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The period map of a 4-parameter family of K3 surfaces and the Aomoto-Gel'fand hypergeometric function of type (3,6)

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We show that one of Aomoto-Gel'fand hypergeometric functions ([G-G]) can be interpreted as the period map of a 4-dimensional family of K3 sufaces, of which target is the 4-dimensional Hermitian symmetric bounded domain of type IV. The corresponding system of differential equations has six linearly independent solutions which are quadratically related. (Such systems are recently studied in [SY3].) This fact confers an algebro-geometric decoration to Aomoto-Gel'fand functions as the relation between the elliptic modular function and the corresponding equation does to the Gauss hypergeometric function. Details will be given in [MSY].

We describe a family of K3 surfaces. Let

$$\ell_j = \{(t^1,t^2,t^3)\in \mathbb{C}P^2 \mid v_{1j}t^1 + v_{2j}t^2 + v_{3j}t^3 = 0\}$$

$(0 \leq j \leq 6)$ be six lines in general position in the complex projective plane $\mathbb{C}P^2$ with homogeneous coordinates (t^1,t^2,t^3) and let $S(\ell)$ be the minimal smooth model of the two-fold cover $S'(\ell)$ of $\mathbb{C}P^2$ branching along the line configuration $\ell = \{\ell_1,\ldots,\ell_6\}$. For a fixed ℓ, the surface $S(\ell)$ is a K3 surface, i.e. There is a unique holomorphic 2-form

$$(1)\quad \eta(\ell) = \Pi_{j=1}^{6}(v_{1j}s^1 + v_{2j}s^2 + v_{3j})^{-1/2}ds^1\wedge ds^2,$$

up to constant multiplication, and the rank of the second homology group $H_2(S(\ell),\mathbb{Z})$ is 22. In this case, there are 16 linearly independent algebraic cycles; 15 exceptional curves coming from the 15 double points of $S'(\ell)$ and a section when considered $S(\ell)$ as an elliptic surface over $\mathbb{C}P^1$. We can take a system $\gamma_1'(\ell),\ldots,\gamma_6'(\ell) \in H_2(S(\ell),\mathbb{Z})$ of six (transcendental) cycles orthogonal to the algebraic cycles such that there exists another system $\gamma_1(\ell),\ldots,\gamma_6(\ell) \in H_2(S(\ell),\mathbb{Z})$ which is dual to γ_j' $(1 \leq j \leq 6)$ i.e. $\gamma_i'\cdot\gamma_j = \delta_{ij}$ (Kronocker's symbol) and that its intersection matrix $(\gamma_i'\cdot\gamma_j')$ $(1 \leq i,j \leq 6)$ takes the fixed form $I = (I_{ij})$, which is symmetric, integral and with signature $(2+,4-)$.

The vector $\omega(\ell) = (\omega_1(\ell),\ldots,\omega_6(\ell))$

$$\omega_j(\ell) = \int_{\gamma_j(\ell)}\eta(\ell) \qquad (1 \leq j \leq 6)$$

is called the period of $S(\ell)$ and it satisfies the Riemann relation and the Riemann inequality as follows

$$(2) \quad \Sigma_{i,j} I_{ij}\omega_i(\ell)\omega_j(\ell) = 0$$

$$(3) \quad \Sigma_{i,j} I_{ij}\omega_i(\ell)\bar{\omega}_j(\ell) > 0.$$

Now we let ℓ vary in the space M of configurations of six projective lines and let the cycles $\gamma_j(\ell)$ depend continuously on ℓ. Then the correspondence sending ℓ to the ratio $\omega_1(\ell):\cdots:\omega_6(\ell)$ gives a multi-valued map

$$\varphi: M' \to Q \subset \mathbb{C}P^5$$

where Q is a quadratic hypersurface defined as follows

$$Q = \{(z^1,\ldots,z^6) \in \mathbb{C}P^5 \mid \Sigma_{i,j} I_{ij} z^i z^j = 0\}.$$

The multi-valuedness of φ is expressed by a subgroup of

$$\Gamma = \{X \in GL(6,\mathbb{Z}) \mid {}^tXIX = I\}/\{\pm 1\}.$$

If we express elements of the space M by a 3 by 6 matrix (v_{ij}) $(1 \leq i \leq 3,\ 1 \leq j \leq 6)$ then the map φ is invariant under the action of $SL(3,\mathbb{C})$ on the left and the action $H = (\mathbb{C}^\times)^6$ from the right.

Therefore φ is defined on the 4-dimensional factor space $X = SL(3,6)\backslash M/H$. Let us choose, for example, a system of local coordinates $(x^1,\dots,x^4)$ of X as follows:

$$(v_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x^1 & x^2 \\ 0 & 0 & 1 & 1 & x^3 & x^4 \end{pmatrix}$$

i.e.

$$(4) \quad \ell_1 = \{t^1 = 0\},\ \ell_2 = \{t^2 = 0\},\ \ell_3 = \{t^3 = 0\},\ \ell_4 = \{t^1+t^2+t^3 = 0\},$$
$$\ell_5 = \{t^1+x^1t^2+x^3t^3 = 0\},\ \ell_6 = \{t^1+x^2t^2+x^4t^3 = 0\}.$$

Then by the theory developed in [SY3], there is a system of linear differential equation

$$(5) \quad \frac{\partial^2 u}{\partial x^i \partial x^j} = g_{ij}\frac{\partial^2 u}{\partial x^1 \partial x^4} + \Sigma_{k=1}^{3} a_{ij}^k \frac{\partial u}{\partial x^k} + a_{ij}^0 u, \quad 1 \leq i,j \leq 6$$

of rank (= dimension of the solution space) six such that the ratio of a system of linearly independent solution (called a projective solution) is exactly φ, that the quadratic form $g = \Sigma_{i,j} g_{ij} dx^i dx^j$ is conformal to the pull back of the canonical <u>flat</u> conformal structure on $Q \subset \mathbb{C}P^5$ and that other coefficients a_{ij}^k and a_{ij}^0 $(1 \leq i,j,k \leq 6)$ are determined by g.

Here we recall the framework of the Aomoto-Gel'fand hypergeometric differential equation associated with the Grassmannian $G_{3,6}$ and see that our system (5) is a special case of these.

Let $M(k,n)$ be the set of k by n matrices $v = (v_{ij})$ and consider the integral

$$(6) \quad \Phi(v) = \int_\Delta \Pi_{j=1}^{n} (\Sigma_{i=1}^{k} v_{ij} t^i)^{\alpha_j - 1} dt$$

where Δ is a region of the (k-1)-dimensional sphere $S^{k-1} \subset \mathbb{R}^k$ and dt is the induced measure of the standard measure of the Euclidean space $\mathbb{R}^k$ onto S^{k-1} and $\Sigma_{j=1}^{n} \alpha_j = n - k$. The function $\Phi(v)$ is

invariant under the action of $SL(k,\mathbb{C})$ from the left and the action of $H_n = (\mathbb{C}^\times)^n$ from the right so that it satisfies

$$(7)\quad \Sigma_{i=1}^{k} v_{ij} \frac{\partial}{\partial v_{ij}} \Phi = (\alpha_j - 1)\Phi \qquad (H_n\text{- invariance})$$

for $1 \leqq j \leqq n$ and

$$(8)\quad \Sigma_{j=1}^{n} v_{ij} \frac{\partial}{\partial v_{kj}} \Phi = -\delta_i^k \Phi \qquad (SL(k,\mathbb{C})\text{-invariance})$$

for $1 \leqq i,k \leqq k$. Important equalities are

$$(9)\quad \frac{\partial^2}{\partial v_{ip}\partial v_{jq}} \Phi = \frac{\partial^2}{\partial v_{iq}\partial v_{jp}} \Phi$$

for $1 \leqq i,j \leqq k$, $1 \leqq p,q \leqq n$. The system of linear differential equations (7),(8) and (9) is called <u>the Aomoto-Gel'fand hypergeometric equation of type</u> (k,n) ([Aom] and [Gel]) and denoted by $E(k,n;\alpha_1,\ldots,\alpha_n)$. It is a holonomic system, that is, its rank is finite. The systems $E(2,4;\alpha_1,\ldots,\alpha_4)$ and $E(2,n;\alpha_1,\ldots,\alpha_n)$ $(n \geqq 5)$ can be naturally considered to be the Gauss hypergeometric equation and the Appell-Lauricella hypergeometric system F_D in $n-3$ variables, respectively.

The system $E(3,6;\alpha_1,\ldots,\alpha_6)$ reduces to the following system with unknown u if one uses the independent variables $x^1,\ldots,x^4$ appeared in the normalization (4).

$$\begin{aligned}
(\alpha_2+\alpha_3+\alpha_4-1+\theta_1+\theta_2+\theta_3+\theta_4)\theta_1 u &= x^1(\theta_1+\theta_3+1-\alpha_5)(\theta_1+\theta_2+\alpha_2)u \\
(\alpha_2+\alpha_3+\alpha_4-1+\theta_1+\theta_2+\theta_3+\theta_4)\theta_2 u &= x^2(\theta_2+\theta_4+1-\alpha_6)(\theta_1+\theta_2+\alpha_2)u \\
(\alpha_2+\alpha_3+\alpha_4-1+\theta_1+\theta_2+\theta_3+\theta_4)\theta_3 u &= x^3(\theta_1+\theta_3+1-\alpha_5)(\theta_1+\theta_2+\alpha_3)u \\
(\alpha_2+\alpha_3+\alpha_4-1+\theta_1+\theta_2+\theta_3+\theta_4)\theta_4 u &= x^4(\theta_1+\theta_3+1-\alpha_6)(\theta_1+\theta_2+\alpha_3)u \\
x^1(\theta_1+\theta_3+1-\alpha_5)\theta_2 u &= x^2(\theta_2+\theta_4+1-\alpha_6)\theta_1 u \\
x^3(\theta_1+\theta_3+1-\alpha_5)\theta_4 u &= x^4(\theta_2+\theta_4+1-\alpha_6)\theta_3 u \\
x^1(\theta_1+\theta_2+\alpha_2)\theta_3 u &= x^3(\theta_3+\theta_4+\alpha_3)\theta_1 u \\
x^2(\theta_1+\theta_2+\alpha_2)\theta_4 u &= x^4(\theta_3+\theta_4+\alpha_3)\theta_2 u \\
x^2x^3\theta_1\theta_4 u &= x^1x^4\theta_2\theta_3 u
\end{aligned}$$

where $\theta_i = x^i\partial/\partial x^i$. This system can be written in the form of (5): as follows

$$(10)_\alpha \qquad \frac{\partial^2 u}{\partial x^i \partial x^j} = G_{ij}\frac{\partial^2 u}{\partial x^1 \partial x^4} + \Sigma_{k=1}^{3} A_{ij}^k(\alpha)\frac{\partial u}{\partial x^k} + A_{ij}^0(\alpha)u,$$

$1 \leqq i,j \leqq 6$. The coefficients $G_{ij} = G_{ji}$'s of the principal part are independent of the α_j's and are given as follows:

$$G_{14} = G_{23} = 1,\ g_{12} = \frac{x^4-x^3}{x^1-x^2},\ g_{13} = \frac{x^4-x^2}{x^1-x^3},$$

$$G_{24} = \frac{x^3-x^1}{x^2-x^4},\ G_{34} = \frac{x^2-x^1}{x^3-x^4},$$

$$G_{11} = \frac{x^2x^3-x^4}{x^1(1-x^1)} - \frac{x^3(x^4-x^2)}{x^1(x^1-x^3)} - \frac{x^2(x^4-x^3)}{x^1(x^1-x^2)},$$

$$G_{22} = \frac{x^1x^4-x^3}{x^2(1-x^2)} - \frac{x^1(x^3-x^4)}{x^2(x^2-x^1)} - \frac{x^4(x^3-x^1)}{x^2(x^2-x^4)},$$

$$G_{33} = \frac{x^1x^4-x^2}{x^3(1-x^3)} - \frac{x^1(x^2-x^4)}{x^3(x^3-x^1)} - \frac{x^4(x^2-x^1)}{x^3(x^3-x^4)},$$

$$G_{44} = \frac{x^2x^3-x^1}{x^4(1-x^4)} - \frac{x^3(x^1-x^2)}{x^4(x^4-x^3)} - \frac{x^2(x^1-x^3)}{x^4(x^4-x^2)}.$$

On the other hand, it is clear from the representations (1) and (6) that our system (5) is equivalent to the system $(10)_\alpha$ with $\alpha_j = \frac{1}{2}$ for $1 \leqq j \leqq 6$. Therefore we know that the coefficients g_{ij} of (5) are equal to G_{ij}, so that the quadratic form $g = \Sigma_{i,j}G_{ij}dx^i dx^j$ is <u>conformally flat</u>.

The algebro-geometric interpretation of the system $E(3,6;\frac{1}{2},\ldots,\frac{1}{2})$ given above shows that the system $E(3,6;\alpha_1,\ldots,\alpha_6)$, out of many systems $E(k,n;\alpha_1,\ldots,\alpha_n)$ $(k \geqq 3)$, is a valuable analogy of the Gauss hypergeometric equation that has the fruitful relation with the elliptic modular function. To conclude this paper, we list up the correspondence between our situation of K3 surfaces and the situation of elliptic curves:

- Configuration ℓ of six lines in $\mathbb{C}P^2$ $\longleftrightarrow$ System of four points $p = (p_1,\ldots,p_4)$ on $\mathbb{C}P^1$
- K3 surface $S(\ell)$ $\longleftrightarrow$ Elliptic curve $E(p)$ obtained by the two fold cover of $\mathbb{C}P^1$ branching at p

- Holomorphic 2-form $\eta(\ell)$ on $S(\ell)$ $\longleftrightarrow$ Holomorphic 1-form $\eta(p)$ on $E(p)$
- Transcendental cycles $\gamma_i'(\ell) \in H_2(S(\ell),\mathbb{Z})$ $\longleftrightarrow$ Standard basis $\gamma_1(p)$ and $\gamma_2(p)$ of $H_1(E(p),\mathbb{Z})$
- Intersection form I of the $\gamma_i'(\ell)$'s $\longleftrightarrow$ Intersection form $\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}$ of $\gamma_1(p)$ and $\gamma_2(p)$
- Periods $\omega_i(\ell) = \int_{\gamma_i(\ell)} \eta_i(\ell)$ $\longleftrightarrow$ Periods $\omega_i(p) = \int_{\gamma_i(p)} \eta_i(p)$
- The Riemann inequality (2) $\longleftrightarrow$ $\mathrm{Im}\, \omega_1(p)/\omega_2(p) > 0$
- Period map $\varphi: \ell \to \omega(\ell) \in Q$ $\longleftrightarrow$ Period map $p \to \omega_1(p)/\omega_2(p) \in H = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$
- Group Γ $\longleftrightarrow$ Group $PSL(2,\mathbb{Z})$
- System $E(3,6;\frac{1}{2},\ldots,\frac{1}{2})$ $\longleftrightarrow$ System $E(2,4;\frac{1}{2},\ldots,\frac{1}{2})$
- System (5) under the normalization (4) $\longleftrightarrow$ The hypergeometric differential equation

$$x(1-x)u'' + (1-2x)u' - \frac{1}{4}u = 0$$

under the normalization $p_1 = 0$, $p_1 = \infty$, $p_3 = 1$, $p_4 = x$.

<u>Note</u>: Such correspondence can be also found using various families of curves of higher genera in place of K3 surfaces, which are studied in [D-M], [Ter], [Mat] and [SY3].

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