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Recent Progress in Semilinear Elliptic Equations

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In this expository paper I wish to describe some recent progress in semilinear elliptic equations. I shall concentrate on such equations on the entire Euclidean spaces, as well as on bounded domains with homogeneous Neumann boundary conditions.

This paper consists of three parts. In Part I, I shall discuss the conformal Gaussian curvature equation in $\mathbb{R}^2$ (with the canonical metric). In particular, some very recent results obtained in a current joint project of K.-S. Cheng and myself which give complete classifications of all possible solutions of this equation in various cases will be included and the proofs will be sketched. In Part II, I shall survey the Matukuma equation, Eddington equation, the conformal scalar curvature equation in $\mathbb{R}^n$, $n \geq 3$, and related equations. Since this part is closely related to my recent survey article [N1] and Professor T. Kusano's lecture, I shall be brief to avoid unnecessary repetitions. In fact, both Part I and Part II should be considered as an update of Section 1 ~ Section 3 in [N1]. Finally in Part III, I shall report some recent results on a semilinear Neumann problem arising in pattern formation in mathematical biology. Here we shall use two different variational approaches: the well-known Mountain-Pass Lemma (a minimax theorem) and a constrained minimization technique first devised by Z. Nehari for a two-point boundary value problem of an ordinary differential equation in early 1960's. We shall show that the solution given by the Mountain-Pass Lemma is in fact the global minimizer of the Nehari's minimization problem. This fact, first noted by W.-Y. Ding and myself in 1985 after we completed our joint papers [DN1,2], was described in public by myself in the 22nd Midwest PDE Conference held at the University of Chicago in 1987 as well as in my course "Topics in PDE" given in the Fall of 1987 at the University of Minnesota with complete proof, and is published here for the first time in the literature.

This paper is essentially a collection of the notes of various lectures I gave at Tokyo and Kyoto in the summer of 1988. I wish to thank Professors Takashi Suzuki and Shoji Yotsutani for the invitations, and the Japan Association for Mathematical Sciences for the financial support which made my visit to Japan possible. Part of this paper was written while I was visiting the University of Tokyo and Miyazaki University, and I wish to thank the staff of those institutions for their warm hospitality.

Part I. The Conformal Gaussian Curvature Equation on $\mathbb{R}^2$.

We intend to give a brief survey of recent progress on the equation

$$\Delta u + Ke^{2u} = 0 \tag{I.1}$$

in $\mathbb{R}^2$, where $\Delta = \sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2}$ and K is a given smooth function on $\mathbb{R}^2$.

Equation (I.1) arises in Riemannian geometry and is known as the conformal Gaussian curvature equation. The readers are referred to a recent survey article [N1; Section 2] for the background and the long history of this equation. However, to motivate this present update on (I.1) a few important results reported in [N1; Section 2] are collected here.

Theorem I.A. (Sattinger) *If $K \le 0$ in $\mathbb{R}^2$ and $K \le -C|x|^{-2}$ at ∞, then equation (I.1) does not possess any solution on $\mathbb{R}^2$.*

Theorem I.B. (Ni [N2]) *If $K \le 0$ and $\not\equiv 0$ in $\mathbb{R}^2$ with $K(x) \ge -C|x|^{-\ell}$ at ∞ for some constant $\ell > 2$, then for every sufficiently small $\alpha > 0$, equation (I.1) possesses a solution u_α in $\mathbb{R}^2$ such that*

(I.2) $u_\alpha(x) = \alpha \log |x| + O(1)$ near ∞ .

Observe that as far as existence is concerned, Theorem I.B already complements Theorem I.A. Nevertheless, Theorem I.B was improved by McOwen [M] as follows.

Theorem I.C. Under the same hypotheses of Theorem I.B (assuming that $K(0) < 0$), we have that for every $0 < \alpha < (\ell-2)/2$, there exists a solution u_α of (I.1) on $\mathbb{R}^2$ such that (I.2) holds. If in addition that $K \leq -C|x|^{-\ell}$ at ∞ , then equation (I.1) admits no solution in $\mathbb{R}^2$ satisfying (I.2) with $\alpha \geq (\ell-2)/2$.

It turns out that Theorem I.C is still not sharp, even in the special case $K \sim -|x|^{-\ell}$ at ∞ for some $\ell > 2$. As we shall see soon that Theorem I.E below gives complete knowledge of all possible solutions of equation (I.1) on $\mathbb{R}^2$ in this case. (In this paper we use the notation "$f \sim g$ at ∞ " to denote that "there exist two positive constants C_1, C_2 such that $C_1 f \geq g \geq C_2 f$ at ∞ ".)

What happens if $K \leq 0$ decays at ∞ faster than any polynomial decay? It was already noticed by Ni [N2] that if $K \leq 0$ in $\mathbb{R}^2$ and $K \sim -\exp(-2|x|^2)$ at ∞ , then, besides the solutions given by Theorem I.B and I.C (which have logarithmic growth at ∞), equation (I.1) actually has a solution $U \sim |x|^2$ at ∞. In fact, to understand the relation between these two kinds of solutions was one of the motivations of my current joint work with K.-S. Cheng [CN1]. We shall see that (Proposition I.H below) the solution U is the maximal solution of (I.1) in $\mathbb{R}^2$.

Our aim in [CN1] is to investigate the structure of the set of all possible solutions of equation (I.1), and, our main results are as follows. (The function K will be assumed to be locally Holder continuous in $\mathbb{R}^2$ throughout the entire paper.)

Theorem I.D. Suppose that

(i) $K \leq 0$ in $\mathbb{R}^2$ and there exists a sequence of bounded smooth simply-connected domains $\{\Omega_i\}$ such that $\mathbb{R}^2 = \bigcup_1^\infty \Omega_i$, $\bar{\Omega}_i \subseteq \Omega_{i+1}$, and $K < 0$ on $\partial\Omega_i$, $i = 1, 2, \ldots$, and

(ii) equation (I.1) possesses a solution v on $\mathbb{R}^2$. Then the function

$$\text{(I.3)} \qquad U(x) = \sup\{u(x) \mid u \text{ is a solution of (I.1) on } \mathbb{R}^2\}$$

is well-defined everywhere in $\mathbb{R}^2$ and is a solution of (I.1) on $\mathbb{R}^2$.

Note that hypothesis (i) in Theorem I.D holds if $K \leq 0$ in $\mathbb{R}^2$ and $K < 0$ at ∞. The function U given by (I.3) is *the maximal solution* of (I.1).

Theorem I.E. *Suppose that* $K \leq 0$ *and* $\not\equiv 0$ *in* $\mathbb{R}^2$, *and that* $K \sim -|x|^{-\ell}$ *at* ∞ *for some* $\ell > 2$.

(i) *For each* $\alpha \in (0,(\ell-2)/2)$, (I.1) *possesses a unique solution* u_α *satisfying* (I.2).

(ii) *Let* u *be a solution of* (I.1) *on* $\mathbb{R}^2$. *Then either* $u \equiv U$ *where* U *is given by* (I.3) *or* $u \equiv u_\alpha$ *for some* $\alpha \in (0,(\ell-2)/2)$ *where* u_α *is given by* (i) *above*.

(iii) *The asymptotic behavior of* U *at* ∞ *is given by*

$$U(x) = \frac{\ell-2}{2}\log|x| - \log\log|x| + O(1) \quad \text{at } \infty . \tag{I.4}$$

Theorem I.F. *Suppose that* $K \leq 0$ *and* $\not\equiv 0$ *in* $\mathbb{R}^2$, *and that* K *has compact support in* $\mathbb{R}^2$. *Then*

(i) *for every harmonic function* h *and every constant* $\alpha > 0$, *there exists a unique solution* u *of* (I.1) *with*

$$u(x) = h(x) + \alpha \log|x| + O(1) \quad \text{at } \infty ; \tag{I.5}$$

(ii) *for every solution* u *of* (I.1) *on* $\mathbb{R}^2$, *there exist a harmonic function* h *and a constant* $\alpha > 0$ *such that* (I.5) *holds*.

Theorem I.F says that in case $K \lesssim 0$, $\not\equiv 0$, on $\mathbb{R}^2$ and has compact support, the "solution set" of (I.1) may be indexed by h and α with h being harmonic and $\alpha > 0$, and is therefore completely understood. This in particular indicates that in this case equation (I.1) does not possess a maximal solution on the entire $\mathbb{R}^2$. The case $K \lesssim 0$, $\not\equiv 0$, in $\mathbb{R}^2$ with $K \sim -|x|^{-\ell}$ at ∞ for some $\ell > 2$, Theorem I.E asserts that the "solution set" of (I.1) can be indexed by $\alpha \in (0,(\ell-2)/2)$ with only one exception (namely, the maximal solution U), thus is also completely understood. Moreover, as a consequence of this result and some further arguments we are able to obtain the following curious conclusion concerning the symmetry of solutions.

Corollary I.G. Under the hypotheses of Theorem I.E, if in addition we assume that K is radially symmetric, then all solutions of (I.1) on $\mathbb{R}^2$ are radially symmetric.

Note that no other conditions on K (such as the monotonicity assumption) are imposed in Corollary I.G. (Compare e.g. results in [GNN2] and [LN1].) Furthermore, combining Corollary I.G and Theorem I.E, we see that Theorem I.C is not sharp even for radial K's.

Remark. Geometrically, for a given (arbitrary) smooth function K on $\mathbb{R}^2$, a theorem of Kazdan and Warner ([KW], p. 210, Corollary 3.3) guarantees the existence of a metric g on $\mathbb{R}^2$ of which the Gaussian curvature is K. Of course g may not be conformal to the standard metric on $\mathbb{R}^2$. For instance, if K satisfies the hypotheses of Theorem I.A, then the corresponding g cannot possibly be conformal to the standard metric on $\mathbb{R}^2$. The geometric significance of Theorem I.E is that under suitable conditions on K it gives a complete classification (and understanidng) of all metrics on $\mathbb{R}^2$ which realize K as their Gaussian curvature and are conformal to the standard one. Theorem I.F admits the same geometric interpretation although the asymptotic behaviors of those (conformal) metrics are not quite as clear.

In the case where K decays exponentially at ∞, although a complete classification of all the possible solutions of equation (I.1) is yet to be obtained, we do have the following

Proposition I.H. *Suppose that* $K \le 0$ *in* $\mathbb{R}^2$ *and that* $K \sim -\exp(-2|x|^2)$ *at* $x = \infty$. *Then the maximal solution* U *of equation* (I.1) *has the following property*

$$U(x) = |x|^2 + O(1) \quad \text{at} \quad x = \infty .$$

In fact, we do have some further information concerning the structure of the solution set of equation (I.1) in this case. We refer the interested readers to [CN1] for details.

I wish to devote the rest of this section to a very brief description of some of the ideas used in the proofs of Theorems I.D and I.E.

The key observation in proving Theorem I.D is that the boundary value problem

$$\begin{cases} \Delta u + Ke^{2u} = 0 & \text{in } \Omega_i , \\ u = \infty & \text{on } \partial\Omega_i , \end{cases}$$

has a solution u_i with the property $u_1 > u_2 > \ldots > u_i > u_{i+1} > \ldots > v$. Then it is not hard to show that u_i converges to U (defined by (I.3)).

In deriving asymptotic behaviors of maximal solutions, the following properties of maximal solutions are often useful:

(i) (Monotonicity) Suppose that $0 \geq K \geq \tilde{K}$ on $\mathbb{R}^2$ and that the hypotheses of Theorem I.D hold. Let U ($\tilde{U}$ resp.) be the maximal solution of $\Delta u + Ke^{2u} = 0$ ($\Delta u + \tilde{K}e^{2u} = 0$ resp.) in $\mathbb{R}^2$. Then $U \geq \tilde{U}$.

(ii) If K is radially symmetric, then so is the corresponding maximal solution U.

(iii) Let u be a solution of (I.1) and U be the maximal solution. Then either $u < U$ everywhere in $\mathbb{R}^2$ or $u \equiv U$ in $\mathbb{R}^2$.

From the monotonicity property above it is easy to see that part (iii) of Theorem I.E is reduced to the special case

$$(I.6) \qquad \Delta u + \tilde{K}e^{2u} = 0$$

in $\mathbb{R}^2$ where $\tilde{K}$ is radial and $\tilde{K}(x) \equiv -|x|^{-\ell}$ at ∞, say, for $|x| \geq R$. The advantage of dealing with (I.6) instead is that $\tilde{K}$ is explicit. Thus, setting $s = \log|x|$ and

$$w(s) = \tilde{U}(x) - \frac{\ell-2}{2}\log|x| ,$$

we obtain $w_{ss} = Ce^{2w}$ in $s > \log R$. Now it is not hard to show that $w(s) = -\log s + O(1)$ at $s = \infty$. Thus (I.4) holds.

The existence part of Theorem I.E (i) is due to McOwen [M] while the uniqueness part follows from the classical Liouville Theorem on bounded subharmonic functions on $\mathbb{R}^2$.

It remains to show part (ii) of Theorem I.E. Let u be a solution of (I.1) on $\mathbb{R}^2$ and $u \not\equiv U$. To show that $u \equiv u_\alpha$ for some $\alpha \in (0,(\ell-2/2)$, the crucial step is to establish the following estimate: <u>there exists</u> $\epsilon > 0$ <u>such that</u>

$$(I.7) \qquad u(x) \leq (\frac{\ell-2}{2} - \epsilon)\log|x| \quad \underline{\text{at}} \quad \infty .$$

Assuming this, we proceed as follows. First observe that (I.7) guarantees that the function

$$v(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2}[-K(y)e^{2u(y)}]\log|x-y|dy$$

is well-defined on $\mathbb{R}^2$ and that

$$\text{(I.8)} \qquad v(x) = \{\frac{1}{2\pi}\int_{\mathbb{R}^2}[-K(y)e^{2u(y)}]dy\}\cdot\log|x| + O(1)$$

at ∞. Moreover, $\Delta(u-v) = 0$ and $u-v = O(\log|x|)$ at ∞. Thus we have $u-v \equiv$ constant, and therefore

$$u(x) = \alpha \log|x| + O(1)$$

at ∞, for some $\alpha > 0$ (given by (I.8)). The fact that $\alpha < (\ell-2)/2$ follows from the non-existence part of Theorem I.C.

We still have to prove (I.7). It is easy to see that $\psi \equiv U-u$ cannot be a constant. Since $\Delta\bar{\psi} \geq 0$, it follows that $\bar{\psi}(r) \geq \epsilon \log r$ at ∞ for some $\epsilon > 0$. Thus

$$\text{(I.9)} \qquad \bar{u}(r) \leq \bar{U}(r) - \epsilon \log r \leq (\frac{\ell-2}{2} - \epsilon)\log r$$

for r large. (Here $\bar{\psi}, \bar{u}$ and $\bar{U}$ denote the spherical mean of ψ, u and U respectively.) Defining

$$u^*(r) = \max_{|x|=r} u(x) \ , \ u_*(r) = \min_{|x|=r} u(x) \ ,$$

by a theorem of Hayman [H] we see that there exists a constant C such that

$$\text{(I.10)} \qquad u^*(r) \le u_*(r) + C$$

holds for all r large except for a "small" set of r's since u is subharmonic and has "slow" growth at ∞ (i.e. $u(x) = O(\log|x|)$ at ∞) . In particular, this implies that there exists a sequence $r_k \to \infty$ such that

$$\text{(I.11)} \qquad u^*(r_k) \le (\frac{\ell-2}{2} - \epsilon)\log r_k + C$$

in view of (I.9). Since u^* is also subharmonic (i.e. satisfies the mean-value inequality), the maximum principle guarantees that (I.11) holds for all $r \ge r_1$. This completes the proof.

<u>Remark</u>. In fact, as an intermediate step in the proof of (I.10), Hayman actually established the following inequality

$$u^*(r) \leq \bar{u}(r) + o(1)$$

<u>for all</u> r <u>near</u> ∞ ((2.10), p. 79 in [H]). Then (I.7) follows from this and (I.9) immediately.

In a number of cases above we could use the weighted Sobolev space approach instead. However, the maximum principle approach adopted here not only seems simpler but also gives us a unified treatment to our results. Moreover, the techniques and methods used in this paper also apply to the conformal scalar curvature equation

$$\Delta u + Ku^{\frac{n+2}{n-2}} = 0$$

in $\mathbb{R}^n$, $n \geq 3$. We refer the interested readers to [CN2] for further details.

Part II. Matukuma Equation, Eddington Equation and the Conformal Scalar Curvature Equation.

1. Matukuma Equation and Eddington Equation.

In 1930, Matukuma, a Japanese astrophysicist, proposed the following mathematical model to describe the dynamics of a globular cluster of stars

$$\Delta u + \frac{1}{1+|x|^2} u^p = 0 \ , \ x \in \mathbb{R}^3 \ , \tag{II.1}$$

where $p > 1$. His aim was to improve an earlier model in 1915 of Eddington

$$\Delta u + \frac{1}{1+|x|^2} e^{2u} = 0 \ , \ x \in \mathbb{R}^3 \ . \tag{II.2}$$

Due to their physical background, positive radial entire solutions of (II.1) and (II.2) are of particular interests. Again, we refer the readers to [N1; Section 3] for a more detailed description of the background of (II.1), (II.2) and the first mathematical results of Ni and Yotsutani [NY] and Li and Ni [LN1]. Here we wish to describe some further progress made by Li and Ni [LN1] very recently concerning the symmetry of finite total mass solutions of (II.1).

A positive solution u of (II.1) in $\mathbb{R}^3$ is said to have <u>finite total mass</u> if

$$\int_{\mathbb{R}^3} \frac{1}{1+|x|^2} u^p(x)dx < \infty .$$

Our main results are as follows.

Theorem II.A. Let $2 < p < 5$. Then every bounded positive entire solution u of (II.1) with finite total mass is radially symmetric about the origin and $u_r < 0$ in $r > 0$. Furthermore,

$$\lim_{r\to\infty} ru(r) = k > 0 .$$

Theorem II.B. Let $p \geq 5$. Then every bounded positive entire solution of (II.1) has infinite total mass.

Needless to say, the above results extend to $\mathbb{R}^n$, $n \geq 3$ and to more general equations than (II.1).

The method of proof consists of two parts. First, we show that for any $p > 1$, a bounded positive entire solution of (II.1) must decay at ∞ like $|x|^{-1}$ (or $|x|^{2-n}$ for general $n \geq 3$). The second ingredient is a refinement of a symmetry result in [GNN2] (Theorem 1", p. 380). As an easy corollary of this, we have

Theorem II.C. Every bounded positive solution u of the equation

$$\Delta u + K(|x|)f(u) = 0$$

in $\mathbb{R}^n$, where $f \geq 0$ is nondecreasing in $u > 0$, $K \geq 0$ is strictly decreasing in $r \geq 0$ and $K(r) \leq r^{\tau}$ at $r = \infty$ with $\tau < -(n+1)$, must be radially symmetric.

The novelty here seems to be that no decay rate on solutions is imposed. In fact, u does not even have to tend to zero at ∞. Naturally, it is the decay of the coefficient K which makes this possible.

2. The Conformal Scalar Curvature Equation.

The following equation

$$\Delta u + Ku^{\frac{n+2}{n-2}} = 0 \tag{II.3}$$

in $\mathbb{R}^n$, $n \geq 3$, is known as the conformal scalar curvature equation in $\mathbb{R}^n$. Due to its geometric application, we are only interested in positie solutions of (II.3). Although equation (II.3) also has a long history, it only became a subject of intensive study in early 1980's since the publication of [N3]. Indeed, there have been enormous work done in connection with (II.3) in the literature by many mathematicians including several Japanese analysts, Kusano, Naito, Kawano, Satsuma, Yotsutani, and their colleagues. Once again, we only intend to give an update on the progress made after the survey [N1; Section 1] was written.

In the so-called "fast-decay" case, that is,

$$\text{(II.4)} \qquad |K(x)| \leq C|x|^{-\ell} \quad \text{at} \quad \infty$$

for some $\ell > 2$, we now have a good understanding of (II.3). It has been established by Ni, Kawano, Naito that under condition (II.4), for every small constant $c > 0$, equation (II.3) possesses a positive entire solution u_c which tends to c at ∞. This leads to a natural question: <u>Does</u> (II.3) <u>possess a positive entire solution which tends to zero under condition</u> (II.4)? This question has been studied by P.-L. Lions as well as many Japanese mathematicians (see Kusano's article [K]). A good progress was made by Kawano, Satsuma and Yotsutani [KSY] recently. In [KSY], a nice sufficient condition was found under the additional hypothesis that K <u>is radial</u>, <u>locally Holder continuous and</u> $K(0) = 0$. On the other hand, in [LN2], the following result was proved.

<u>Theorem II.D</u>. <u>Suppose that</u> (II.4) <u>holds and that the function</u>

$x \cdot \nabla K(x)$ <u>never changes sign in</u> $\mathbb{R}^n$. <u>Then equation</u> (II.3) <u>does not possess any bounded positive solution</u> u <u>in</u> $\mathbb{R}^n$ <u>with</u> $\liminf_{x\to\infty} u = 0$.

The proof contains two main ingredients. The first one is an estimate which guarantees that any such solution will have to actually tend to zero at ∞ like $|x|^{2-n}$. Then we apply a generalized Rellich-Pohozaev identity to the solution and use the estimate above to control the "boundary terms" to obtain a contradiction. We should point out that Theorem II.D above is an extension of Theorem 1.9 in [N1], and that all the theorems mentioned in this section admit extensions to more general equations.

Part III. Variational Methods in a Semilinear Neumann Problem.

We shall consider the following Neumann problem

$$\text{(III.1)} \qquad \begin{cases} d\Delta u - u + u^p = 0 & \text{in } \Omega , \\ u > 0 & \text{in } \Omega , \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega , \end{cases}$$

where $d > 0$, $p > 1$ are constants, Ω is a bounded smooth domain in $\mathbb{R}^n$ and ν is the unit outer normal to $\partial\Omega$. Problem (III.1) is (mathematically) equivalent to an (elliptic) chemotaxis system (the Keller-Segal model); moreover, it also arises naturally in an activator-inhibitor system due to Gierer and Meinhardt. For these biological applications, we are interested in solutions of (III.1) which exhibit "spiky" patterns. For the details of the background of (III.1), the readers are referred to [LNT] and the references therein.

It turns out that (III.1) is very different from its Dirichlet counterpart

$$\text{(III.2)} \qquad \begin{cases} d\Delta u - u + u^p = 0 & \text{in } \Omega , \\ u > 0 & \text{in } \Omega , \\ u = 0 & \text{on } \partial\Omega . \end{cases}$$

While the Dirichlet problem (III.2) has long been studied and we have had a good understanding about it, significant progress on the Neumann problem (III.1) was not made until the recent papers [NT], [LNT] and [LnN] were published. In this section, we would like to describe some of the results in those papers as well as in the current joint project of I. Takagi and myself.

Main results on (III.1) may be summarized as follows.

Theorem III.A. *Let* $p < \frac{n+2}{n-2}$. *Then* (III.1) *possesses only trivial solution* $u \equiv 1$ *if* d *is sufficiently large*.

Theorem III.B. *Let* $p < \frac{n+2}{n-2}$. *Then for each* $d > 0$ *there exists a solution* u_d *of* (III.1) *with the following properties*:

(i) $u_d \to 0$ *in measure as* $d \to 0$;

(ii) *there exists a constant* C , *independent of* $d > 0$, *such that*

$$1 < \|u_d\|_{L^\infty} < C$$

for all $d > 0$;

(iii) *for each* $q \in [1,\infty)$, *there exist constants* $C_i(q)$, $i = 1,2$, *independent of* $d > 0$, *such that*

$$C_1(q)d^{n/2} \le \int_\Omega u_d^q \le C_2(q)d^{n/2}$$

<u>for all</u> $d > 0$;

(iv) $\lambda_1(u_d) < 0 \le \lambda_2(u_d)$ <u>for all</u> $d > 0$, <u>where</u> $\lambda_j(u_d)$ <u>is the</u> j-th <u>eigenvalue of</u> (III.1) <u>linearized at</u> u_d ; i.e.

$$\begin{cases} (d\Delta - 1 + pu_d^{p-1})\varphi + \lambda_j(u_d)\varphi = 0 & \text{in } \Omega , \\ \frac{\partial \varphi}{\partial \nu} = 0 & \text{on } \partial\Omega ; \end{cases}$$

(v) $u_d \not\equiv 1$ <u>if</u> $d < (p-1)/\lambda_2$ <u>where</u> λ_2 <u>is the second eigenvalue of</u> Δ <u>on</u> Ω <u>with zero Neumann boundary data</u>;

(vi) <u>for any</u> $\eta > 0$, <u>set</u> $\Omega_{\eta,d} \equiv \{x\in\Omega \mid u_d(x) > \eta\}$. <u>Then there exists a positive integer</u> m <u>which depends only on</u> Ω, p <u>and</u> η (<u>but independent of</u> d) <u>such that</u> $\Omega_{\eta,d}$ <u>may be covered by at most</u> m <u>balls of radius</u> $\sqrt{d}$;

(vii) <u>there exist positive constants</u> $\tilde{C}$, γ , <u>independent of</u> $d > 0$, <u>such that</u>

$$\inf_\Omega u_d \le \tilde{C} \exp(-\gamma/\sqrt{d})$$

<u>for all</u> $d > 0$.

The proof of Theorem III.A is in [LNT] and will be omitted here. The proof of Theorem III.B may be found in

[LNT] and [LnN]. However, we would like to describe a new proof of (iv) above which makes use of Nehari's variational approach, and is more transparent than the original proof in [LnN]. To describe the proof, we need to sketch the original existence proof of u_d .

The existence proof is based on the well-known Mountain-Pass Lemma of Ambrosetti and Rabinowitz [AR]. To simplify our notations, we first set

$$f(u) = \begin{cases} u^p & \text{if } u \geq 0 , \\ 0 & \text{if } u \leq 0 , \end{cases}$$

and $F(u) = \int_0^u f(t)dt$. Then, in $H_1(\Omega)$, we define the variational functional

$$J_d(u) = \frac{d}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega u^2 - \int_\Omega F(u) .$$

and we look for the critical points of J_d . It is standard to show that the value c_d defined by

$$\text{(III.3)} \qquad c_d = \inf_{h\in\Gamma} \max_{t\in[0,1]} J_d(h(t))$$

where Γ is the class of all continuous paths connecting 0 and e in $H_1(\Omega)$ (here e is an arbitrary but fixed positive function with $J_d(te) \leq 0$, for all $t \geq 1$, in $H_1(\Omega)$), is a positive critical value of J_d, and thus gives a positive solution u_d of (1.1). Since (1.1) always has a constant solution, namely, $u \equiv 1$, and we know that it is the only solution of (1.1) if d is sufficiently large, we conclude easily that the "Mountain-Pass solution" $u_d \equiv 1$ for d large. To show that $u_d \not\equiv 1$ for d small, we first observe that $J_d(1) = (\frac{1}{2} - \frac{1}{p+1})|\Omega|$ which is independent of d. Then we estimate the value c_d, <u>we show that</u>

$$c_d \sim d^{n/2} \qquad \text{(III.4)}$$

<u>for</u> $d > 0$ <u>small</u>. This is the first crucial estimate in the proof.

We now turn to Nehari's approach [Ne]. In $H_1(\Omega)$ we define

$$g_d(u) = \int_\Omega [d|\nabla u|^2 + u^2 - uf(u)]dx$$

and the "solution manifold"

$$M_d = \{u \in H_1(\Omega) \mid u > 0 \text{ in } \Omega \text{ and } g_d(u) = 0\} .$$

It is routine to check that g_d is a C^1-functional.

Theorem III.C. The number

$$m_d = \operatorname*{Inf}_{u \in M_d} J_d(u) \tag{III.5}$$

is assumed on M_d (i.e. a global minimizer exists) and the minimizer is a solution of (III.1). In fact $m_d = J_d(u_d) = c_d$, i.e. the "Mountain-Pass solution" u_d is a global minimizer on M_d.

The proof consists of several steps.

Lemma 1. Suppose that m_d defined in (III.5) is assumed at $v \in M$, then $J_d'(v) = 0$.

Proof. It is easy to see that there exists $\mu \in \mathbb{R}$ such that

$$J_d'(v)\varphi = \mu g_d'(v)\varphi \quad \text{for all} \quad \varphi \in H_1(\Omega) ,$$

that is, we have, for all $\varphi \in H_1(\Omega)$

$$\int_\Omega [d\nabla v \cdot \nabla\varphi + v\varphi - f(v)\varphi] = \mu \int_\Omega [2(\nabla v \cdot \nabla\varphi + v\varphi) - (f(v)\varphi + vf'(v)\varphi)]$$

Choosing $\varphi = v$, we obtain

$$0 = \int_\Omega [d|\nabla v|^2+v^2-vf(v)] = \mu\int_\Omega [2(|\nabla v|^2+v^2)-(vf(v)+v^2f'(v))]$$
$$= \mu \int_\Omega [2vf(v)-vf(v)-v^2f'(v)]$$
$$= \mu\int_\Omega v^2[\frac{f(v)}{v} - f'(v)] < 0$$

since $v \in M$ and f is convex in $\mathbb{R}^+$. Thus $\mu = 0$.

<u>Lemma 2.</u> <u>For each</u> $\varphi > 0$ <u>in</u> Ω, <u>the function</u> $\alpha(t) = J_d(t\varphi)$, $t \geq 0$, <u>has a unique positive maximum</u>; <u>and</u>, $\alpha(t_o) = \max_{t\geq 0} \alpha(t)$ <u>if and only if</u> $t_o\varphi \in M_d$. <u>Furthermore, every such ray</u> $\{t\varphi \mid t > 0\}$ <u>intersects</u> M_d <u>at exactly one point</u>.

<u>Proof</u>.
$$\alpha(t) = \frac{t^2}{2} \int_\Omega (d|\nabla\varphi|^2 + \varphi^2) - \int_\Omega F(t\varphi) .$$

$$\alpha'(t) = t \int_\Omega (d|\nabla\varphi|^2 + \varphi^2) - \int_\Omega \varphi f(t\varphi) .$$

Thus, $\alpha'(t_o) = 0$ if and only if

$$\text{(III.6)} \qquad \int_\Omega [d|\nabla(t_o\varphi)|^2 + (t_o\varphi)^2] = \int_\Omega t_o\varphi f(t_o\varphi)$$

i.e. $g(t_o\varphi) = 0$. That is, $\alpha'(t_o) = 0$ is equivalent to $t_o\varphi \in M_d$. Next, we observe that such a t_o is unique. For, if we divide both sides of (III.6) by t_o^2 , we have

$$\int_\Omega (d|\nabla\varphi|^2 + \varphi^2) = \int_\Omega \varphi^2 \frac{f(t_o\varphi)}{t_o\varphi} .$$

and the right-hand side is strictly increasing in $t_o > 0$ (since $f(t)/t$ is strictly increasing in $t > 0$) while the left-hand side is a constant. Since it is clear that $\alpha(t) > 0$ for $t > 0$ small, and that $\alpha(t) \to -\infty$ as $t \to \infty$, our assertions follow.

<u>Lemma 3</u>. <u>The critical value</u> c_d <u>defined by</u> (III.3) <u>is independent of the choice of</u> e (<u>as long as</u> $e > 0$ <u>in</u> Ω <u>and</u> $J_d(te) \leq 0$ <u>for all</u> $t \geq 1$) . <u>Furthermore</u>, <u>there exists a path</u> $h \in \Gamma$ <u>such that</u>

$$c_d = \max_{t\in[0,1]} J_d(h(t)) .$$

<u>Proof</u>. Let e_1, e_2 both be positive in Ω and $J_d(te_i) \leq 0$ for all $t \geq 0$, and $i = 1,2$. Define
$\Gamma_i = \{h \in C([0,1],H_1(\Omega)) \mid h(0) = 0 , h(1) = e_i\}$, $i = 1,2$
and

$$c_d^i = \inf_{h\in\Gamma_i} \max_{t\in[0,1]} J_d(h(t)) , \ i = 1,2 .$$

Consider the set $V^+ = \{a_1e_1 + a_2e_2 \mid a_1,a_2 \geq 0\}$ and the two-dimensional subspace V of $H_1(\Omega)$ spanned by e_1,e_2. Let S be a circle on V with radius R so large that $R > \max\{\|e_1\|,\|e_2\|\}$ and $J_d \leq 0$ on $S \cap V^+$ (this follows from the fact that for any $0 \neq e$ in V^+ $J(te) < 0$ for sufficiently large $t > 0$ and a standard compactness argument) where $\|\cdot\|$ denotes the norm in $H_1(\Omega)$. Now, let γ be an arbitrary path in Γ_1 connecting 0 and e_1. Define $\bar{\gamma}$ to be the path that consists of γ, the line segment with endpoints e_1 and $Re_1/\|e_1\|$, the circular arc $S \cap V^+$, and the line segment with endpoints $Re_2/\|e_2\|$ and e_2. Then it is easy to see that $\bar{\gamma} \in \Gamma_2$, and since J_d is positive on $\{u\in H_1(\Omega) \mid \|u\| = \delta\}$ for small $\delta > 0$, it follows that

$$\max_{u\in\gamma} J_d(u) = \max_{u\in\bar{\gamma}} J_d(u) .$$

This implies $c_d^1 \geq c_d^2$. Similar arguments show that $c_d^2 \geq c_d^1$, and our first assertion is established. From the above arguments, it is clear that we can now choose e to be Tu_d where u_d is a "mountain-pass solution" (thus $u_d > 0$ by the maximum principle, see [LNT] for a detailed proof) and

T is chosen so that $J_d(tu_d) \le 0$ for all $t \ge T$. And, the "minimizing" path γ may be chosen to be the line segment with endpoints 0 and Tu_d.

Lemma 4. $c_d = J_d(u_d) = m_d$.

Proof. Since u_d is a solution of (III.1), $u_d \in M_d$. Thus $m_d \le c_d = J_d(u_d)$. Suppose that $m_d < c_d$. Then there exists $v \in M_d$ such that $J_d(v) < c_d$. By Lemma 2 above $J_d(v) = \max_{t>0} J_d(tv)$. By Lemma 3 we may choose e to be Tv where $T > 0$ is so large that $J_d(tv) \le 0$ for all $t \ge T$. Thus, choosing γ to be the line segment connecting 0 and Tv, we obtain

$$c_d \le \max_{u\in\gamma} J_d(u) = \max_{t\ge 0} J_d(tv) = J_d(v) < c_d ,$$

a contradiction. This completes the proof of Theorem III.C.

Remark. The arguments used in Lemma 3 above were first used by Ding and Ni [DN1; p. 288, Proposition 2.14].

Using Theorem III.C, we can give an easy and conceptually transparent proof of part (iv) of Theorem III.B. The first inequality $\lambda_1(u_d) < 0$ is easy to see. To prove that

$\lambda_2(u_d) \geq 0$, we use the following variational characterization of $\lambda_2(u_d)$:

$$\lambda_2(u_d) = \sup_{w\in H_1(\Omega)} \inf \left\{ \frac{\int_\Omega [d|\nabla v|^2+v^2-v^2f'(u_d)]}{\int_\Omega v^2} \,\middle|\, 0\neq v\in H_1(\Omega),\ \int_\Omega vw = 0 \right\}$$

Since $J_d(u_d)$ is the global minimum of J_d on M_d , it is natural to choose w to be the normal of M_d at u_d which is $g_d'(u_d)$ since M_d is a level surface of the functional g_d . Let T be the tangent space of M_d at u_d , i.e. $T = \{\varphi \in H_1(\Omega) \mid g_d'(u_d)\varphi = 0\}$, then for a fixed $\varphi \in T$, setting $\beta(t) = J_d(u_d+t\varphi)$ for $t \geq 0$, we have $\beta'(0) = 0$ and $\beta''(0) \geq 0$ since J_d attains its minimum on M_d at u_d . We now compute,

$$\beta(t) = \frac{1}{2}\int_\Omega [d|\nabla(u_d+t\varphi)|^2 + (u_d+t\varphi)^2] - \int_\Omega F(u_d+t\varphi) \ .$$

$$\beta'(t) = \int_\Omega [d\nabla(u_d+t\varphi)\cdot\nabla\varphi + (u_d+t\varphi)\varphi] - \int_\Omega f(u_d+t\varphi)\varphi \ .$$

$$\beta''(t) = \int_\Omega (d|\nabla\varphi|^2 + \varphi^2) - \int_\Omega \varphi^2 f'(u_d+t\varphi) \ .$$

Thus

$$0 \leq \beta''(0) = \int_\Omega (d|\nabla\varphi|^2 + \varphi^2) - \int_\Omega \varphi^2 f'(u_d) .$$

Since $\varphi \in T$ is arbitrary, we conclude that

$$\inf\left\{\frac{\int_\Omega [d|\nabla\varphi|^2+\varphi^2-f'(u_d)\varphi^2]}{\int_\Omega \varphi^2} \;\middle|\; 0 \neq \varphi \in T\right\} \geq 0 .$$

This implies that $\lambda_2(u_d) \geq 0$.

The element $w = g_d'(u_d)$ may be computed as follows.

$$\begin{aligned} g_d'(u_d)\varphi &= \int_\Omega \{2(d\nabla u_d \cdot \nabla\varphi + u_d\varphi) - [f(u_d)\varphi + u_d f'(u_d)\varphi]\} \\ &= \int_\Omega [2f(u_d)\varphi - f(u_d)\varphi - u_d f'(u_d)\varphi] \\ &= \int_\Omega [f(u_d)-u_d f'(u_d)]\varphi \end{aligned}$$

where the second equality follows from the fact that $\beta'(0) = 0$. Thus $w = f(u_d)-u_d f'(u_d)$.

Part (iv) is a useful result, for instance, part (v) is an easy consequence of it. However, we shall not repeat the proof here.

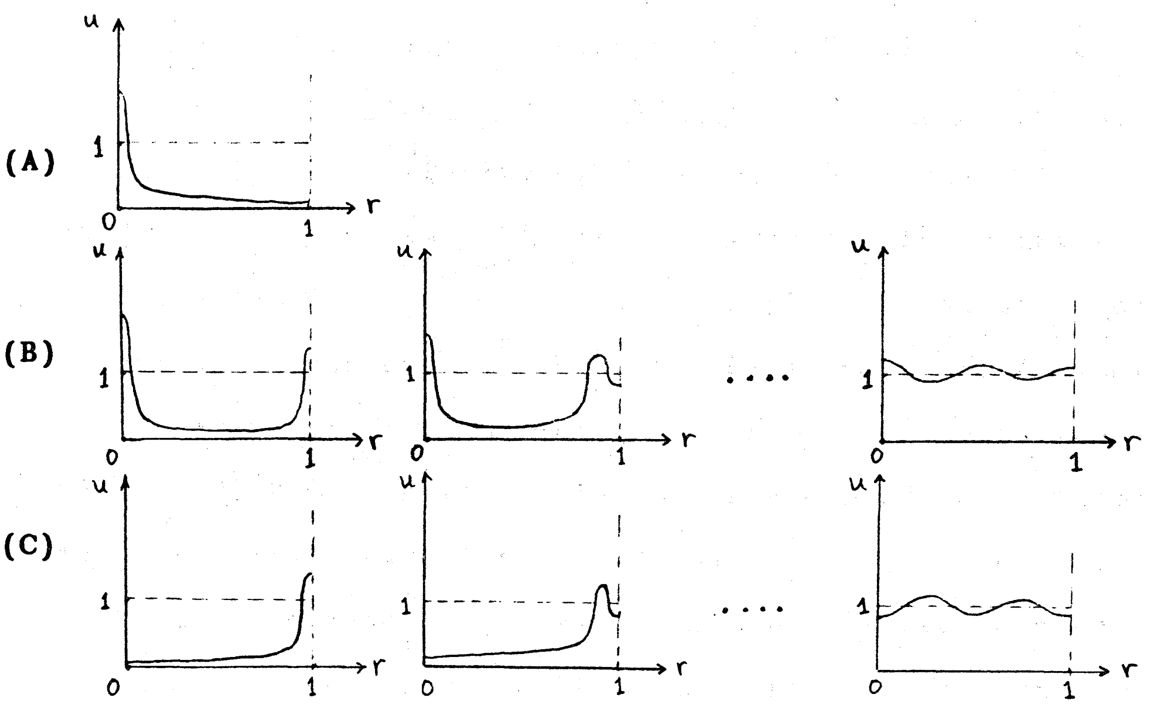
Remarks

(1) It is easy to see, from integrating the equation (III.1) directly, that the set $\{x \in \Omega \mid u_d(x) > 1\}$ is non-empty for d small (since $u_d \not\equiv 1$ for d small). This, together with (i) in Theorem III.B, imply that *the* C^1 *norm of* u_d *cannot possibly be bounded independent of* d, *for* d *small*. Nevertheless the L^∞ norm of u_d is bounded independent of d as is guaranteed by (ii). This indicates that u_d should have "peaks" of finite amplitude when d is sufficiently small and thus exhibits "spiky" pattern.

(2) Since $p > 1$ in (III.1), no nontrivial solution of (III.1) can be stable (although *we do expect* the Gierer-Meinhardt system and Keller-Segal system to have stable nontrivial solutions). In particular, u_d is unstable (which is equivalent to $\lambda_1(u_d) < 0$, guaranteed by (iv)). However, Part (iv) of Theorem III.B says that $\lambda_2(u_d) \geq 0$ which, in some sense, seems to suggest *that* u_d, *although unstable*, *is the* "*most stable*" *nontrivial solution of* (III.1). This implies that the unstable manifold of u_d is of dimension one only. Since there is one conserved quantity in the (parabolic) Keller-Segal system, u_d *seems to give rise to a stationary solution of the Keller-Segal system which is likely to be stable*.

(3) The estimate given by (v) in Theorem III.B is, in general, best possible. This follows from Theorem 4, p. 218 in [T].

(4) (vi) is actually very useful in determining the "shape" of u_d in the radial case. First we remark that <u>Theorem III.B holds true without any change if</u> Ω <u>is a ball and if we restrict ourselves to the class of radial functions</u>. If we examine all the possible radial solutions of (III.1) in a ball, it is not hard to see that they may be categorized as follows (see [N4], [LnN]):



Note that <u>only</u> (A) <u>exhibits a spiky pattern</u>, and the others exhibit either boundary layer phenomena or combinations of spikes and boundary layers. <u>It is easy to see from Part</u> (vi) <u>of Theorem III.B that</u> u_d <u>in this case must be</u> (A).

However, if we do not restrict ourselves to only radial functions in case Ω is a ball, the solution u_d of (III.1) guaranteed by Mountain-Pass Lemma (corresponding to the critical value c_d) which minimizes the "generalized energy" J_d , <u>must be non-radial if</u> d <u>is sufficiently small</u>. This indicates a <u>fundamental</u> difference between Neumann problems and Dirichlet problems. (Recall that a (any) solution of the Dirichlet problem (III.2) must be radial ([GNN1]).) This result may be found in a current joint project of Ni and Takagi which will appear soon.

(5) <u>In case</u> Ω <u>is an annulus</u>, <u>we claim that</u> u_d is <u>also non-radial if</u> d <u>is sufficiently small</u>. For, again if we restrict ourselves to radial functions in $H_1(\Omega)$, then the same arguments in proving Theorem III.B may be carried through without change except now we have $n = 1$. In particular, the estimate (III.4) now reads $c_d^* \sim d^{1/2}$, where c_d^* is the critical value of J_d given by the Mountain-Pass Lemma when restricted to the class of radial functions in $H_1(\Omega)$. Thus, for d small, $c_d \neq c_d^*$ and the corresponding critical points must also be different. Therefore u_d cannot be radial. Notice that this observation applies equally well to the Dirichlet problem (III.2). However, the existence of non-radial solutions to (III.2) (i.e. the Dirichlet problem), in case Ω is an annulus, was established earlier by C.V. Coffman [C].

(6) In the "super-critical" case $p > (n+2)/(n-2)$, our progress is rather limited. However, we do know that in the radial case (Ω is either a ball or an annulus) (III.1)

possesses a nontrivial radial solution if d is sufficiently small, and that (III.1) has no nonconstant radial solution if d is sufficiently large. This part seems to agree with the "sub-critical" case $p < (n+2)/(n-2)$. We would also like to give some partial results just to indicate the difference between these two cases.

Let Ω be the unit ball. After a change of scale, a radial solution of (III.1) satisfies

$$\begin{cases} u'' + \frac{n-1}{r} u' - u + u^p = 0 \ , \ p > (n+2)/(n-2) \ , \\ u'(0) = u'(1/\sqrt{d}) = 0 \ . \end{cases}$$

It is not difficult to show that <u>there exists a positive constant</u> α, <u>independent of</u> $d > 0$, <u>such that</u>

$$\inf_{\Omega} u \geq \alpha$$

<u>for all radial solutions</u> u <u>of</u> (III.1) <u>with</u> $u(0) > 1$. This marks a <u>basic</u> difference between the behavior of solutions of these two cases $p < n^*$ and $p > n^*$. <u>It eliminates the possibility of the existence of a radial spiky solution which approaches zero in measure as</u> d <u>approaches zero in the super-critical case</u> $p > n^*$.

The critical case $p = (n+2)/(n-2)$ is a bit more delicate. Some of our methods do carry over to this case; however, we shall not discuss this case here.

(7) Equation (III.1) may be viewed as a singular perturbation problem when d is sufficiently small. Methods and techniques developed in that field could be helpful here in locating the spikes of a particular solution of (III.1) for general domain Ω. However, we are not able to do this using singular perturbation techniques, even in the radial case (when Ω is the unit ball) which we already know from Theorem III.B (vi) (see Remark (4) above) that (III.1) possesses a solution which has only one spike and it is located at the origin.

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