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Kyoto University
On Higher Differentiability and Partial Regularity of the Minimizers in the Calculus of Variations

KAZUHIRO HORIHATA
Keio University

1. Introduction

In this paper we shall treat with the following problem in the calculus of variations: Let $n$ and $N$ be positive integers and suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with the $C^2$-class boundary. Then we consider the functional,

$$ I[v] \equiv \sum_{\alpha, \beta = 1}^{n} \sum_{i,j=1}^{N} \int_{\Omega} a_{i,j}^{\alpha,\beta}(x, v) D_{\alpha}v^i D_{\beta}v^j dx $$

for $v : \Omega \mapsto \mathbb{R}^N,$

where $D_{\alpha}v^i = \frac{\partial v^i}{\partial x_{\alpha}}$ ($\alpha = 1, \cdots, n$, $i = 1, \cdots, N$) and $a_{i,j}^{\alpha,\beta}$ ($\alpha, \beta = 1, \cdots, n$, $i, j = 1, \cdots, N$) are continuously differentiable functions in $\Omega \times \mathbb{R}^N$ satisfying the following: There exist positive numbers $\lambda$ and $\Lambda$ ($0 < \lambda \leq \Lambda < +\infty$) such that $a_{i,j}^{\alpha,\beta}$ ($\alpha, \beta = 1, \cdots, n$, $i, j = 1, \cdots, N$) satisfy for $(x, v) \in \Omega \times \mathbb{R}^N$

$$ \lambda |\zeta|^2 \leq \sum_{i,j=1}^{N} \sum_{\alpha, \beta = 1}^{n} a_{i,j}^{\alpha,\beta}(x, v) \zeta_{\alpha} : \zeta_{\beta}^j \leq \Lambda |\zeta|^2 $$

for $(x, v) \in \mathbb{R}^{n \times N}$ and $\forall \zeta \in \mathbb{R}^{n \times N},$

$$ a_{i,j}^{\alpha,\beta} = a_{j,i}^{\beta,\alpha}. $$

In addition, since the coefficients $a_{i,j}^{\alpha,\beta}$ belong to $C^1(\Omega \times \mathbb{R}^N ; \mathbb{R})$, for positive numbers $K_1$ and $K_2$ there exists a positive number $L(K_1, K_2)$ such that

$$ \max_{1 \leq i,j \leq N} \max_{|z| \leq K_1} |a_{i,j}^{\alpha,\beta}(x, z)| + \max_{1 \leq i,j \leq N} \max_{|z| \leq K_2} \left| \frac{\partial a_{i,j}^{\alpha,\beta}}{\partial e}(x, z) \right| 

+ \max_{1 \leq i,j,k \leq N} \max_{|z| \leq K_1} |a_{i,j,k}^{\alpha,\beta}(x, z)| \leq L(K_1, K_2) $$

where $a_{i,j,k}^{\alpha,\beta}(x, z) \equiv \frac{\partial a_{i,j}^{\alpha,\beta}}{\partial z_k}(x, z)$.
and
\[ \frac{\partial a_{i,j}^{\alpha}}{\partial e'}(x, z) \] denotes the derivative in a direction of a vector \( e \) in \( \mathbb{R}^n \).

This implies the existence of at least a minimizer of the functional \( I \) in the Sobolev space \( H^{1,2}(\Omega; \mathbb{R}^N) \) and \( I \) is lower semicontinuous with respect to the weak topology of \( H^{1,2}(\Omega; \mathbb{R}^N) \) (see [Mo]) under an appropriate boundary condition.

First, we show that the first-derivatives of minimizers satisfies a modulus of uniform continuity in the norm \( L^2_{loc}(\Omega; \mathbb{R}^N) \).

Secondly, we mention a convergence theorem and a partial regularity result of the weak differentials of minimizers. However, we remark that the former theorem was proved in [Gm], [HKL] and [Mm].

We use the summation convention that Latin indices run from 1 to \( N \) and Greek indices run from 1 to \( n \).

We conclude this introduction by recalling other notational conventions:

\[
B_R(x_0) \equiv \{ x \in \mathbb{R}^N : |x - x_0| < R \}. 
\]

For a set \( A \subset \mathbb{R}^N \), we denote by \( mesA \) and \( |A| \) the \( n \)-dimensional Lebesgue measure of \( A \).

For \( u \in L^1(B_R(x_0); \mathbb{R}^N) \), we define
\[
ux_0, R = \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} u(x) dx. 
\]

For a sufficiently small number \( d \), we define an open set
\[
\Omega_d = \Omega - \{ x \in \Omega : dist(x, \partial \Omega) \leq d \}, 
\]
where \( dist(x, \partial \Omega) \) means the Euclidean metric between \( x \) and \( \partial \Omega \).

For a set \( A \) in \( \mathbb{R}^n \), \( H^k(A) \) denotes the \( k \)-dimensional Hausdorff measure of \( A \) (for the definition, see [Gm]).

\( e_i \) \( (i = 1, \cdots, n) \) means the unit vector in \( \mathbb{R}^N \) parallel to the \( x_i \)-axis. We define a translate operator \( \Delta_m \) \( (m = 1, 2, \cdots, n) \) by
\[
(\Delta_m f)(x) = f(x + he_m) - f(x), \quad f \in L^p(\Omega; \mathbb{R}^N). 
\]

2. Main Result

Under the above preparations, we can describe

**Theorem 1.** Let \( u \) be a minimizer of the functional \( I \) in \( H^{1,2}(\Omega; \mathbb{R}^N) \) and let us suppose that \( u \) is a bounded, namely there exists some positive constant \( M \) such that \( ess.\sup u \leq M. \) Then, for any fixed domain \( \hat{\Omega} \) compactly contained in \( \Omega \), there exists positive number \( \alpha = \alpha (n, N, \lambda, \Lambda, M, L) \) \( (0 < \alpha \leq 1) \) and \( C = C(n, N, \lambda, \Lambda, \Omega, M, L) \) such that for \( h > 0 \) with \( h \leq dist(\hat{\Omega}; \partial \Omega) \) \( u \) satisfies
\[
\int_\Omega |\Delta_m(\nabla u(x))|^2 dx \leq C \cdot h^\alpha \quad \text{for} \quad \forall m (m = 1, 2, \cdots, n) 
\]
THEOREM 2. Suppose that $\{u_i\}_{i \geq 1}$ is a sequence of minimizers of $I$ in the space $H^{1,2}(\Omega; R^N)$ such that $\{u_i\}_{i \geq 1}$ converges strongly to a function $u_0$ in $L^2_{loc}(\Omega; R^N)$. Then the function $u_0$ belongs to $H^{1,2}_{loc}(\Omega; R^N)$ and moreover a suitable subsequence of $\{u_i\}_{i \geq 1}$ converges strongly to $u_0$ in $H^{1,2}_{loc}(\Omega; R^N)$.

THEOREM 3. Let $u$ be a minimizer of the functional $I$ in $H^{1,2}(\Omega; R^N)$. Then, for a singular set defined by

$$S = \{x \in \Omega : \# \lim_{\epsilon \to 0} |(Du)_{x,\epsilon}| \} \cup \{x \in \Omega : \lim_{\epsilon \to 0} |(Du)_{x,\epsilon}| = +\infty\}$$

the following

$$H^{(\beta)}(S) = 0$$

holds for any positive number $\beta$ satisfying $n - 2\alpha < \beta < n$.

Remark. In the following proof, the letter $C_i$ $(i = 1, \cdots, 14)$ means a various constant depending only on $n, N, \lambda, \Lambda, \Omega, \tilde{\Omega}, M$ and $L$.

PROOF OF THEOREM 1

First, a minimizer $u$ is a weak solution of the Euler-Lagrange equations of the functional $I$, $u$ satisfies

$$2 \int_{\Omega} a^{\alpha,\beta}_{i,j}(u(x), x) D_\alpha u^i(x) D_\beta \phi^j(x) dx$$

$$+ \int_{\Omega} a^{\alpha,\beta}_{i,j}(u(x), x) D_\alpha u^i(x) D_\beta u^j(x) \phi^k(x) dx = 0$$

for $\forall \phi(x) \in C^\infty(\Omega; R^N)$.

Next, let $\delta$ be a positive number satisfying $\delta < \frac{1}{8} dist(\tilde{\Omega}, \partial \Omega)$. For each number $h (0 < h < \delta)$, the parallel transition along with $x_m -$ axis $(m = 1, \cdots, n)$ leads to

$$2 \int_{\Omega} a^{\alpha,\beta}_{i,j}(u(x+he_m), x+he_m) D_\alpha u^i(x+he_m) D_\beta \phi^j(x) dx$$

$$+ \int_{\Omega} a^{\alpha,\beta}_{i,j}(u(x+he_m), x+he_m) D_\alpha u^i(x+he_m) D_\beta u^j(x+he_m) \phi^k(x) dx = 0$$

for $\forall \phi(x) \in C^\infty(\Omega_\delta; R^N)$.

Then we have

$$2 \int_{\Omega} a^{\alpha,\beta}_{i,j}(u(x), x) D_\alpha \Delta_m u^i(x) D_\beta \phi^j(x) dx$$

$$= 2 \int_{\Omega} [a^{\alpha,\beta}_{i,j}(x, u(x)) - a^{\alpha,\beta}_{i,j}(u(x+he_m), x+he_m)] D_\alpha u^i(x+he_m) D_\beta \phi^j(x) dx$$

$$- \int_{\Omega} a^{\alpha,\beta}_{i,j}(x+he_m, u(x+he_m)) D_\alpha u^i(x+he_m) D_\beta u^j(x+he_m) \phi^k(x) dx$$

$$+ \int_{\Omega} a^{\alpha,\beta}_{i,j}(x, u(x)) D_\alpha u^i(x) D_\beta u^j(x) \phi^k(x) dx$$

(2.5)
after subtracting (2.4) from (2.3).

We now substitute $\Delta_m u(x)\zeta^2(x)$ into $\phi(x)$ in (2.5), where $\zeta(x) \in C^\infty(\Omega_\delta; \mathbb{R})$ is defined by

$$\zeta(x) = \begin{cases} 1 & : \Omega_{4\delta} \\ 0 & : \Omega / \Omega_{3\delta} \end{cases} \quad \text{with} \quad |D\zeta(x)| \leq \frac{2}{\delta}.$$  

Then we have

$$\begin{align*} &2 \int_{\Omega} a_{i,j}^\alpha(x) D_\alpha (\Delta_m u^i(x)) D_\beta (\Delta_m u^j(x)) \zeta^2(x) \, dx \\
&+ 4 \int_{\Omega} a_{i,j}^\alpha(x) D_\alpha (\Delta_m u^i(x)) D_\beta \zeta(x) \Delta_m u^j(x) \zeta(x) \, dx \\
&= 2 \int_{\Omega} [a_{i,j}^\alpha(x, u(x)) - a_{i,j}^\alpha(x + he_m, x + he_m)] D_\alpha u^i(x + he_m) \\
&\quad [D_\beta (\Delta_m u^j(x)) \zeta^2(x) + 2 \Delta_m u^j(x) D_\beta \zeta(x) \zeta(x)] \, dx \\
&- \int_{\Omega} a_{i,j,k}^\alpha(x, u(x)) D_\alpha u^i(x + he_m) D_\beta u^j(x + he_m) \Delta_m u^k(x) \zeta^2(x) \, dx \\
&\quad + \int_{\Omega} a_{i,j,k}^\alpha(x, x) D_\alpha u^i(x) D_\beta u^j(x) \Delta_m u^k(x) \zeta^2(x) \, dx \\
\end{align*}$$  

(2.7)

Here, we estimate the left-hand side of (2.7), which we call $(L)$, from below. First, by using (1.2), we have

$$\begin{align*} (L) &\geq \int_{\Omega} |D(\Delta_m u(x))|^2 |\zeta^2(x)| \, dx \\
&- 4nN\Lambda \int_{\Omega} |D(\Delta_m u(x))| |\zeta(x)| |\Delta_m u(x)||D\zeta(x)| \, dx. \\
\end{align*}$$  

(2.8)

Second, applying the Schwarz inequality to the second term of (2.8) with $\epsilon = \frac{\lambda}{2nN\Lambda}$, we have

$$\begin{align*} (L) &\geq 2\lambda \int_{\Omega} |\Delta_m (D u(x))|^2 |\zeta^2(x)| \, dx \\
&- 2\epsilon nN\Lambda \int_{\Omega} |\Delta_m (D u(x))|^2 |\zeta^2(x)| \, dx \\
&- \frac{2nN\Lambda}{\epsilon} \int_{\Omega} |D\zeta(x)|^2 |\Delta_m u(x)|^2 \, dx \\
&\geq \lambda \int_{\Omega} |\Delta_m (D u(x))|^2 |\zeta^2(x)| \, dx \\
&- 2 \frac{2(nN\Lambda)^2}{\lambda} \int_{\Omega} |D\zeta^2(x)||\Delta_m u(x)|^2 \, dx. \\
\end{align*}$$  

(2.9)

On the other hand, we perform the estimates of the right-hand side of (2.7), which we call
\[(R) = -2 \int_{0}^{1} \frac{d\alpha_{i,j}^{\beta}}{dt}(x + \text{the}_{m}, u(x) + t\Delta_{m}u(x))dt \quad D_{0}u^{i}(x + \text{he}_{m})
\]
\[
[\Delta_{m}u^{j}(x)](2(x) + 2\Delta_{m}u^{j}(x)D_{\beta}\zeta(x)\zeta(x)]dx
\]
\[
- \int_{\Omega} a_{i,j}^{\alpha}
\]
\[
D_{\alpha}u^{i}(x + \text{he}_{m})D_{\beta}u^{j}(x + \text{he}_{m})\Delta_{m}u^{k}(x)\zeta(x)^{2}dx
\]
\[
(2.10)
\]
\[
+ \int_{\Omega} a_{i,j}^{\alpha}
\]
\[
D_{\alpha}u^{i}(x)D_{\beta}u^{j}(x)\Delta_{m}u^{k}(x)\zeta(x)^{2}dx
\]

By using (1.4) and the boundedness of \(u\), and applying the Schwarz inequality to (2.10), we have

\[(R) \leq C_{1} \int_{\Omega}(h + |\Delta_{m}u(x)|)\zeta(x)dx
\]

Thus, by combining (2.9) with (2.11), we have the following:

\[
\int_{\Omega_{4\delta}}|\Delta_{m}(Du(x))|^{2}dx \leq C_{2} \int_{\Omega_{3\delta}}|\Delta_{m}u(x)|^{2}dx
\]

Here, it is a well-known fact that a minimizer \(u\) satisfies a so-called Caccioppoli inequality (see [Gm]): There exists a positive constant \(C\), depending only on \(n, N, \lambda, \Lambda, M\) such that

\[
\int_{B_{R}}|Du(x)|^{2}dx \leq \frac{C}{R^{2}} \int_{B_{2R}}|u(x) - u_{R}|^{2}dx
\]

holds for any ball \(B_{2R} \subset \subset \Omega\) with \(0 < R < \delta\). A direct application of the above inequality to Gering inequality due to F.W. Gering [Ge] (see also [Gm]) leads to the following: There exists a positive number \(p (p > 2)\), which can be supposed to satisfy \(p < 4\) and \(C\) depending only on \(n, N, \lambda, \Lambda, \Omega, M\) such that \(Du(x)\) belongs to \(L_{\text{loc}}^{p}(\Omega; R^{N})\) and moreover

\[
\left(\frac{1}{|\tilde{\Omega}|} \int_{\tilde{\Omega}}|Du(x)|^{p}dx\right)^{\frac{1}{p}} \leq \left(\frac{1}{|\Omega|} \int_{\Omega}|Du(x)|^{2}dx\right)^{\frac{1}{2}}
\]

holds for \(\forall \tilde{\Omega} \subset \subset \Omega\).

Thus, we apply Hölder inequality to the second term of the right-hand side of (2.12) and we have

\[
\int_{\Omega_{4\delta}}|\Delta_{m}(Du(x))|^{2}dx \leq C_{4} \int_{\Omega_{2\delta}}(h + |\Delta_{m}u(x)| + |\Delta_{m}u(x)|^{2})dx
\]

\[
(2.14)
\]

\[
+ C_{4} \left[ \int_{\Omega_{2\delta}}|D_{m}u(x)|^{p}dx \right]^{\frac{2}{p}} \left[ \int_{\Omega_{2\delta}}|\Delta_{m}u(x)|^{2}dx \right]^{\frac{p-2}{p}}.
\]
In addition, by using (2.13),(2.14) and the boundedness of $u$, we have
\[
\int_{\Omega_{4\delta}} |\Delta_{m}(Du(x))|^{2} dx \leq C_{4} \int_{\Omega_{2\delta}} [h + |\Delta_{m}u(x)| + |\Delta_{m}u(x)|^{2}] dx
\]
(2.16)  \[+ C_{5} \left[ \int_{\Omega_{2\delta}} |\Delta_{m}u(x)|^{\frac{p}{1-p}} dx \right]^{\frac{p-2}{p}}.\]

Since $2 < p < 4$ implies $\frac{p}{p-2} > 2$ it follows from the boundedness of $u(x)$ that
\[
\int_{\Omega_{4\delta}} |\Delta_{m}(Du(x))|^{2} dx \leq C_{4} \int_{\Omega_{2\delta}} [h + |\Delta_{m}u(x)| + |\Delta_{m}u(x)|^{2}] dx
\]
(2.17)  \[+ C_{6} \left[ \int_{\Omega_{2\delta}} |\Delta_{m}u(x)|^{2} dx \right]^{\frac{p-2}{p}}.\]

Also, from Newton-Leibnitz formula and a Caccioppoli inequality, we obtain
\[
\int_{\Omega_{2\delta}} |\Delta_{m}u(x)|^{2} dx \leq C_{7} h^{2} \int_{\Omega_{\delta}} |Du(x)|^{2} dx \leq C_{8} h^{2}
\]
(2.18)

Consequently, from (2.17) and (2.18), we deduce
\[
\int_{\Omega_{4\delta}} |\Delta_{m}(Du(x))|^{2} dx \leq C_{8} h^{2(p-2)}.
\]
(2.19)

Also, for any fixed unit vector $e$ one can easily prove
\[
\int_{\Omega_{4\delta}} |\Delta_{e}(Du(x))|^{2} dx \leq C_{8} h^{2(p-2)}
\]
(2.20)

**Proof of Theorem 2**

From (2.19), we obtain an equi-continuity of a sequence of minimizers in $H^{1,2}_{loc}(\Omega; \mathbb{R}^{N})$. Also, it follows from (2.14) that a sequence of minimizers satisfies a uniform boundedness in $H^{1,2}_{loc}(\Omega; \mathbb{R}^{N})$. Thus we obtain the assertion of this theorem from *Rellich - Kondrachev theorem*, (see [Ad]).

**Proof of Theorem 3**

The proof of this Theorem is based on estimate (2.20) and the following lemma due to [Gi] (see also [Gm]).

\[1\text{The estimate (2.19) and (2.20) play an important role in the proofs of the Theorem 2 and Theorem 3.}\]
LEMMA 3.1.

Let $v$ be a function in $L^1_{loc}(\Omega)$ and $\beta$ be any number satisfying $n - 2\alpha < \beta < n$. Set

\[(2.21)\quad E_\beta = \{x \in \Omega : \limsup_{\rho \to 0} \rho^{-\beta} \int_{B_\rho(x)} |v(y)| \, dy > 0\}.
\]

Then, we have

\[(2.22)\quad H^{(\beta)}(E_\beta) = 0.
\]

First, to apply Lemma 3.1 to the proof of Theorem 3 we construct a support function defined as follows: For $\rho_k = \delta(\frac{1}{2})^{k+1}$ ($k = 1, 2, \cdots$) with $\delta = \text{dist}(\tilde{\Omega}, \partial\Omega)$ and a sequence $\{e_k\}_{k \geq 1}$ of unit vectors in $\mathbb{R}^n$ we define

\[(2.23)\quad \varphi_k(y) = \rho_k^{-(n-\beta)-\epsilon} |Du(y + \rho_ke_k) - Du(y)|^2 \quad \text{with} \quad \epsilon = \frac{1}{2} (2\alpha - (n - \beta)).
\]

When we set

\[(2.24)\quad \phi_k(y) = \sum_{j=1}^{k} \varphi_j(y),
\]

one easily finds that the function $\phi_k(y)$ is a non-decreasing function for $k$ and the following

\[(2.25)\quad \int_{\tilde{\Omega}} \phi_k(y) \, dy = \sum_{j=1}^{k} \int_{\tilde{\Omega}} \varphi_j(y) \, dy
\]

follows from (2.20) and assumption of $2\alpha - (n - \beta) = \beta - (n - 2\alpha) > 0$.

Thus $\{\phi_k\}_{k \geq 1}$ is a sequence of measurable functions and moreover, putting $\phi_\infty(y) = \lim_{k \to \infty} \phi_k(y)$, we obtain from Beppo-Levi Theorem

\[(2.26)\quad \int_{\tilde{\Omega}} \phi_\infty(y) \, dy = \lim_{k \to \infty} \int_{\tilde{\Omega}} \phi_k(y) \, dy \leq C_{10}.
\]

Consequently, $\phi_\infty(y)$ is an integrable function on $\tilde{\Omega}$ and

\[(2.27)\quad \varphi_k(y) \leq \phi_\infty(y) \quad \text{for any} \quad k \quad \text{and almost all} \quad y \in \tilde{\Omega}.
\]

To complete the proof of theorem, it is sufficient to show

\[(2.28)\quad S \subset E_\beta \quad \text{namely, if} \quad x_0 \notin E_\beta, \quad \text{then} \quad x_0 \notin S.
\]
Now we fix \(x_0 \notin E_\beta\), Then we show that the function
\[
(2.29) \quad r \mapsto (Du)_{x_0, r} = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} (Du)(y)dy
\]
is a continuous and bounded function in the open interval \((0, \delta)\) with \(\delta = \text{dist}(x_0, \partial\Omega)\).

At first, we shall estimate \(|(Du)_{x_0, R_i} - (Du)_{x_0, R_{i+1}}|\) \((i = 1, 2, \ldots)\). Also, by integrating the following \((2.30)\) over \(B_{R_i}(x_0)\) \(R_i = \frac{\delta}{2}(\frac{1}{2})^i\) \((i = 1, \ldots)\),
\[
|(Du)_{x_0, R_i} - (Du)_{x_0, R_{i+1}}|
\]
\[
(2.30) \quad \leq |(Du)_{x_0, R_i} - (Du)(x)| + |(Du)_{x_0, R_{i+1}} - (Du)(x)|.
\]
we obtain
\[
|B_{R_i}| \cdot |Du_{x_0, R_i} - Du_{x_0, R_{i+1}}|
\]
\[
(2.31) \quad \leq \int_{B_{R_i}} |Du_{x_0, R_i} - Du(x)|dx + \int_{B_{R_i}} |Du_{x_0, R_{i+1}} - Du(x)|dx.
\]
Next, dividing \((2.31)\) by \(|B_{R_i}|\) and by using Hölder inequality, we have
\[
(2.32) \quad |Du_{x_0, R_i} - Du_{x_0, R_{i+1}}|
\]
\[
\leq \frac{1}{|B_{R_i}|} \int_{B_{R_i}} |Du_{x_0, R_i} - Du(x)|dx + \frac{1}{|B_{R_i}|} \int_{B_{R_i}} |Du_{x_0, R_{i+1}} - Du(x)|dx
\]
\[
\leq \frac{1}{|B_{R_i}|} \int_{B_{R_i}} \left( \frac{1}{|B_{R_i}|} \int_{B_{R_i}} Du(y)dy - Du(x) \right)dx + \frac{1}{|B_{R_i}|} \int_{B_{R_i}} \left( \frac{1}{|B_{R_{i+1}}|} \int_{B_{R_{i+1}}} Du(y)dy - Du(x) \right)dx
\]
\[
\leq \frac{1}{|B_{R_i}|^2} \int_{B_{R_i}} dx \int_{B_{R_i}} |Du(y) - Du(x)|dy + \frac{1}{|B_{R_i}| \cdot |B_{R_{i+1}}|} \int_{B_{R_i}} dx \int_{B_{R_{i+1}}} |Du(y) - Du(x)|dy
\]
\[
\leq \frac{1 + 2^n}{|B_{R_i}|^2} \int_{B_{R_i}} dx \int_{B_{R_i}} |Du(y) - Du(x)|dy.
\]
\[
\leq \frac{1 + 2^n}{|B_{R_i}|^2} \left[\int_{B_{R_i}} dx \int_{B_{R_i}} dy \right]^{\frac{1}{2}} \left[\int_{B_{R_i}} dx \int_{B_{R_i}} |Du(y) - Du(x)|^2 dy \right]^{\frac{1}{2}}
\]
\[
\leq \frac{1 + 2^n}{|B_{R_i}|} \left[\int_{B_{R_i}} dx \int_{B_{R_i}} |Du(y) - Du(x)|^2 dy \right]^{\frac{1}{2}}.
\]
Here, we extend \(Du(x)\) to be zero outside \(B_{R_i}\) and successively rewrite it to be \(Du(x)\) for convenience. Then we continue the estimates of \((2.32)\) as follows: From the change of variables,
\[
\begin{align*}
\bar{x} &= x, \\
\bar{y} &= y - x
\end{align*}
\]
we obtain

$$
\frac{(1+2^n)}{|B_{R_i}|} \int_{B_{R_i}} dz \int_{B_{S_i}} |(Du)(y) - (Du)(x)|^2 dy \right]^{1/2} \tag{2.33}
$$

$$
= \frac{(1+2^n)}{|B_{R_i}|} \int_{B_{R_i}} dz \int_{B_{S_i}(x)} |(Du)(y) - (Du)(x)|^2 dy \right]^{1/2}
$$

$$
= \frac{(1+2^n)}{|B_{R_i}|} \int_{B_{R_i}} d\bar{x} \int_{B_{2R_i}(0)} |(Du)(\bar{x} + \bar{y}) - (Du)(\bar{x})|^2 dy \right]^{1/2}.
$$

By using Fubini Theorem and successively the mean value theorem, there exists a vector $\bar{y}_i^* \in R^n$ with $0 < |\bar{y}_i^*| < 2R_i$ such that

$$
(2.34) \quad (2.33) = \left[ \frac{c_{11}}{|B_{R_i}|} \int_{B_{R_i}} |(Du)(\bar{x} + \bar{y}_i^*) - (Du)(\bar{x})|^2 d\bar{x} \right].
$$

From (2.32) and (2.34), we obtain

$$
(2.35) \quad |(Du)_{x_0,R_i} - (Du)_{x_0,R_{i+1}}| \leq C_{12} \left[ \frac{R_i^{n-\beta+\epsilon}}{|B_{R_i}|} \int_{B_{R_i}} \frac{|(Du)(\bar{x} + \bar{y}_i^*) - (Du)(\bar{x})|^2}{|\bar{y}_i^*|^{n-\beta+\epsilon}} d\bar{x} \right]^{1/2}.
$$

Next we shall show that $\{Du_{x_0,r}\} (r > 0)$ is a Cauchy filter. Let $r$ and $R (r < R)$ be positive numbers sufficiently small and then we can take positive integer $j$ and $i (i \leq j)$ such that $R_{j+1} < r \leq R_j$ and $R_{i+1} < R \leq R_i$. We estimate $|Du_{x_0,r} - Du_{x_0,R}|$ by dividing it into the following three terms:

$$
(2.36) \quad |Du_{x_0,r} - Du_{x_0,R}| \leq |Du_{x_0,r} - Du_{x_0,R_j}| + |Du_{x_0,R_j} - Du_{x_0,R_i}| + |Du_{x_0,R} - Du_{x_0,R_i}|.
$$

Thus, by the same way as above, for $0 < r < R < \delta$, the following holds:

$$
(2.37) \quad |(Du)_{x_0,r} - (Du)_{x_0,R}| \leq C_{12} \sum_{k=i}^{j} R_k^{\frac{n}{2}} \left[ \frac{R_k^{n-\beta+\epsilon}}{|B_{R_k}|} \int_{B_{R_k}} \frac{|(Du)(\bar{x} + \bar{y}_k^*) - (Du)(\bar{x})|^2}{|\bar{y}_k^*|^{n-\beta+\epsilon}} d\bar{x} \right]^{1/2}.
$$

We obtain from $\beta - (n - 2\alpha) \geq 0$,

$$
\sum_{k=i}^{j} R_k^{\frac{n}{2}} \leq R_i^{\frac{n}{2}} \frac{1 - \left(\frac{j}{i}\right)^{1/2}}{1 - \left(\frac{j}{i}\right)^{1/2}}
$$

By noting

$$
\frac{|(Du)(\bar{x} + \bar{y}_k^*) - (Du)(\bar{x})|^2}{|\bar{y}_k^*|^{n-\beta+\epsilon}} \leq \phi_\infty(x) \quad a.e \quad \bar{x} \in \tilde{\Omega} \quad k = 1, 2, \ldots.
$$

we can continue to estimate (2.37) as follows:

$$
|(Du)_{x_0,r} - (Du)_{x_0,R}|
$$
(2.38) \[ \leq C_{14} R^{\frac{1}{2}} \left[ \operatorname{ess.sup}_{k>0} R_k^{-\beta} \int_{B_{R_k}} \phi_{\infty}(y) dy \right]^{\frac{1}{2}}. \]

Also, from (2.28), there exists a constant $K$ such that

(2.39) \[ |(Du)_{x_0,r} - (Du)_{x_0,R}| \leq C_{14} K R^{\frac{\beta - (n - 2\alpha)}{2}}. \]

This shows that \{(Du)_{x_0,r}\}_{r>0} is a Cauchy filter. Thus, \( \lim_{R \to +0} (Du)_{x_0,R} \) surely exists. Also, from (2.39), we obtain

(2.40) \[ |(Du)_{x_0,r} - (Du)_{x_0,R}| \leq C_{14} K^{\frac{1}{2}} 2 R^{\frac{\beta - (n - 2\alpha)}{2}}. \]

Then

(2.41) \[ \lim_{R \to +0} |(Du)_{x_0,R}| \leq |(Du)_{x_0,\delta/4}| + C_{14} K^{\frac{1}{2}} (\delta/4)^{\frac{\beta - (n - 2\alpha)}{2}}. \]

Consequently, \( \lim_{R \to +0} (Du)_{x_0,R} \) exists and is finite. This shows \( x_0 \notin S \).

References

[Gi]. Giusti E, Precisazione delle funzione $H^{1,p}$ e singolarita delle soluzioni deboli di sistemi ellittici non lineari, Boll U.M.I 2.