Notes on Imperfect Repair by Fumio OHI Osaka University

0. Introduction

A component with failure time distribution \( F(t) = 1 - e^{-\Lambda(t)} \) is repaired at failure. Each repair results in minimal repair or perfect repair. Let \( N \) be a positive integer valued random variable denoting the number of repairs performed till the perfect repair, i.e., \( N = k \) means the event that \( k-1 \) minimal repairs were performed and the repair for the \( k \)-th failure was perfect and the component returned to the good-as-new-state. After the perfect repair the process is renewed. Time for repair is assumed to be negligible.

The dynamic process of the component is governed by a non-homogeneous Poisson process \( \{N(t), t \geq 0\} \) with mean value function \( \Lambda(t) \). Then \( T_N \) means the time that the component returns to the good-as-new-state, where \( T_k = \inf \{t | N(t) = k\} \) \( k \geq 1 \), the time that the \( k \)-th failure occurs supposed that the repairs for the previous \( k-1 \) failures were minimal.

In this paper we study monotonic properties of \( T_N \) and other stochastic quantities, e.g., steady-state-distributions and so on. Our results may be of interest in renewal theory as well as in reliability theory.

1. Preliminaries

\( \{N(t), t \geq 0\} \) : a non-homogeneous Poisson Process with differentiable mean value function \( \Lambda(t) \),

\[
\Pr[N(t) = k] = e^{-\Lambda(t)} \frac{(\Lambda(t))^k}{k!},
\]

\[
\lambda(t) = \frac{d}{dt} \Lambda(t),
\]
Theorem 1. For $k \geq 0$, $\ell \geq 1$,

1. $\Pr[T_{k+\ell} - T_k > x | T_k = y] = \Pr[N(x+y) - N(y) < y],$

2. $\Pr[T_{k+\ell} - T_k > x] = \int_0^\infty \Pr[N(x+y) - N(y) < \ell]d\Pr[T_k \leq y],$

3. $E[T_{k+\ell} - T_k | T_k = y] = \int_0^\infty \Pr[N(x+y) - N(y) < \ell]dx,$

4. $E[T_{k+\ell} - T_k] = \int_0^\infty \int_0^\infty \Pr[N(x+y) - N(y) < \ell]dx d\Pr[T_k \leq y].$ \hspace{1cm} □

Corollary 2. Letting $\ell = 1$ in the previous theorem, for $k \geq 1$,

1. $\Pr[T_{k+1} - T_k > x | T_k = y] = e^{-\Lambda(x+y)} - e^{-\Lambda(y)} = \frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}},$

2. $\Pr[T_{k+1} - T_k > x] = \int_0^\infty \frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} d\Pr[T_k \leq y],$

3. $E[T_{k+1} - T_k | T_k = y] = \int_0^\infty \frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} dx,$

4. $E[T_{k+1} - T_k] = \int_0^\infty \int_0^\infty \frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} dx d\Pr[T_k \leq y].$ \hspace{1cm} □

In the sequel we use the following lemmas, of which proof is easy and then omitted.

Lemma 3.

Let $g(x) \uparrow$, $g(x) \geq 0$ for $x \geq 0$, $f(x) \uparrow$, $f(x) \geq 0$ for $x \geq 0$. If two distribution functions $F_1$ and $F_2$ satisfy $F_1(0^-) = F_2(0^-) = 0$ and $F_1(x) \leq F_2(x)$ for $x \geq 0$, then

$$\int_0^\infty g(x) dF_1(x) \leq \int_0^\infty g(x) dF_2(x) \quad \text{and} \quad \int_0^\infty f(x) dF_1(x) \geq \int_0^\infty f(x) dF_2(x),$$

supposing that the integrations finitely exist. \hspace{1cm} □

Lemma 4. For $\lambda \geq \mu$, $\sum_{k=0}^n e^{-\lambda} \frac{\lambda^k}{k!} \leq \sum_{k=0}^n e^{-\mu} \frac{\mu^k}{k!}$ holds for $\forall n \geq 0$. \hspace{1cm} □

Theorem 5. For $k \geq 0$, $\ell \geq 1$,

1. $1-e^{-(t)} : IFR \Rightarrow \Pr[T_{k+\ell} - T_k > x] \uparrow_k$ for $\forall x \geq 0$.
(2) \(1-e^{-A(t)}: \text{DMRL} \Rightarrow E[T_{k+\ell}+\ell-T_k] \leq k\),

(3) \(1-e^{-A(t)}: \text{NBU} \Rightarrow \Pr[T_{k+\ell}+\ell-T_k > x] \leq \Pr[T_{\ell} > x] \text{ for } \forall x \geq 0\),

(4) \(1-e^{-A(t)}: \text{NBUE} \Rightarrow E[T_{k+\ell}+\ell-T_k] \leq \ell E[T_1]\).

**Proof.** We notice that \(T_k \uparrow k\) a.s.

(1) \(1-e^{-A(t)}: \text{IFR} \iff A(x+y)-A(y) \tau_y \Rightarrow \Pr[N(x+y)-N(y)<\ell] \uparrow y\) (by Lemma 4)

\(\Rightarrow \Pr[T_{k+\ell}+\ell-T_k > x] \uparrow k\) (by Theorem 1 (2) and Lemma 3).

(2) \(E[T_{k+\ell}+\ell-T_k] = \sum_{i=1}^{\ell-1} E[T_{k+i}+1-T_{k+i}]\) is decreasing in \(k\), since each term of the right hand side is decreasing in \(k\) by Corollary 2 and Lemma 3.

(3) By Lemma 4, \(\Pr[N(x+y)-N(y)<\ell] \leq \Pr[N(x)<\ell] = \Pr[T_{\ell} > x]\). Then (3) is obvious.

(4) \(E[T_{k+\ell}+\ell-T_k] = \sum_{i=1}^{\ell-1} E[T_{k+i}+1-T_{k+i}] \leq E[T_1]\) by the assumption and Corollary 2 (4). Then we have the inequality.

\(\Box\)

2. Monotonic Properties of \(T_k\)

Let \(N\) be a positive integer valued r.v. independent with \([N(t), t \geq 0]\). In this section we study monotonic properties of \(T_N\).

**Theorem 6.** (1) Suppose that \(Z_i (i \geq 1)\) are i.i.d. r.v.'s with common distribution function same to the one of \(T_1\), and are independent with \(N\).

(1) \(1-e^{-A(t)}: \text{NBU} \Rightarrow \Pr[T_N > t] \leq \Pr[\sum_{i=1}^{N} Z_i > t]\).

(2) \(1-e^{-A(t)}: \text{NBUE} \Rightarrow E[T_N] \leq E[N] E[T_1]\).

**Proof.** (2) is obvious from Theorem 5 (4).

(1) It is sufficient to prove \(\Pr[T_k > t] \leq \Pr[Z_1 + \ldots + Z_k > t]\) for \(k \geq 1\). We prove the inequality by the mathematical induction on \(k\).

\[
\Pr[T_{k+1} > t] = \int \Pr[T_{k+1} > t | T_k = x] dPr[T_k = x] \\
= \int \Pr[T_{k+1} - T_k > t-x | T_k = x] dPr[T_k = x] \\
\leq \int \Pr[Z_{k+1} > t-x] dPr[T_k = x] \quad \text{(by Corollary 2 (1) and th}]
\]
assumption) \[ \leq \int Pr[Z_{k+1} > t-x] dPr[Z_1 + \ldots + Z_k \leq x] \] (by the inductive assumption and Lemma 3) \[ = Pr[Z_1 + \ldots + Z_{k+1} > t]. \]

3. Stochastic Comparisons of $T_N$ and $T_{N'}$

Let \( \tilde{F}_N(t) = Pr[T_N > t] = \sum_{k=0}^{\infty} Pr[N(t) = k] Pr[N > k] \),

\[ f_N(t) = \frac{d}{dt} Pr[T_N \leq t] = \sum_{k=0}^{\infty} Pr[N(t) = k] \lambda(t) Pr[N = k+1], \]

\[ \lambda_N(t) = \frac{f_N(t)}{F_N(t)} . \]

In this section $N$ and $N'$ are positive integer valued r.v.'s independent with $(N(t), t \geq 0)$

**Theorem 7.** (1) \[ \frac{Pr[N = \ell]}{Pr[N = k]} \leq \frac{Pr[N' = \ell]}{Pr[N' = k]} \text{ for } k < \ell \Rightarrow \frac{f_N(x + \Delta)}{f_N(x)} \leq \frac{f_{N'}(x + \Delta)}{f_{N'}(x)} \]

for $x \geq 0$ and $\Delta \geq 0$.

(2) \[ \frac{Pr[N = k+1]}{Pr[N > k]} \leq \frac{Pr[N' = k+1]}{Pr[N' > k]} \text{ for } k \geq 1 \Leftrightarrow \frac{Pr[N \geq \ell]}{Pr[N \geq k]} \geq \frac{Pr[N' \geq \ell]}{Pr[N' \geq k]} \text{ for } k < \ell \]

\[ \Rightarrow \frac{\tilde{F}_N(t + \Delta)}{\tilde{F}_N(t)} \geq \frac{\tilde{F}_{N'}(t + \Delta)}{\tilde{F}_{N'}(t)} \text{ for } t \geq 0, \Delta \geq 0 \Leftrightarrow \lambda_N(t) \leq \lambda_{N'}(t) \text{ for } t \geq 0. \]

(3) \[ Pr[N \geq \ell] \geq Pr[N' \geq \ell] \text{ for } \ell \geq 1 \Rightarrow \tilde{F}_N(t) \geq \tilde{F}_{N'}(t) \text{ for } t \geq 0 . \]

**Proof.** (1) 

\[
\frac{\sum_{k=0}^{\infty} Pr[N(x + \Delta) = k] \lambda(x + \Delta) Pr[N = k+1]}{\sum_{k=0}^{\infty} Pr[N(x) = k] \lambda(x) Pr[N = k+1]} \leq \frac{\sum_{k=0}^{\infty} Pr[N(x + \Delta) = k] \lambda(x + \Delta) Pr[N' = k+1]}{\sum_{k=0}^{\infty} Pr[N(x) = k] \lambda(x) Pr[N' = k+1]}
\]

(2) The equivalent relations of (2) is obvious.

\[
\frac{\tilde{F}_N(t + \Delta)}{\tilde{F}_{N'}(t + \Delta)} \leq \frac{\sum_{k=0}^{\infty} Pr[N(t + \Delta) = k] Pr[N(t + \Delta) = \ell]}{\sum_{k=0}^{\infty} Pr[N(t) = k] Pr[N(t) = \ell]} \Rightarrow \frac{Pr[N > k]}{Pr[N' > k]} \leq \frac{Pr[N > \ell]}{Pr[N' > \ell]}
\]
The relation (3) is easily proved by using Lemma 3. □

We present simple bounds for the distribution and the expectation of $T_N$.

**Corollary 8.** Let $q_m = \inf_k \Pr[N=k+1|N=k]$, $q_M = \sup_k \Pr[N=k+1|N=k]$, and $N_m$ and $N_M$ be positive integer valued r.v.'s independent with \( N(t), t \geq 0 \) such that $\Pr[N_m = k] = q_m^{k-1}(1-q_m)$, $\Pr[N_M = k] = q_M^{k-1}(1-q_M)$. Since $\Pr[N_m > k] \leq \Pr[N > k] \leq \Pr[N_M > k]$ for $k \geq 1$, by Theorem 7 we have $\Pr[T_{N_m} > t] \leq \Pr[T_N > t] \leq \Pr[T_{N_M} > t]$ for $t \geq 0$ and $E[T_{N_m}] \leq E[T_N] \leq E[T_{N_M}]$. □

**Remark 9.** Theorem 7 (1) (2) (3) show that stochastically-larger-relations between $N$ and $N'$ are preserved to the same stochastic relations between $T_N$ and $T_{N'}$, without any assumption on $1-e^{-\Lambda(t)}$. □

**Theorem 10.** $1-e^{-\Lambda(t)}: \text{DMRL, } \frac{\sum_{k=j}^{\infty} \Pr[N \geq k]}{E[N]} \leq \frac{\sum_{k=j}^{\infty} \Pr[N' \geq k]}{E[N]}$ for $j \geq 1$.

Proof. Since $1-e^{-\Lambda(t)}$ is DMRL, $E[T_{j-j-1}]$ is decreasing in $j$ by Theorem 5 (2). Then using Lemma 3,

$E[T_{N_m}] = \sum_{j=1}^{\infty} E[T_{j-j-1}] \Pr[N \geq j] \leq \sum_{j=1}^{\infty} E[T_{j-j-1}] \Pr[N' \geq j] = \frac{E[T_{N_m}]}{E[N']}$. □

**Remark 11.** The following relation holds.

$\Pr[N \geq t] \geq \Pr[N' \geq t]$ for $k \leq t$ $\Rightarrow$ $\Pr[N \geq t] \geq \frac{\Pr[N \geq t]}{\Pr[N \geq k]}$ for $k \leq t$.

$\frac{\sum_{k=j}^{\infty} \Pr[N \geq k]}{E[N]} \geq \frac{\sum_{k=j}^{\infty} \Pr[N' \geq k]}{E[N']}$. (1)

$\Pr[N \geq k] \geq \Pr[N' \geq k]$ for $k \geq 1$. (2)

There is generally no relation between (1) and (2). □

**Remark 12.** It is easily verified that if $\Pr[N \leq 2] = \Pr[N' \leq 2] = 1$, $\Pr[N = 2] \geq \Pr[N' = 2]$ and $1-e^{-\Lambda(t)}$ is NBU, then $E[T_N] \leq E[T_{N'}]$. □
Lemma 13. For $a_1 \geq a_2 \geq \ldots \geq a_1^+ + \ldots + a_n^+ n \geq \frac{a_1^+ + \ldots + a_{n+1}^+}{n+1}$ holds for $n \geq 1$. The proof is easy and omitted. □

Theorem 14. $1 - e^{-\Lambda(t)}$ is DMRL and $Pr[N=k] \geq Pr[N'=k]$ for $k \geq 1$

\[ \Rightarrow E[\frac{T_N}{N}] \leq E[\frac{T_{N'}}{N'}]. \]

Proof. Since $1 - e^{-\Lambda(t)}$ is DMRL, $E[T_{j+T_{j-1}}]$ is decreasing in $j$ by Theorem 6 (2). Then by Lemma 13, $E[\frac{T_{k+T_{k}}}{}$ is decreasing in $k$. Then Theorem 14 is obvious by Lemma 3. □

4. Stochastic Comparisons of Steady-State-Distributions

$\{N_j^j(t), t \geq 0\} (j \geq 1)$: independent non-homogeneous Poisson processes with common mean value function $\Lambda(t)$, $T_k^j = \inf \{t | N_j^j(t) = k\}$

$N_j (j \geq 1)$: independent positive integer valued r.v.'s with common distribution same to the one of $N$,

i.e., $\{N_j^j(t), t \geq 0\} (j \geq 1)$ are replicas of $\{N(t), t \geq 0\}$ and $N_j (j \geq 1)$ are replicas of $N$. We assume that $\{N_j^j(t), t \geq 0\} (j \geq 1)$ are independent with $N_j (j \geq 1)$.

We define a counting process $\{M(t), t \geq 0\}$ as $M(t) = \sum_{j=1}^{n-1} N_j^j + N^n(t - \sum_{j=1}^{n-1} T_{N_j^j})$ if $\sum_{j=1}^{n-1} T_{N_j^j}^j \leq t \leq \sum_{j=1}^{n} T_{N_j^j}^j$.

We notice that $T_{N_j}^j (j \geq 1)$ are i.i.d. random variables with common distribution function $F_N(t)$, $1 - F_N(t) = \sum_k^\infty Pr[N(t) = k]Pr[N > k]$. In this section we consider stochastic quantities with respect to $\{M(t), t \geq 0\}$, which means the number of repairs performed in $[0,t]$.

Let's define $Z(t) = \sum_{j=1}^{n} T_{N_j}^j - (t - \sum_{j=1}^{n-1} T_{N_j}^j)$ if $\sum_{j=1}^{n-1} T_{N_j}^j \leq t \leq \sum_{j=1}^{n} T_{N_j}^j$.
which means the time to the next failure from the time epoch t.

**Theorem 15.**

\[
\lim_{t \to \infty} \Pr[ Z(t) > x ] = \int_0^\infty \frac{F_N(y) \cdot e^{-\Lambda(x+y)}}{E[T_N]} e^{-\Lambda(y)} \, dy.
\]

**Proof.** Simple calculation verifies the above equality. Since

\[
\Pr[ Z(t) > x ] = \int_0^\infty \Pr[ T_N(x-y+1) > t-y+x, t-y < T_N ] \, dy \quad \text{for } 0 \leq x < T_N,
\]

where \( F_N^{(n-1)} \) is the \((n-1)\)-hold convolution of \( F_N \), then by the basic renewal theory we have

\[
\lim_{t \to \infty} \Pr[ Z(t) > x ] = \frac{1}{E[T_N]} \int_0^\infty \Pr[ T_N(y+1) > y+x, y < T_N ] \, dy.
\]

Noticing that \( \Pr[ T_N(y+1) > y+x, y < T_N ] = F_N(y) \Pr[ N(y+x) - N(y) = 0 ] \), the theorem is proved. \( \square \)

We write the steady-state-distribution of \( Z(t) \) as \( H_N \), i.e.,

\[
H_N(x) = 1 - H_N(x) = \int_0^\infty \frac{F_N(y) \cdot e^{-\Lambda(x+y)}}{E[T_N]} e^{-\Lambda(y)} \, dy.
\]

**Theorem 16.** (1) The density function \( h_N(x) \) and the failure rate function \( r_N(x) \) of \( H_N(x) \) are

\[
h_N(x) = \int_0^\infty \frac{F_N(y) \cdot e^{-\Lambda(x+y)}}{E[T_N]} e^{-\Lambda(y)} \cdot \lambda(x+y) \, dy,
\]

\[
r_N(x) = \frac{\int_0^\infty F_N(y) \cdot e^{-\Lambda(x+y)} \cdot \lambda(x+y) \, dy}{\int_0^\infty F_N(y) \cdot e^{-\Lambda(y)} \, dy}.
\]

(2) \( \frac{Pr[N > t]}{Pr[N > k]} \leq \frac{Pr[N' > t]}{Pr[N' > k]} \) for \( k < t \), \( 1 - e^{-\Lambda(t)} \) has PF-density

\[
\Rightarrow \frac{h_N(t+\Delta)}{h_N(t)} \geq \frac{h_N'(t+\Delta)}{h_N'(t)} \quad \text{for } t > 0, \Delta > 0.
\]

(3) \( \frac{Pr[N > t]}{Pr[N > k]} \leq \frac{Pr[N' > t]}{Pr[N' > k]} \) for \( k < t \), \( 1 - e^{-\Lambda(t)} \) : IFR

\[
\Rightarrow r_N(x) \leq r_N'(x) \quad \text{for } \forall x > 0 \quad \Rightarrow H_N(x) \leq H_N'(x) \quad \text{for } \forall x > 0.
\]
Proof. Differentiating $H_N(x)$, (1) is easily obtained.

\[
\begin{vmatrix}
\int_0^\infty F_N(y)e^{-\Lambda(x+y)}\frac{\lambda(x+y)}{e^{-\Lambda(y)}}\,dy & \int_0^\infty F_N(y)e^{-\Lambda(x+y)}\frac{\lambda(x+y)}{e^{-\Lambda(y)}}\,dy \\
\int_0^\infty F_N(y)e^{-\Lambda(x+y)}\frac{\lambda(x+y)}{e^{-\Lambda(y)}}\,dy & \int_0^\infty F_N(y)e^{-\Lambda(x+y)}\frac{\lambda(x+y)}{e^{-\Lambda(y)}}\,dy
\end{vmatrix}
\]

\[
= \int_{y_1}^{y_2} \int_{F_N(y_1)}^{F_N(y_2)} \left[ \frac{e^{-\Lambda(x+y_1)}}{e^{-\Lambda(y_1)}} \cdot \lambda(x+y_1) \right] \left[ \frac{e^{-\Lambda(x+y_2)}}{e^{-\Lambda(y_2)}} \cdot \lambda(x+y_2) \right] \,dx 
\]

\[
\geq 0.
\]

Using Basic Composition Theorem, (3) is proved similarly to the proof of (2). \qed

We define

\[
Z^*_t(t) = \sum_{j=1}^n T_{N_j} - t \quad \text{if} \quad \sum_{j=1}^{n-1} T_{N_j} \leq t < \sum_{j=1}^n T_{N_j}
\]

which means the time to the next perfect repair from the time epoch $t$.

It is well known that

\[
\lim_{t \to \infty} \Pr[Z^*_t(t) \leq x] = \frac{1}{E[T_N]} \int_0^x F_N(u) \,du.
\]

We write the right hand side of the above equality as $H^*_N(x)$.

**Theorem 17.** \begin{align*}
\Pr[N > \ell] & \leq \Pr[N' > \ell] \quad \text{for } k < \ell \\
\Rightarrow \frac{H^*_N(t+\Delta)}{H^*_N(t)} & \leq \frac{H^*_{N'}(t+\Delta)}{H^*_{N'}(t)} \quad \text{for } t > 0, \Delta > 0 \Rightarrow H^*_N(t) \geq H^*_{N'}(t) \quad \text{for } t > 0.
\end{align*}

**Proof.**

\[
\begin{aligned}
\int_{t+\Delta}^{\infty} F_N(u) \,du & \geq \int_{t+\Delta}^{\infty} F_{N'}(u) \,du \\
\int_{t}^{\infty} F_N(u) \,du & \geq \int_{t}^{\infty} F_{N'}(u) \,du
\end{aligned}
\]

\[
\Rightarrow \int_{t}^{\infty} F_N(u) \,du \cdot \int_{t+\Delta}^{\infty} F_{N'}(u) \,du \leq 0.
\]

Noticing that $\frac{\lambda(x+y)}{e^{-\Lambda(y)}} \,dy = H^*_N(x)$, we have by Theorem 17 and Lemma 3:
Theorem 18. \[ \frac{\Pr[N > t]}{\Pr[N > k]} \leq \frac{\Pr[N' > t]}{\Pr[N' > k]} \quad \text{for} \quad k \leq t, \quad 1 - e^{-A(t)} : \text{DMRL} \]
\[ \Rightarrow \int_0^\infty R_N(x) \, dx \geq \int_0^\infty R_{N'}(x) \, dx. \]

References.