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An envelope appearing in semi-infinite programming
半無限計画法に現れる包絡線

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We gave in [7] second-order necessary optimality conditions for an abstract optimization problem with a generalized inequality constraint. Our condition involves an extra term besides the second derivative of the ordinary Lagrange function. Moreover, we showed that an envelope may be formed by the generalized inequality constraint and that the extra term is closely related the second-order directional derivative of the envelope. The aim of this talk is to give a necessary optimality condition for a semi-infinite programming problem by applying the result of [7] to this problem. We will, moreover, show that the strengthened condition of the necessary condition becomes a sufficient condition under some additional assumptions.

1. Introduction. We deal with the following semi-infinite programming problem:

(SIP) minimize $f(x)$ subject to $g(x,t) \leqq 0 \quad \forall t \in T$,

where T is a compact set in a metric space, f and g are functions defined on $\mathbb{R}^n$ and $\mathbb{R}^n \times T$, respectively. Throughout this paper, we assume that f is of C^2-class and that g, $\partial g/\partial x$ and $\partial^2 g/\partial x^2$ are continuous on $\mathbb{R}^n \times T$.

When one is concerned with second-order optimality conditions for (SIP), one must take account of two important facts. One is that <u>an envelope may be formed by the infinitely many inequality constraints $g(x,t) \leqq 0 \ \forall t$</u>. The other is that <u>there is a gap between the second derivatives of the envelope and</u>

g(x,t). For example, though the second derivatives of strait lines are zero, they may form an envelope with positive second derivative. Though some investigators were aware of this gap, see e.g. Ben-Tal and et al [1, p.23], they could not bridge the gap. When an envelope is formed by the constraints, usual second-order necessary optimality conditions for (SIP) have given no information, see e.g. Ben-Tal and et al [1], Ben-Tal and Zowe [2], Fiacco and Kortanek [3], Hettich [4], Hettich and Jongen [5], Ioffe [6], Lempio and Zowe [11], Linnemann [12] and Shapiro [16].

The purpose of this talk is to give second-order necessary and sufficient optimality conditions for (SIP) which agree with the above-mentioned facts.

In this talk, we treat (SIP) as a minimization problem in the space of continuous functions $C(T)$. Indeed, (SIP) is equivalent to

$$\text{minimize} \quad f(x) \quad \text{subject to} \ -G(x) \in C_+(T), \tag{1.1}$$

where G is a mapping from $\mathbb{R}^n$ to $C(T)$ defined by $G(x)(t) := g(x,t)$ and $C_+(T) := \{u \in C(T);\ u(t) \geqq 0 \ \forall t\}$. Then the mapping G is of C^2-class and its first and second derivatives are given by $(G'(x)y)(t) = (\partial G(x,t)/\partial x)y$, $(G''(x)(y,y))(t) = y^T(\partial^2 G(x,t)/\partial x^2)y$, respectively.

We gave in [7] second-order necessary optimality conditions for an abstract optimization problem including (1.1). Our necessary condition for (SIP) is obtained by applying the results of [7] to (1.1). We express the conditions in a familiar form with finite number of Lagrange multipliers.

The canonical pairing between V and its topological dual V^* is denoted by $\langle \cdot,\cdot \rangle$. The first and second Frèchet derivative of g at x are denoted by $g'(x)$ and $g''(x)$, respectively. Moreover, $g''(x)(y,z)$ denotes the corresponding bilinear mapping from X^2 to V. The adjoint operator of $g'(x)$ is denoted by $g'(x)^*$, that is, $\langle g'(x)^* v^*, y\rangle = \langle v^*, g'(x)y\rangle$. Let A be a subset of X. Then the conical hull and convex hull of A are denoted by $\mathrm{cone}A$ and $\mathrm{co}A$, respectively. The closure of $\mathrm{cone}A$ is denoted by $\mathrm{cl}(\mathrm{cone}A)$. The support function of A is defined by $\delta^*(x^* \mid A) := \sup\{\langle x^*, x\rangle;\ x \in A\}$. The polar cone of K is defined by $K^o := \{v^* \in V^*;\ \langle v^*, v\rangle \leqq 0\ \ \forall v \in K\}$.

2. Preliminary results. Let us consider the following abstract optimization problem with a generalized inequality constraint:

(P) minimize $f(x)$ subject to $g(x) \in K$,

where X and V are Banach spaces, $f: X \to \mathbb{R}$ and $g: X \to V$ are of C^2-class and K is a closed convex cone with non-empty interior in V. This problem was studied in [7].

Definition 2.1. ([7]) For any $u, v \in V$,

$$K(u,v) := \{w \in V;\ \exists\, o(1) \in V \text{ s.t. } s^2 w + sv + u + o(1) \in K\ \ \forall s > 0\},$$

where $o(1)$ is an arbitrary element of V satisfying $o(1) \to 0$ as $s \to +0$. In particular, we denote $K(g(x^*), g'(x^*)y)$ by $K(y)$.

Definition 2.2. ([13,14,17]) The system $g(x) \in K$ is said to be regular at x^* if there exists $z \in X$ such that $g(x^*) + g'(x^*)z \in \mathrm{int}K$.

Definition 2.3. ([7]) A vector $y \in X$ is called a critical direction at x^* if both $f'(x^*)y = 0$ and $g'(x^*)y \in \mathrm{cl}(\mathrm{cone}(K - g(x^*)))$ hold.

Theorem 2.1. ([7,Theorem 5.1]) Let x^* be a local minimum solution of (P). Suppose that the constraint system is regular at x^*. Then, for each critical direction y satisfying $K(y) \neq \phi$, there exists $v^* \in K^o$ such that

$$f'(x^*) + g'(x^*)^* v^* = 0, \tag{2.1}$$

$$f''(x^*)(y,y) + \langle v^*, g''(x^*)(y,y)\rangle - 2\delta^*(v^* \mid K(y)) \geqq 0, \tag{2.2}$$

$$\langle v^*, g(x^*)\rangle = 0, \quad \langle v^*, g'(x^*)y\rangle = 0. \tag{2.3}$$

The following theorem characterizes $K(u,v)$ when $K = C_+(K)$, see [8,9].

Theorem 2.2. Let $u \in C_+(T)$ and v satisfy that $v(t) \geqq 0$ whenever $u(t) = 0$. Let T_0 denote the set of all $t \in T$ for which there exists a sequence $\{t_n\} \subset T$ satisfying (2.4) below

$$u(t_n) > 0, \; t_n \to t \text{ and } -v(t_n)/u(t_n) \to +\infty \text{ as } n \to +\infty. \tag{2.4}$$

Then $w \in K(u,v)$ if and only if $w(t) \geqq E(t)$ for all $t \in T$, where $E(t)$ is defined by

$$\begin{cases} \sup\{\limsup v(t_n)^2/4u(t_n); \{t_n\} \text{ satisfies } (2.4)\}, \text{ if } t \in T_0, & (2.5a) \\ 0, \text{ if } u(t) = v(t) = 0 \text{ and } t \notin T_0, & (2.5b) \\ -\infty, \text{ otherwise.} & (2.5c) \end{cases}$$

Definition 2.4. Let $S: \mathbb{R}^n \to \mathbb{R}$ be an arbitrary function. Then the directional derivative of $S(x)$ at x in the direction y is denoted by $S'(x;y)$. The second-order directional derivative of $S(x)$ at x in the direction y is defined by

$$S''(x;y) := \lim_{s \to +0} \{S(x+sy) - S(x) - sS'(x;y)\}/s^2, \tag{2.6}$$

if the limit exists (we admit the value $\pm\infty$). When the limit does not exist, the upper and lower limits are denoted by $\overline{S}''(x;y)$ and $\underline{S}''(x;y)$, respectively.

Theorem 2.3. ([8,Theorem 2.2]) Let x^* be any point of $\mathbb{R}^n$ and let y be any non-zero direction of $\mathbb{R}^n$. Then we have

$$2\overline{S}''(x^*;y) = \max\{y^T(\partial^2 f(x^*,t)/\partial x^2)y + 2E(t;y);\ t\in T\}, \tag{2.7}$$

where $E(t;y)$ is defined via (2.5) by taking

$$u(t) = -g(x^*,t), \quad v(t) = S'(x^*;y) - (\partial g(x^*,t)/\partial x)y. \tag{2.8}$$

3. Second-order necessary conditions for (SIP). Throughout this section, we use the following notations:

$$S(x) := \sup\{g(x,t);\ t\in T\}, \tag{3.1}$$

$$T(x^*) := \{t\in T;\ g(x^*,t) = 0\}, \tag{3.2}$$

$$T(x^*;y) := \{t\in T(x^*);\ (\partial g(x^*,t)/\partial x)y = 0\}, \tag{3.3}$$

$$u(t) := -g(x^*,t), \quad v(t) := -(\partial g(x^*,t)/\partial x)y, \tag{3.4}$$

$E(t;y)$ is defined via (2.5) by using the above u, v. (3.5)

We use $v(t;y)$ instead of $v(t)$, when we emphasize that $v(t)$ depends on y. We may assume that $T(x^*)\neq\phi$ without loss of generality, because (SIP) results in a minimization problem with no constraint if $T(x^*)=\phi$. Moreover, we note that $T(x^*;y)\neq\phi$ for all y whenever $T(x^*)\neq\phi$. We shall first characterize the critical directions for (SIP).

Lemma 3.1 (<u>Critical direction</u>). Let x^* be a feasible point for (SIP). Then $y \in \mathbb{R}^n$ is critical at x^* if and only if both $f'(x^*)y = 0$ and $S'(x^*;y) \leqq 0$ hold.

Lemma 3.2 (<u>Regularity Condition</u>). The system $g(x,t) \leqq 0 \ \forall t \in T$ is regular at x^* if and only if there exists $z \in \mathbb{R}^n$ such that

$$(\partial g(x^*,t)/\partial x)z < 0 \quad \text{for all} \quad t \in T(x^*). \tag{3.6}$$

Lemma 3.3. Let x^* be a feasible solution for (SIP) and let y be an arbitrary critical direction. Then $K(y) \neq \phi$ is equivalent to that either $S'(x^*;y) < 0$ or $\bar{S}''(x^*;y) < +\infty$ holds.

Theorem 3.1. (<u>Necessary condition</u>) Let x^* be a local minimum solution for (SIP). Assume that (3.6) holds for some $z \in \mathbb{R}^n$. Then, for each critical direction y satisfying either $S'(x^*;y) < 0$ or $\bar{S}''(x^*;y) < +\infty$, there exist at most $n+1$ points $t_1, \cdots, t_k \in T(x^*;y)$ and $\lambda_1 \geqq 0, \cdots, \lambda_k \geqq 0$, $k \leqq n+1$, such that

$$\partial f(x^*)/\partial x_j + \sum_{i=1}^{k} \lambda_i \partial g(x^*,t_i)/\partial x_j = 0 \quad \forall j = 1, \cdots, n,$$

$$y^T f''(x^*)y + \sum_{i=1}^{k} \lambda_i \{y^T(\partial^2 g(x^*,t_i)/\partial x^2)y + 2E(t_i;y)\} \geqq 0.$$

4. Second-order sufficient condition for (SIP). First we note that we will also use the notations (3.1)–(3.5) as well as in the previous section. In this section, we shall show that the strengthened condition of the necessary condition in the previous section becomes sufficient for optimality under some additional assumptions. Roughly speaking, Assumption 4.2 below requires that $g(x^*,t)$ and $(\partial g(x^*,t)/\partial x)y$ do not vibrate with respect to t on neighborhoods of their common zero points.

Assumption 4.1. T is a compact convex subset of $\mathbb{R}^r$.

Assumption 4.2. Let y be an arbitrary critical direction. Then, for each $\tau \in T(x^*;y)$ and $d \in \mathbb{R}^r$, there exist α, $\beta \neq 0$ and natural numbers p, q such that the functions u, v are expanded into Taylor series as follows:

$$u(\tau + \varepsilon d + o(\varepsilon)) = \alpha \varepsilon^p + o(\varepsilon^p),$$
$$v(\tau + \varepsilon d + o(\varepsilon)) = \beta \varepsilon^q + o(\varepsilon^q),$$

whenever $\tau + \varepsilon d + o(\varepsilon) \in T$ and $\varepsilon > 0$.

Theorem 4.1. (<u>Sufficient condition</u>) Under Assumptions 4.1 and 4.2, a feasible point $x^* \in \mathbb{R}^n$ is an isolated local minimum for (SIP) if the following conditions are satisfied: (i) there exist $a \geqq 0$, $\rho_1, \cdots, \rho_a \in T(x^*;y)$ and $\mu_1 > 0, \cdots\cdots, \mu_a > 0$ such that

$$\partial f(x^*)/\partial x_j + \Sigma_i \mu_i \partial g(x^*, \rho_i)/\partial x_j = 0, \quad j=1,\cdots,n, \tag{4.1}$$

(ii) $$\max\{E(t;y);\ t \in T\} < +\infty \quad \forall y \in \mathbb{R}^n, \tag{4.2}$$

(iii) for each critical direction $y \neq 0$, there exist $b \geqq 0$, $\tau_1, \cdots, \tau_b \in T(x^*;y)$ and $\lambda_1 > 0, \cdots, \lambda_b > 0$ such that

$$\partial f(x^*)/\partial x_j + \Sigma_i \lambda_i \partial g(x^*, \tau_i)/\partial x_j = 0, \quad j= 1,\cdots,n, \tag{4.3}$$

$$y^T f''(x^*)y + \Sigma_i \lambda_i \{y^T(\partial^2 f(x^*, \tau_i)/\partial x^2)y + 2E(\tau_i;y)\} > 0. \tag{4.4}$$

5. Concluding remark. Let x^* be a local minimum solution of (SIP). Then, it seems to the author that it has long been believed that, for each direction y satisfying

$$(\partial f(x^*)/\partial x)y = 0, \quad \max\{(\partial g(x^*,t)/\partial x)y;\ t \in T(x^*)\} = 0, \tag{5.1}$$

the second-order directional derivative of the ordinary Lagrange function would be nonnegative for suitable multipliers. Strictly speaking, there would exist finite number of multipliers $\lambda_1 \geqq 0, \cdots, \lambda_q \geqq 0$ and $\tau_1, \cdots, \tau_q \in T(x^*;y)$ such that

$$\partial f(x^*)/\partial x_j + \sum_{k=1}^{q} \lambda_k \partial g(x^*, \tau_k)/\partial x_j = 0 \qquad j=1,\cdots,n, \tag{5.2a}$$

$$y^T(\partial^2 f(x^*)/\partial x^2)y + \sum_{k=1}^{q} \lambda_k y^T(\partial^2 g(x^*, \tau_k)/\partial x^2)y \geqq 0. \tag{5.2b}$$

But this expectation is not necessarily true. When an envelope is formed from the infinitely many inequality constraints, the second-order directional derivative of the ordinary Lagrange function may be negative for some critical direction y, no matter how the multipliers are chosen. Furthermore, it is easily verified by the separation theorem that the existence of $\{\lambda_i\}$ and $\{\tau_i\}$ satisfying (5.2) is equivalent to the inconsistency of the following inequality system:

$$(\partial f(x^*)/\partial x)z + y^T(\partial^2 f(x^*)/\partial x^2)y < 0, \tag{5.3a}$$

$$(\partial g(x^*,t)/\partial x)z + y^T(\partial^2 g(x^*,t)/\partial x^2)y \leqq 0, \ \forall t \in T(x^*;y). \tag{5.3b}$$

Hence the inequality system (5.3) may has a solution z, when an envelope is formed.

Ben-Tal and Zowe [2,Theorem 11.1], Hettich and Jongen [5,Theorem 4.2], Ioffe [6,Theorem 4], Lempio and Zowe [11,Theorem 3.2] and Shapiro [16,Theorem 5.2] gave sufficient optimality conditions for semi-infinite programming problems. But they missed the envelope-like effect, so that their conditions seem to be too strong to be satisfied.

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