}u^{k}(x)\zeta^{2}(x)dx \]

(2.10)

By using (1.4) and the boundedness of \( u \), and applying the Schwarz inequality to (2.10), we have

\[ (R) \leq C_{1} \int_{\Omega} (h + |\Delta u(x)|)\zeta(x)dx . \]

(2.11)

Thus, by combining (2.9) with (2.11), we have the following:

\[ \int_{\Omega_{4\delta}} |\Delta_{m}(Du(x))|^{2}dx \leq C_{2} \int_{\Omega_{3\delta}} |\Delta_{m}u(x)|^{2}dx \]

\[ + C_{2} \int_{\Omega_{3\delta}} (h + |\Delta_{m}u(x)|)dx \]

(2.12)

Here, it is a well-known fact that a minimizer \( u \) satisfies a so-called Caccioppoli inequality (see [Gm]): There exists a positive constant \( C \), depending only on \( n, N, \lambda, \Lambda, M \) such that

\[ \int_{B_{2R}} |Du(x)|^{2}dx \leq \frac{C}{R^{2}} \int_{B_{2R}} |u(x) - u_{R}|^{2}dx \]

(2.13)

holds for any ball \( B_{2R} \subset\subset \Omega \) with \( 0 < \forall_{R} < \delta \). A direct application of the above inequality to Gering inequality due to F.W. Gering [Ge] (see also [Gm]) leads to the following: There exists a positive number \( p (p > 2) \), which can be supposed to satisfy \( p < 4 \) and \( C \) depending only on \( n, N, \lambda, \Lambda, \Omega, M \) such that \( Du(x) \) belongs to \( L^{p}_{loc}(\Omega; R^{N}) \) and moreover

\[ \left( \frac{1}{|\tilde{\Omega}|} \int_{\tilde{\Omega}} |Du(x)|^{p}dx \right)^{\frac{1}{p}} \leq \left( \frac{1}{|\Omega|} \int_{\Omega} |Du(x)|^{2}dx \right)^{\frac{1}{2}} \]

(2.14)

holds for \( \forall_{\tilde{\Omega}} \subset\subset \Omega \).

Thus, we apply Hölder inequality to the second term of the right-hand side of (2.12) and we have

\[ \int_{\Omega_{4\delta}} |\Delta_{m}(Du(x))|^{2}dx \leq C_{4} \int_{\Omega_{3\delta}} (h + |\Delta_{m}u(x)| + |\Delta_{m}u(x)|^{2}) \]

\[ + C_{4} \left[ \int_{\Omega_{2\delta}} |D_{m}u(x)|^{p}dx \right]^{\frac{2}{p}} \left[ \int_{\Omega_{2\delta}} |\Delta_{m}u(x)|^{\frac{2p}{p-2}}dx \right]^{\frac{p-2}{p}}. \]

(2.15)
In addition, by using (2.13), (2.14) and the boundedness of $u$, we have
\[
\int_{\Omega_{4\delta}} |\Delta_{m}(Du(x))|^2 \, dx \leq C_4 \int_{\Omega_{2\delta}} [h + |\Delta_{m}u(x)| + |\Delta_{m}u(x)|^2] \, dx
\]
\[
+ C_5 \left[ \int_{\Omega_{2\delta}} |\Delta_{m}u(x)|^{\frac{p}{2}} \, dx \right]^\frac{p-2}{p}.
\]
(2.16)

Since $2 < p < 4$ implies $\frac{p}{p-2} > 2$ it follows from the boundedness of $u(x)$ that
\[
\int_{\Omega_{4\delta}} |\Delta_{m}(Du(x))|^2 \, dx \leq C_4 \int_{\Omega_{2\delta}} [h + |\Delta_{m}u(x)| + |\Delta_{m}u(x)|^2] \, dx
\]
\[
+ C_6 \left[ \int_{\Omega_{2\delta}} |\Delta_{m}u(x)|^2 \, dx \right]^\frac{p-2}{p}.
\]
(2.17)

Also, from Newton-Leibnitz formula and a Caccioppoli inequality, we obtain
\[
\int_{\Omega_{2\delta}} |\Delta_{m}u(x)|^2 \, dx \leq C_7 h^2 \int_{\Omega_{4\delta}} |Du(x)|^2 \, dx \leq C_8 h^2.
\]
(2.18)

Consequently, from (2.17) and (2.18), we deduce
\[
\int_{\Omega_{4\delta}} |\Delta_{m}(Du(x))|^2 \, dx \leq C_8 h^2 (p-2).
\]
(2.19)

Also, for any fixed unit vector $e$ one can easily prove
\[
\int_{\Omega_{4\delta}} |\Delta_{e}(Du(x))|^2 \, dx \leq C_8 h^2 (p-2).
\]
(2.20)

**Proof of Theorem 2**

From (2.19), we obtain an equi-continuity of a sequence of minimizers in $H^{1,2}_{loc}(\Omega; R^N)$. Also, it follows from (2.14) that a sequence of minimizers satisfies a uniform boundedness in $H^{1,2}_{loc}(\Omega; R^N)$. Thus we obtain the assertion of this theorem from *Rellich - Kondrachev theorem*, (see [Ad]).

**Proof of Theorem 3**

The proof of this Theorem is based on estimate (2.20) and the following lemma due to [Gi] (see also [Gm])

\[1\] The estimate (2.19) and (2.20) play an important role in the proofs of the Theorem 2 and Theorem 3.
Lemma 3.1.
Let \( v \) be a function in \( L_{loc}^{1}(\Omega) \) and \( \beta \) be any number satisfying \( n - 2\alpha < \beta < n \). Set

(2.21) \[ E_{\beta} = \{ x \in \Omega : \limsup_{\rho \to 0} \rho^{-\beta} \int_{B_{\rho}(x)} |v(y)| \, dy > 0 \} . \]

Then, we have

(2.22) \[ H^{(\beta)}(E_{\beta}) = 0 . \]

First, to apply Lemma 3.1 to the proof of Theorem 3 we construct a support function defined as follows: For \( \rho_{k} = \delta \left( \frac{1}{2} \right)^{k+1} (k = 1, 2, \cdots) \) with \( \delta = \text{dist}(\tilde{\Omega}, \partial \Omega) \) and a sequence \( \{ e_{k} \}_{k \geq 1} \) of unit vectors in \( \mathbb{R}^{n} \) we define

(2.23) \[ \varphi_{k}(y) = \rho_{k}^{-(n-\beta)-\epsilon} |Du(y + \rho_{k}e_{k}) - Du(y)|^{2} \quad \text{with} \quad \epsilon = \frac{1}{2}(2\alpha - (n - \beta)). \]

When we set

(2.24) \[ \phi_{k}(y) = \sum_{j=1}^{k} \varphi_{j}(y), \]

one easily finds that the function \( \phi_{k}(y) \) is a non-decreasing function for \( k \) and the following

(2.25) \[ \int_{\tilde{\Omega}} \phi_{k}(y) \, dy = \sum_{j=1}^{k} \int_{\tilde{\Omega}} \varphi_{j}(y) \, dy \]

(2.26) \[ \int_{\tilde{\Omega}} \phi_{\infty}(y) \, dy = \lim_{k \to \infty} \int_{\tilde{\Omega}} \phi_{k}(y) \, dy \leq C_{10} . \]

Consequently, \( \phi_{\infty}(y) \) is an integrable function on \( \tilde{\Omega} \) and

(2.27) \[ \varphi_{k}(y) \leq \phi_{\infty}(y) \quad \text{for any} \quad k \quad \text{and almost all} \quad y \in \tilde{\Omega} . \]

To complete the proof of theorem, it is sufficient to show

(2.28) \[ S \subset E_{\beta} \quad \text{namely, if} \quad x_{0} \notin E_{\beta} , \quad \text{then} \quad x_{0} \notin S . \]
Now we fix \( x_0 \notin E_\beta \), Then we show that the function

\[
(2.29) \quad \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} (Du)(y)dy
\]

is a continuous and bounded function in the open interval \((0, \delta)\) with \( \delta = dist(x_0, \partial \Omega) \).

At first, we shall estimate \(|(Du)_{x_0,R_i} - (Du)_{x_0,R_{i+1}}|\) \((i = 1, 2, \cdots)\). Also, by integrating the following (2.30) over \( B_{R_i}(x_0) \) \( R_i = \frac{1}{2} \left( \frac{1}{2} \right)^i \) \((i = 1, \cdots)\),

\[
|(Du)_{x_0,R_i} - (Du)_{x_0,R_{i+1}}|
\]

(2.30)

\[
\leq |(Du)_{x_0,R_i} - (Du)(x)| + |(Du)_{x_0,R_{i+1}} - (Du)(x)|.
\]

we obtain

\[
|B_{R_i}| \cdot |Du_{x_0,R_i} - Du_{x_0,R_{i+1}}|
\]

(2.31)

\[
\leq \int_{B_{R_i}} |Du_{x_0,R_i} - Du(x)|dx + \int_{B_{R_i}} |Du_{x_0,R_{i+1}} - Du(x)|dx.
\]

Next, dividing (2.31) by \(|B_{R_i}|\) and by using Hölder inequality, we have

(2.32)

\[
|Du_{x_0,R_i} - Du_{x_0,R_{i+1}}|
\]

\[
\leq \frac{1}{|B_{R_i}|} \int_{B_{R_i}} |Du_{x_0,R_i} - Du(x)|dx + \frac{1}{|B_{R_i}|} \int_{B_{R_i}} |Du_{x_0,R_{i+1}} - Du(x)|dx
\]

\[
\leq \frac{1}{|B_{R_i}|} \int_{B_{R_i}} \frac{1}{|B_{R_i}|} \int_{B_{R_i}} Du(y)dy - Du(x)|dx + \frac{1}{|B_{R_i}|} \int_{B_{R_i}} \frac{1}{|B_{R_{i+1}}|} \int_{B_{R_{i+1}}} Du(y)dy - Du(x)|dx
\]

\[
\leq \frac{1}{|B_{R_i}|^2} \int_{B_{R_i}} dx \int_{B_{R_i}} |Du(y) - Du(x)|dy + \frac{1}{|B_{R_i}| |B_{R_{i+1}}|} \int_{B_{R_i}} dx \int_{B_{R_{i+1}}} |Du(y) - Du(x)|dy
\]

\[
\leq \frac{1 + 2^n}{|B_{R_i}|^2} \int_{B_{R_i}} dx \int_{B_{R_i}} |Du(y) - Du(x)|dy.
\]

Here, we extend \( Du(x) \) to be zero outside \( B_{R_i} \) and successively rewrite it to be \( Du(x) \) for convenience. Then we continue the estimates of (2.32) as follows: From the change of variables,

\[
\begin{align*}
\bar{x} &= x, \\
\bar{y} &= y - x
\end{align*}
\]
we obtain
\[\frac{(1+2^n)}{|B_{R_i}|} \int_{B_{R_i}} \int_{B_{R_i}} |(Du)(y) - (Du)(x)|^2 dy \ dx \tag{2.33}\]
\[= \frac{(1+2^n)}{|B_{R_i}|} \int_{B_{R_i}} \int_{B_{2R_i}(x)} |(Du)(y) - (Du)(x)|^2 dy \ dx \tag{2.33}\]
\[= \frac{(1+2^n)}{|B_{R_i}|} \int_{B_{R_i}} \int_{B_{2R_i}(0)} |(Du)(\bar{x} + \bar{y}) - (Du)(\bar{x})|^2 dy \ dx \tag{2.33}\]

By using Fubini Theorem and successively the mean value theorem, there exists a vector \(\bar{y}_i^* \in R^n\) with \(0 < |\bar{y}_i^*| < 2R_i\) such that

\[(2.34) \quad (2.33) = \left[ \frac{c_{11}}{|B_{R_i}|} \int_{B_{R_i}} |(Du)(\bar{x} + \bar{y}_i^*) - (Du)(\bar{x})|^2 d\bar{x} \right].\]

From (2.32) and (2.34), we obtain

\[| (Du)_{x_0,R_i} - (Du)_{x_0,R_{i+1}} | \leq C_{12} \left[ \frac{R_i^{n-\beta+\epsilon}}{|B_{R_i}|} \int_{B_{R_i}} \frac{|(Du)(\bar{x} + \bar{y}_i^*) - (Du)(\bar{x})|^2}{|\bar{y}_i^*|^{n-\beta+\epsilon}} d\bar{x} \right]^\frac{1}{2}. \tag{2.35}\]

Next we shall show that \(\{Du_{x_0,r}\} (r > 0)\) is a Cauchy filter. Let \(r\) and \(R (r < R)\) be positive numbers sufficiently small and then we can take positive integer \(j\) and \(i (i \leq j)\) such that \(R_{j+1} < r \leq R_j \) and \(R_{i+1} < R \leq R_i\). We estimate \(|Du_{x_0,r} - Du_{x_0,R}|\) by dividing it into the following three terms:

\[|Du_{x_0,r} - Du_{x_0,R}| \leq \sum_{k=i}^{j} R^\frac{\epsilon}{k2} \int_{B_{R_k}} \frac{|(Du)(\bar{x} + \bar{y}_k^*) - (Du)(\bar{x})|^2}{|\bar{y}_k^*|^{n-\beta+\epsilon}} d\bar{x} \tag{2.37}\]

We obtain from \(\beta - (n - 2\alpha) \geq 0\),

\[\sum_{k=i}^{j} R^\frac{\epsilon}{k2} \leq R^\frac{\epsilon}{i2} \leq R^\frac{\epsilon}{i2} \frac{1 - (\frac{j}{2})^{j-0\frac{\epsilon}{2}}}{1 - (\frac{j}{2})^{j-0\frac{\epsilon}{2}}} \]

By noting
\[\frac{|(Du)(\bar{x} + \bar{y}_k^*) - (Du)(\bar{x})|^2}{|\bar{y}_k^*|^{n-\beta+\epsilon}} \leq \phi_\infty(x) \quad a.e \quad \bar{x} \in \tilde{\Omega} \text{ and } k = 1, 2, \ldots\]

we can continue to estimate (2.37) as follows:

\[| (Du)_{x_0,r} - (Du)_{x_0,R} | \leq \phi_\infty(x) \quad a.e \quad \bar{x} \in \tilde{\Omega} \text{ and } k = 1, 2, \ldots.\]
Also, from (2.28), there exists a constant $K$ such that

$$
| (Du)_{x_0,r} - (Du)_{x_0,R} | \leq C_{14} K^{\frac{1}{2}} R \frac{\beta-(n-2\alpha)}{2}
$$

This shows that $\{ (Du)_{x_0,r} \}_{r>0}$ is a Cauchy filter. Thus, $\lim_{R \to 0} (Du)_{x_0,R}$ surely exists. Also, from (2.39), we obtain

$$
| (Du)_{x_0,r} - (Du)_{x_0,R} | \leq C_{14} K^{\frac{1}{2}} R \frac{\beta-(n-2\alpha)}{2}
$$

Then

$$
\lim_{R \to 0} | (Du)_{x_0,R} | \leq | (Du)_{x_0,\delta/4} | + C_{14} K^{\frac{1}{2}} (\delta/4)^{\frac{\beta-(n-2\alpha)}{2}}
$$

Consequently, $\lim_{R \to 0} (Du)_{x_0,R}$ exists and is finite. This shows $x_0 \notin S$.

References

[Gi]. Giusti E, Precisazione delle funzione $H^{1,p}$ e singolarita delle soluzioni deboli di sistemi ellittici non lineari, Boll U.M.I 2.