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Str. of the Moduli Sp. of SL-eq's on a Riem. Surface and Monod. Preserving Deformation.

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§1. SL-eq's on a Riem. surface.

M: cpt Riem. surface of genus $g \geq 0$.

$\xi \in H^1(M, \mathcal{O}^*)$: holo. line b'dle over M.

$\mathcal{U} = \{(U_j, x_j)\}$: a coord. covering of M.

$(\xi_{jk}) \in Z^1(\mathcal{U}, \mathcal{O}^*)$: a repr. 1-cocycle for ξ.

<u>Def.</u> (1) <u>a GL-eq. on $\mathcal{U}$ for ξ</u> is a collection $\star = \{\star_j\}$ of loc. def. diff. eq's

$$\star_j \qquad \frac{d^2 f_j}{dx_j^2} + P_j \frac{df_j}{dx_j} + Q_j f_j = 0 \quad \text{in } U_j \quad (\text{loc. expression})$$

s.t.
$$f_j = \xi_{jk} f_k \quad \text{in } U_j \cap U_k$$
$$\Rightarrow \quad \star_j = \star_k \quad \text{as diff. eq's in } U_j \cap U_k.$$

(2) $\{\underline{\text{GL-eq's on } M \text{ for } \xi}\} := \varinjlim_{\mathcal{U}} \{\text{GL-eq's on } \mathcal{U} \text{ for } \xi\}$

<u>Def.</u> <u>an SL-eq. on M for ξ</u> is a GL-eq. on M for ξ s.t.

$\exists$ repr. GL-eq. $\star$ on $\mathcal{U}$ s.t.

$$P_j \equiv 0 \quad \text{in } U_j \quad (\forall j).$$

<u>Lemma</u> $\xi \in H^1(M, \mathcal{O}^*)$

$\exists$ SL-eq's for $\xi \iff c_1(\xi) = 1 - g$.

From now on, fix $\xi \in H^1(M, \mathcal{O}^*)$ s.t. $c_1(\xi) = 1-g$.

$\theta_{jk} := \{x_j ; x_k\}$ in $U_j \cap U_k$

$(\theta_{jk}) \in Z^1(\mathcal{U}, \mathcal{O}(K^2))$.

$$\mathcal{Q} := \{ Q = (Q_j) \in C^0(\mathcal{U}, \mathcal{M}(K^2)) \;;\; \delta(Q_j) = (\theta_{jk}) \}$$

<u>Lemma</u>

$$\begin{array}{ccc} \{\text{SL-eq's (or op's)}\} & = & \mathcal{Q} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ L = (L_j) & \longleftrightarrow & Q = (Q_j) \end{array}$$

where

$$L_j = -\frac{d^2}{dx_j^2} + Q_j \quad \text{in } U_j .$$

<u>Rem</u> an SL-op. L is considered as a diff. op. $L : \mathcal{M}(\xi) \to \mathcal{M}(\xi \otimes K^2)$.

<u>Rem</u> The sp. $\mathcal{Q}$ parametrizing SL-op's is an $H^0(M, \mathcal{M}(K^2))$-<u>affine</u> sp. However, if a nice (= proj.) coord. cover. $\mathcal{U} = \{(U_j, x_j)\}$ is chosen, then $\mathcal{Q}$ is viewed as a <u>lin.</u> sp. i.e.

Fix a <u>proj. str.</u> subord. to the complex str. of $M \Rightarrow \theta_{jk} \equiv 0$.

$$\begin{array}{ccc} \{\text{SL-op's}\} & = & H^0(M, \mathcal{M}(K^2)) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ L = \left(-\frac{d^2}{dx_j^2} + Q_j\right) & \longleftrightarrow & Q = (Q_j) \end{array}$$

<u>Rem</u> Loc. properties of a GL-eq. (eg. reg. sing. pt. char. exp's. e.t.c.) are def. through its loc. expr.'s.

reg. sing.
- generic $\Leftrightarrow$ $\theta :=$ difference of char. exp's $\notin \mathbb{Z}$
- non-generic $\Updownarrow$ $\theta \in \mathbb{Z}$
  - logarithmic
  - non-logarithmic ($\Rightarrow \theta \in \mathbb{Z}_{\geq 2}$) $=$ apparent

Def. L : a GL-eq. $p \in M$: an app. sing. pt. of L

① p : <u>multiplicity</u> $N \in \mathbb{N} \underset{\text{def}}{\Longleftrightarrow} \theta = N+1$ (= differ. of char. exp's)

p : <u>ground state</u> $\Longleftrightarrow$ mult. $= 1$.

② $p_1, \cdots, p_k \in M$: mutually differ. app. sing. pts of L

$N_1, \cdots, N_k \in \mathbb{N}$: mult's

$\Rightarrow \quad n := \sum_{i=1}^{k} N_i$: the # of app. sing. pts of L <u>counted with multiplicities</u>

L : <u>ground state</u> $\underset{\text{def}}{\Longleftrightarrow}$ p_j : g.s. ($\forall j$).

§2. Analytic sp's of SL-eq's on a Riem. surface (Part 1).

<u>FSL-eq.</u> = Fuchsian SL-eq.

m = # of reg. sing. pts

k = # of app. sing. pts

n = # of app. sing. pts c.w. mult's.

☆ $E(\ell) \underset{\text{def}}{:=} \left\{ \begin{array}{l} \text{FSL-eq's on } M \text{ with exactly } \ell \\ \underline{\text{ordered}} \text{ reg. sing. pts.} \end{array} \right\}$

Study <u>analytic space str's</u> of $E(m+k)$ and its subsp's.

$C(\ell) \underset{\text{def}}{:=} \{ (p_1, \cdots, p_\ell) \in M^\ell \; ; \; \exists i, j \text{ st. } i \neq j, \; p_i = p_j \}$

$B(\ell) := M^\ell \setminus C(\ell)$.

(D.G.1)
$$\begin{tikzcd} & & E(m+k) \arrow[dl, "\pi"'] \arrow[dd, "\varpi"] \\ M^m \times M^k \supset & B(m+k) \arrow[dr, "p"'] & \\ & & B(m) \subset M^m \end{tikzcd}$$

$$\pi: \begin{array}{ccc} E(m+k) & \longrightarrow & B(m+k) \\ \Psi & & \Psi \\ Q & \longmapsto & \text{the ordered reg. sing. pts of } Q \end{array}$$

$$p: \begin{array}{ccc} B(m+k) & \longrightarrow & B(m) \\ \Psi & & \Psi \\ (p_1 \cdots p_m, q_1 \cdots q_k) & \longmapsto & (p_1, \cdots, p_m) \end{array} \quad \begin{array}{l} : \text{surj.} \\ \\ \text{proj. into first } m \text{ factors} \end{array}$$

<u>Complex str. of $E(\ell)$</u>

Given $\mathbb{P} = (p_1, \cdots, p_\ell) \in B(\ell)$, let

$F(\ell)_{\mathbb{P}} := \Gamma(M, \mathcal{O}(K^2 \otimes [2p_1 + \cdots + 2p_\ell]))$

$\dim F(\ell)_{\mathbb{P}} = (2\ell + 3g - 3)^+$: indep. of $\mathbb{P} \in B(\ell)$ ($\because$ Riem. Roch)

where
$$a^+ = \begin{cases} a & (a > 0) \\ 1 & (a = 0) \\ 0 & (a < 0) \end{cases}$$

$$F(\ell) \underset{def}{:=} \coprod_{\mathbb{P} \in B(\ell)} F(\ell)_{\mathbb{P}}$$

<u>Lemma</u> $F(\ell)$ is naturally a complex vect. b'dle over $B(\ell)$ of rank $(2\ell + 3g - 3)^+$ ($\because$ Kodaira - Spencer)

$\pi_{\ell, j} : B(\ell) \to B(\ell - 1)$, $(p_1, \cdots, p_\ell) \mapsto (p_1, \cdots \hat{p}_j \cdots p_\ell)$

<u>Prop.</u> $E(\ell) = F(\ell) \setminus \bigcup_{j=1}^{\ell} \pi_{\ell,j}^{*} F(\ell - 1)$: a complex mfd

$$\dim E(\ell) = (2\ell + 3g - 3)^+ + \ell.$$

☆ $E(m, k) \underset{def}{:=} \left\{ Q \in E(m+k) ;\ \begin{array}{l} \text{the last } k \text{ sing. pts of } Q \\ \text{are app. sing. pts} \end{array} \right\}$

Given $N = (N_1, \cdots, N_k) \in \mathbb{N}^k$, let

☆ $E(m, k; N) \underset{def}{:=} \left\{ Q \in E(m+k) ;\ \begin{array}{l} \text{the } (m+j)^{th} \text{ sing. pt. of } Q \\ \text{is an app. sing. pt of} \\ \text{mult. } N_j \quad (1 \leq j \leq k) \end{array} \right\}$

Rem $E(m.k) = \bigsqcup_{N \in \mathbb{N}^k} E(m,k;N)$

Given $\mathbb{P} \in B(m)$, let

☆ $E(\mathbb{P},k;N) \underset{def}{:=} \{ Q \in E(m,k;N) \ ; \ \varpi(Q) = \mathbb{P} \}$

Given $\theta = (\theta_1, \cdots, \theta_m) \in \mathbb{C}^m$, let

☆ $E(m,k;\theta,N) \underset{def}{:=} \left\{ Q \in E(m,k;N) ; \begin{array}{l} \text{the differ. of char. exp's at} \\ \text{the } j^{th} \text{ sing. pt is } \theta_j \ (1 \leq j \leq m) \end{array} \right\}$

☆ $E(\mathbb{P},k;\theta,N) := E(\mathbb{P},k;N) \cap E(m,k;\theta,N)$

Theorem 1 Let $m \geqq \max(2-g, 0)$, $k \geqq 1$, $N \in \mathbb{N}^k$. Then

(1) $E(m,k;N)$: pure dim. analytic sub sp. of $E(m+k)$

$\dim E(m,k;N) = 3(m+g-1)+k$

(2) $\pi : E(m.k;N) \longrightarrow B(m+k)$: surj.

Cor 1 $E(\mathbb{P},k;N)$: pure dim. analytic sp.

$\dim E(\mathbb{P},k;N) = 2m+k+3g-3$

Cor 2 $E(m,k;\theta,N)$: pure dim. analytic sp.

$\dim E(m,k;\theta,N) = 2m+k+3g-3$

$\pi : E(m,k;\theta,N) \longrightarrow B(m+k)$: surj.

Cor 3 $E(\mathbb{P},k;\theta,N)$: pure dim. analytic sp.

$\dim E(\mathbb{P},k;\theta,N) = m+k+3g-3.$

§3. Analytic sp's of SL-eq's on a Riem. surface (Part 2).

Given $\mathbb{r} = (p_1 \cdots p_m, q_1 \cdots q_k) \in B(m+k)$, $N = (N_1, \cdots, N_k) \in \mathbb{N}^k$, let

$$\eta(\mathbb{r};N) := K^{-1}\otimes [N_1 q_1+\cdots+N_k q_k-(p_1+\cdots+p_m)] \in H^1(M,\mathcal{O}^*).$$

<u>Rem</u> $c_1(\eta(\mathbb{r};N)) = |N|-m+2-2g.$

$$D(m,k;N) := \{\mathbb{r}\in B(m+k)\ ;\ h^0(M,\mathcal{O}(\eta(\mathbb{r},N)))\geqq 1\}$$

$$B(m,k;N) := B(m+k)\setminus D(m,k;N).$$

<u>Rem</u> $|N|\leqq m+2g-3 \Rightarrow D(m,k;N)=\phi.$

<u>Lemma</u> $|N|\leqq m+3g-3$ (∵ Grauert)

$\Rightarrow$ $D(m,k;N)$: a proper analytic subset of $B(m+k)$, i.e.

$B(m,k;N)$: a nonempty analytic Z-open subset of $B(m+k)$.

<u>Lemma</u> $p: B(m,k;N)\to B(m)$: surj.

☆ $\mathbb{E}(m,k;N) \underset{def}{:=} \{Q\in E(m,k;N)\ ;\ \pi(Q)\in B(m,k;N)\}$

☆ $\mathbb{E}(\mathbb{P},k;N) \underset{def}{:=} \{Q\in E(\mathbb{P},k;N)\ ;\ \pi(Q)\in B(m,k;N)\}$

☆ $\mathbb{E}(m,k;\theta,N) \underset{def}{:=} \{Q\in E(m,k;\theta,N)\ ;\ \pi(Q)\in B(m,k;N)\}$

☆ $\mathbb{E}(\mathbb{P},k;\theta,N) \underset{def}{:=} \{Q\in E(\mathbb{P},k;\theta,N)\ ;\ \pi(Q)\in B(m,k;N)\}$

$$\begin{tikzcd} & \mathbb{E}(m,k;N) \arrow[dl, "\pi"'] \arrow[dd, "\varpi"] \\ M^m\times M^k\supset B(m,k;N) \arrow[dr, "p"'] & \\ & B(m)\subset M^m \end{tikzcd} \qquad (D.G.\ 2)$$

<u>Theorem 1</u> If $E(\cdots)$ and $B(m+k)$ are replaced by $\mathbb{E}(\cdots)$ and $B(m,k;N)$, resp., then Th & Cor's in § 2 are still O.K.

§4. Reducible and irreducible SL-eq's.

$$\tau_{jk} := \frac{1}{2}\frac{d}{dx_j}\log K_{jk} \quad \text{in } U_j\cap U_k$$

$(\tau_{jk}) \in Z^1(\mathcal{U}, \mathcal{O}(K))$

$\mathcal{P} := \{ P = (P_j) \in C^0(\mathcal{U}, \mathcal{M}(K)) \;;\; \delta(P_j) = (\tau_{jk}) \}$

☆ $V(\ell) \underset{def}{:=} \{ P \in \mathcal{P} \;;\; P$ *has exactly* ℓ *ordered simple poles* $\}$

$A(\ell) := \{ \alpha = (\alpha_1, \cdots, \alpha_\ell) \in \mathbb{C}^\ell \;;\; \alpha_1 + \cdots + \alpha_\ell = g-1,\ \alpha_1 \cdots \alpha_\ell \neq 0 \}$

$\pi : V(\ell) \longrightarrow A(\ell) \times B(\ell), \quad P \longmapsto (\alpha, \mathbb{P})$

$\mathbb{P} = (P_1, \cdots, P_\ell)$: *m-tuple of ordered simple poles of* P

$\alpha = (\alpha_1, \cdots, \alpha_\ell)$: $\quad \alpha_j = \mathrm{Res}\, P$ *at* P_j.

$\Phi : V(\ell) \longrightarrow E(\ell), \quad P = (P_j) \mapsto Q = \left(\frac{dP_j}{dx_j} + P_j^2 \right)$

$$\begin{tikzcd}
V(\ell) \arrow[r, "\Phi"] \arrow[d, "\pi"'] & E(\ell) \arrow[dd, "\pi"] \\
A(\ell) \times B(\ell) \arrow[dr] & \\
& B(\ell)
\end{tikzcd} \qquad (D.G.3)$$

__*Lemma*__ $V(\ell)$ *has a natural* __*complex mfd str.*__ *s.t.*

$\pi : V(\ell) \to A(\ell) \times B(\ell)$ *is a holo. affine b'dle of rk* g.

$\dim V(\ell) = 2\ell + g - 1. \qquad (\because K\text{-}S)$

__*Rem*__ Φ *is a* __*holo*__ *and* __*closed*__ *map.*

__*Lemma*__ $\#(\text{each fiber of } \Phi) \leq 2^\ell$.

Φ *is a* __*finite*__ *holo. map.* (*we can apply* FMT)

__*Def*__ $L = (L_j) \leftrightarrow Q \in E(\ell)$ *is* __*reducible*__

$\Leftrightarrow \quad L_j =$ *a product of first order op's* $(\forall j)$.

☆ $E(\ell)_{red} := \{ Q \in E(\ell) \;;\; Q$ *is reducible* $\}$

$E(\ell)_{irr} := \{ Q \in E(\ell) \;;\; Q$ *is irreducible* $\}$

Given $X \subset E(\ell)$, let

$$X_{red} := X \cap E(\ell)_{red}, \quad X_{irr} := X \cap E(\ell)_{irr}$$

Prop. Let $\ell \geqq \max(1, 2-g)$, then

$$E(\ell)_{red} = \Phi(V(\ell)) : \text{an analytic subsp. of } E(\ell)$$

$$\operatorname{codim} E(\ell)_{red} = \ell + 2g - 2$$

Theorem 1 Let $m \geqq \max(1, 2-g)$, $k \geqq 0$, $N \in \mathbb{N}^k$. Then

$$E(m,k;N)_{red} : \text{an analytic subsp. of } E(m,k;N)$$

$$\operatorname{codim} E(m,k;N)_{red} = m + 2g - 2$$

Cor 1 m, k, N : as above. $\mathbb{P} \in B(m)$.

$$E(\mathbb{P},k;N)_{red} : \text{an analytic subsp. of } E(\mathbb{P},k;N)$$

$$\operatorname{codim} E(\mathbb{P},k;N)_{red} = m + 2g - 2$$

Def $\theta = (\theta_1, \cdots, \theta_m) \in \mathbb{C}^m$, $N = (N_1, \cdots, N_k) \in \mathbb{N}^k$: given.

(θ, N) : generic $\underset{def}{\Longleftrightarrow}$ $\forall\, \delta_i, \varepsilon_j \in \{\pm 1\}$ $(1 \leq i \leq m,\ 1 \leq j \leq k)$

$$\sum_{i=1}^{m} \delta_i \theta_i + \sum_{j=1}^{k} \varepsilon_j (N_j + 1) \neq 2g - 2 - (m+k)$$

Cor 2 m, k, N : as in Th.

(θ, N) : generic $\Longrightarrow$ $E(m,k;\theta,N)_{red} = \phi$

Cor 3 (θ, N) : non-generic

$$E(m,k;\theta,N)_{red} : \text{an analytic subsp. of } E(m,k;\theta,N)$$

$$\operatorname{codim} E(m,k;\theta,N)_{red} \geqq m + 2g - 3$$

Cor 4 (θ, N) : non-generic, $\mathbb{P} \in B(m)$.

$$E(\mathbb{P},k;\theta,N)_{red} : \text{an analytic subsp. of } E(\mathbb{P},k;\theta,N)$$

$$\operatorname{codim} E(\mathbb{P},k;\theta,N)_{red} \geqq m + 2g - 3$$

Rem $E(\cdots)$ can be replaced by $\mathbb{E}(\cdots)$ in the above Th & Cor's.

Rem If $m \geqq \max(1, 3-g)$, then all analytic subsp's app. in the above Th & Cor's have + codim's.

Theorem 2 If $E(\cdots)$ and $\mathbb{E}(\cdots)$ are replaced by $E(\cdots)_{irr}$ and $\mathbb{E}(\cdots)_{irr}$, resp., then Th's 1 in § 2 & § 3 are still O.K.

$$\Lambda(n) := \{(k, N)\ ;\ 0 \leqq k \leqq n,\ N = (N_1, \cdots, N_k) \in \mathbb{N}^k,\ |N| \leqq n\}$$

☆ $\mathbb{E}(m|n)_{irr} = \bigsqcup_{(k,N)\in\Lambda(n)} \mathbb{E}(m, k\,; N)_{irr}$

☆ $\mathbb{E}(\mathbb{P}|n)_{irr} = \bigsqcup_{(k,N)\in\Lambda(n)} \mathbb{E}(\mathbb{P}, k\,; N)_{irr}$

☆ $\mathbb{E}(m|n\,;\theta)_{irr} = \bigsqcup_{(k,N)\in\Lambda(n)} \mathbb{E}(m, k\,; \theta, N)_{irr}$

☆ $\mathbb{E}(\mathbb{P}|n\,;\theta)_{irr} = \bigsqcup_{(k,N)\in\Lambda(n)} \mathbb{E}(\mathbb{P}, k\,; \theta, N)_{irr}$

Rem $\Lambda(n) \ni (n, \mathbb{1}_n)$ corresp. to g.s. eq's, where $\mathbb{1}_n = \overbrace{(1, \cdots, 1)}^{n}$.

§ 5. Complex mfd's of proj. representations & proj. monod. map.

$G = PL(2.\mathbb{C}) = Aut(\mathbb{P}^1)$, $\mathfrak{g} = Lie\ G = sl(2, \mathbb{C})$.

Given $\mathbb{P} = (P_1, \cdots, P_m) \in B(m)$, let $|\mathbb{P}| = \{P_1, \cdots, P_m\}$ and

$[\mathbb{P}]$: the real blow-up of M at $P_1, \cdots, P_m$. (see Fig. 1)

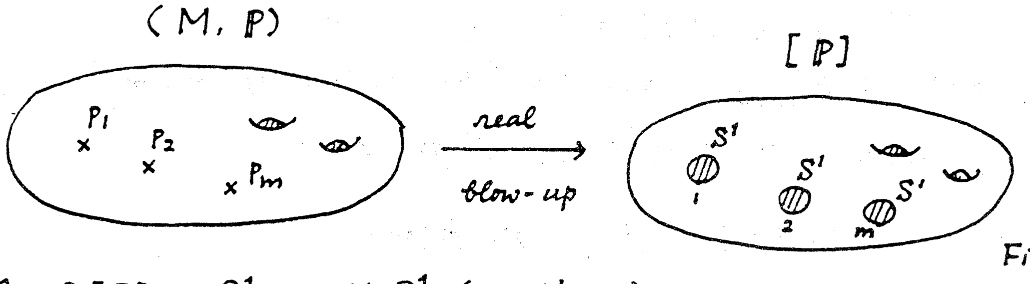

Fig. 1.

Rem $\partial[\mathbb{P}] = S^1 \sqcup \cdots \sqcup S^1$ (m times)

$\hat{R}(\mathbb{P}) := Hom(\pi_1(M \setminus |\mathbb{P}|), G)$, equipped with C-O topology

$\hat{\rho} \in \hat{R}(\mathbb{P})$: irreducible $\underset{def}{\Leftrightarrow}$ $\hat{\rho}(\pi_1) \subset Aut(\mathbb{P}^1)$ has no fixed pt in $\mathbb{P}^1$.

$\hat{R}(\mathbb{P})_{irr} := \{ \hat{\rho} \in \hat{R}(\mathbb{P}) \; ; \; irr. \}$

<u>Rem</u> $\hat{R}(\mathbb{P})$: noncan. ly a <u>complex submfd</u> of the Lie gr. G^{m+2g}.
: admits an action of $Ad(G)$

$\hat{R}(\mathbb{P})_{irr}$: <u>inv.</u> under the act. of $Ad(G)$
nonempty analytic Z-open subset of $\hat{R}(\mathbb{P})$.

Act. of $Ad(G)$ on $\hat{R}(\mathbb{P})_{irr}$ is <u>free</u>. ($\because$ Schur's lemma)

☆ $R(\mathbb{P}) := \hat{R}(\mathbb{P})/Ad(G)$, $R(\mathbb{P})_{irr} := \hat{R}(\mathbb{P})_{irr}/Ad(G)$.

<u>Rem</u>
$$\begin{array}{ccc} R(\mathbb{P}) & = & H^1([\mathbb{P}], G) \\ \Psi & & \Psi \\ \rho & \longleftrightarrow & L(\rho) \end{array}$$
$L(\rho)$: loc. system whose char. repr. is ρ

<u>Prop.</u> $R(\mathbb{P})_{irr}$ carries a natural <u>complex mfd</u> str. s.t.
$\hat{R}(\mathbb{P})_{irr} \xrightarrow{Ad(G)} R(\mathbb{P})_{irr}$ is a holo. principal G-bdle.
$\dim R(\mathbb{P})_{irr} = 3m + 6g - 6$.
$T_\rho R(\mathbb{P})_{irr} \cong H^1([\mathbb{P}]; Ad\, L(\rho))$, $\rho \in R(\mathbb{P})_{irr}$

<u>Rem</u> $R(\mathbb{P})$ is <u>not</u> a complex mfd.

Given $\theta = (\theta_1, \cdots, \theta_m) \in (\mathbb{C} \setminus \mathbb{Z})^m$, let
$$\hat{R}(\mathbb{P};\theta)_{irr} = \left\{ \hat{\rho} \in \hat{R}(\mathbb{P})_{irr} \; ; \; \begin{array}{l} \text{loc. repr. around } P_j \text{ induced by } \rho \\ \text{has eigen-v's } \exp(\pm \pi\sqrt{-1}\,\theta_j) \end{array} \right\}$$

☆ $R(\mathbb{P};\theta)_{irr} := \hat{R}(\mathbb{P};\theta)_{irr} / Ad(G)$.

<u>Prop.</u> $R(\mathbb{P};\theta)_{irr}$ is a complex submfd of $R(\mathbb{P})_{irr}$ s.t.
$\dim R(\mathbb{P};\theta)_{irr} = 2(m + 3g - 3)$.
$$T_\rho R(\mathbb{P};\theta)_{irr} \cong \mathrm{Ker}\left[\begin{array}{l} H^1([\mathbb{P}]; Ad\,L(\rho)) \\ \quad \xrightarrow{j^*} H^1(\partial[\mathbb{P}]; Ad\,L(\rho)|\partial[\mathbb{P}]) \end{array} \right]$$
for $\rho \in R(\mathbb{P};\theta)_{irr}$.

(see the follow. Rem.)

Rem Cohom exact seq.

$$H^0([\mathbb{P}];\mathrm{Ad}L(\rho)) \xrightarrow{j^*} H^0(\partial[\mathbb{P}];\mathrm{Ad}L(\rho)|\partial[\mathbb{P}])$$

$$\underset{0}{\parallel}\ \begin{pmatrix}\text{irr. of } \rho\\ \text{Schur}\end{pmatrix} \qquad\qquad \underset{\mathbb{C}\oplus\cdots\oplus\mathbb{C}\ (m \text{ times})}{\wr\parallel}$$

$$\xrightarrow{\delta^*} H^1([\mathbb{P}],\partial[\mathbb{P}];\mathrm{Ad}L(\rho)) \xrightarrow{i^*} H^1([\mathbb{P}];\mathrm{Ad}L(\rho))$$

$$\xrightarrow{j^*} H^1(\partial[\mathbb{P}];\mathrm{Ad}L(\rho)|\partial[\mathbb{P}])$$

$$T_\rho R(\mathbb{P};\theta)_{irr} \cong \frac{H^1([\mathbb{P}],\partial[\mathbb{P}];\mathrm{Ad}\,L(\rho))}{H^0(\partial[\mathbb{P}];\mathrm{Ad}L(\rho)|\partial[\mathbb{P}])}$$

Def the proj. monod. map PM. Given $\mathbb{P}\in B(m)$, let

$$\begin{array}{rccc} PM: & \mathbb{E}(\mathbb{P}|n)_{irr} & \longrightarrow & R(\mathbb{P})_{irr} \\ & \Psi & & \Psi \\ & Q & \longmapsto & \text{the proj. monod. repr. class for } Q \end{array}$$

$$PM:\ \mathbb{E}(\mathbb{P}|n;\theta)_{irr} \longrightarrow R(\mathbb{P};\theta)_{irr}$$

Rem PM is a holo map.

Theorem 1 $m, k, N\in\mathbb{N}^k$, $\theta\in(\mathbb{C}\setminus\mathbb{Z})^m$: given. $|N|\leqq m+3g-3$

$\Rightarrow$ $\forall \mathfrak{q}\in B(\mathbb{P},k;N)$, $\exists\, U$: open nbd of $\mathfrak{q}$ in $B(\mathbb{P},k;N)$ s.t.

PM is 1 to 1 on $\mathbb{E}(\mathbb{P},k;\theta,N)_{irr}|_\pi U$. (see D.G. 4)

$$\begin{tikzcd} \mathbb{E}(\mathbb{P},k;\theta,N)_{irr}|_\pi U \arrow[d,"\pi"] \arrow[rr, phantom, "\subset"] & & \mathbb{E}(\mathbb{P},k;\theta,N)_{irr} \arrow[dl,"\pi"'] \arrow[dd,"\varpi"] \\ \mathfrak{q}\in U \arrow[r, phantom, "\subset"] & B(\mathbb{P},k;N) \arrow[dr,"p"'] & \\ & & \mathbb{P}\in B(m) \end{tikzcd} \qquad (D.G.4)$$

Theorem 1′ Let $\mathbb{P}\in B(m)$. $n = m+3g-3 \Longrightarrow$

$PM: \mathbb{E}(\mathbb{P}|n)_{irr}\to R(\mathbb{P})_{irr}$, $PM:\mathbb{E}(\mathbb{P}|n;\theta)_{irr}\to R(\mathbb{P};\theta)_{irr}$

are loc. injective holo. map's.

This Th1′ has a lot of implications.

Rem Let $\mathbb{P} \in B(m)$, $n = m+3g-3$. Then, for $(k, N) \in \Lambda(n)$

$$\begin{array}{l} \dim \mathbb{E}(\mathbb{P}, k; N)_{irr} \left\{ \begin{array}{c} < \\ = \end{array} \right\} \dim R(\mathbb{P})_{irr} \\ \dim \mathbb{E}(\mathbb{P}, k; \theta, N)_{irr} \left\{ \begin{array}{c} < \\ = \end{array} \right\} \dim R(\mathbb{P}; \theta)_{irr} \end{array} \quad \text{if } (k, N) \left\{ \begin{array}{c} \neq \\ = \end{array} \right\} (n, \mathbb{1}_n)$$

☆ $\mathbb{E}'(\mathbb{P}|n)_{irr} := \mathbb{E}(\mathbb{P}|n)_{irr} \setminus \mathbb{E}(\mathbb{P}, n; \mathbb{1}_n)_{irr}$

☆ $\mathbb{E}'(\mathbb{P}|n; \theta)_{irr} := \mathbb{E}(\mathbb{P}|n; \theta)_{irr} \setminus \mathbb{E}(\mathbb{P}, n; \theta, \mathbb{1}_n)_{irr}$

Theorem 2 Let $\mathbb{P} \in B(m)$, $n = m+3g-3$.

(i) $PM(\mathbb{E}'(\mathbb{P}|n)_{irr})$ / $PM(\mathbb{E}'(\mathbb{P}|n; \theta)_{irr})$: nowhere dense in $R(\mathbb{P})_{irr}$ / $R(\mathbb{P}; \theta)_{irr}$

(ii) $PM : \mathbb{E}(\mathbb{P}, n; \mathbb{1}_n)_{irr} \longrightarrow R(\mathbb{P})_{irr}$

$PM : \mathbb{E}(\mathbb{P}, n; \theta, \mathbb{1}_n)_{irr} \longrightarrow R(\mathbb{P}; \theta)_{irr}$ loc. inj. open holo map.

(This fact will be said that the R-H prob. is loc. solvable for g.s. eq's with $m+3g-3$ app. sing. pts)

Def A_1, A_2 : analytic sp's. $f : A_1 \to A_2$: holo. map.

f : almost surjective $\Leftrightarrow$ analytic Z-closure of $f(A_1) = A_2$

Theorem 3 (ess. by Ôtsuki) $\mathbb{P} \in B(m)$, $\theta \in (\mathbb{C} \setminus \mathbb{Z})^m$, $n = m+4g-3$

$\Rightarrow$ $PM : \mathbb{E}(\mathbb{P}|n)_{irr} \longrightarrow R(\mathbb{P})_{irr}$: almost surj.

$PM : \mathbb{E}(\mathbb{P}|n; \theta)_{irr} \longrightarrow R(\mathbb{P}; \theta)_{irr}$: surj.

Conj. $n = m+3g-3 \Rightarrow PM : \mathbb{E}(\mathbb{P}, n; \mathbb{1}_n)_{irr} \to R(\mathbb{P})_{irr}$ alm. surj.

(If this is true, then one can say that the R-H prob. is glob. solvable for SL-eq's with at most $m+3g-3$ app. sing. pts c.w.m.)

<u>*Cor to Th 2*</u> $\mathbb{P} \in B(m)$, $n = m + 3g - 3$

$\mathbb{E}(\mathbb{P}, n; \mathbb{1}_n)_{irr}$, $\mathbb{E}(\mathbb{P}, n; \theta, \mathbb{1}_n)_{irr}$: <u>*complex mfds.*</u>

<u>*Amplification*</u> $n = m + 3g - 3$

$$R(m;\theta)_{irr} := \coprod_{\mathbb{P} \in B(m)} R(\mathbb{P};\theta)_{irr}$$

$$\begin{tikzcd}
 & \mathbb{E}(m,n;\theta,\mathbb{1}_n)_{irr} \arrow[dl, "\pi"'] \arrow[d, "\varpi"] \arrow[r, "PM"] & R(m;\theta)_{irr} \arrow[dl] \\
B(m,n;\mathbb{1}_n) \arrow[r, "p"'] & B(m) &
\end{tikzcd} \qquad (D.G.5)$$

NB. $R(m;\theta)_{irr}$ *is a loc. system over* $B(m)$ *whose* <u>*char. homom.*</u> *is given by*

$Br(m) := \pi_1(B(m), \mathbb{P}_0) \longrightarrow Aut(R(\mathbb{P}_0;\theta)_{irr})$, $\ell \longmapsto [\rho \mapsto \rho \circ \ell_*]$

where $B_r(m) \longrightarrow Aut(\pi_1(M \setminus |\mathbb{P}|))$, $\ell \longmapsto \ell_*$

Later, by using Th 2, we shall show

"*Garnier distribution*" *is* $\begin{cases} \text{Frob. integrable} \\ \text{transverse to each fiber of } \varpi \end{cases}$

§6. *Cousin Prob. & the mfd of g.s. eq's.*

By §5, it is natural to assume

$$(**) \qquad \boxed{n = m + 3g - 3}$$

☆ $\mathbb{E}_m(\theta) \underset{def}{:=} \mathbb{E}(m, n; \theta, \mathbb{1}_n)_{irr}$; *more detailed study of* $\mathbb{E}_m(\theta)$.

Given $\mathbb{r} = (P_1, \cdots, P_m, q_1, \cdots, q_n) \in B(m+n)$, *let*

$\xi(\mathbb{r}) \underset{def}{:=} K^2 \otimes [P_1 + \cdots + P_m - (q_1 + \cdots + q_n)] \in H^1(M, \mathcal{O}^*)$.

<u>*Rem*</u> $\xi(\mathbb{r}) = K \otimes \eta(\mathbb{r}, \mathbb{1}_n)^{-1}$ (*see* §3)

$c_1(\xi(\mathbb{r})) = g - 1$.

<u>Advantage of (**)</u> = <u>Fredholm Alternative</u>

$h^0(M, \mathcal{O}(\xi(\mathbb{r}))) = h^1(M, \mathcal{O}(\xi(\mathbb{r})))$ for $\mathbb{r} \in B(m+n)$ $\quad (\because$ R-R$)$

☆ $D_m := \{ \mathbb{r} \in B(m+n) \; ; \; h^0(M, \mathcal{O}(\xi(\mathbb{r}))) \geqq 1 \}$

☆ $X_m := B(m+n) \setminus D_m$: <u>the sp. of sing. pts</u>

<u>Rem</u> $\quad D_m = D(m, n \, ; \mathbb{1}_n), \quad B_m = B(m, n \, ; \mathbb{1}_n)$ (see §3)

$$\begin{tikzcd}
 & \mathbb{E}_m(\theta) \arrow[dl, "\pi"'] \arrow[d, "\varpi"] \arrow[r, "PM"] & R(m;\theta)_{irr} \arrow[dl] \\
X_m \arrow[r, "p"'] & B(m) &
\end{tikzcd}$$

(D.G. 6)

<u>Example</u> (i) $g = 0 \Rightarrow D_m = \phi$

(ii) $g = 1$ $(\Rightarrow m = n)$

$$D_m = \{ \mathbb{r} = (p_1, \cdots, p_m, q_1, \cdots, q_n) \in B(m+n) \; ; \; p_1 + \cdots + p_m \overset{\text{lin. eq.}}{\sim} q_1 + \cdots + q_n \}$$

$$\underset{\text{Abel th}}{=} \{ \mathbb{r} \in B(m+n) \; ; \; \sum_{j=1}^{m} \int_{p_j}^{q_j} \underset{\text{holo 1-form on } M}{\omega} \equiv 0 \; (\text{mod periods}) \}$$

<u>Key Idea</u> Given sing. pts $\mathbb{r} \in X_m$ and "fiber coord's" (= accessory param. of a diff. eq.), then construct a diff. eq. $\in \mathbb{E}_m(\theta)$ as a sol. to the "<u>Cousin Problem</u> (CP)"!

<u>Key Lemma</u> X_m : an analytic Z-open subset $(\neq \phi)$ of $B(m+n)$.

(i) $H^1(M, \mathcal{O}(\xi(\mathbb{r}))) = 0$, $\leftarrow$ solvability of (CP)

(ii) $H^0(M, \mathcal{O}(\xi(\mathbb{r}))) = 0$ $\leftarrow$ uniq. of sol. to (CP)

(iii) hold for <u>all</u> $\mathbb{r} \in X_m$ $\leftarrow$ holo. dep. of sol to (CP) on "deta".

<u>Theorem</u> $\mathbb{E}_m(\theta) \xrightarrow{\pi} X_m$ is holo. affine bdle of rk $m + 3g - 3$.

<u>Rem</u> $\dim \mathbb{E}_m(\theta) = 2(m + 3g - 3) + m$.

§ 7. Fund. 2-form on the mfd of g.s. eq's.

<u>Theorem</u> $\exists\, \Omega$: a closed 2-form on $\mathbb{E}_m(\theta)$ s.t. $\Omega | \varpi^{-1}(\mathbb{P})$ is a symplectic str. on each fiber $\varpi^{-1}(\mathbb{P})$ of $\mathbb{E}_m(\theta) \xrightarrow{\varpi} B(m)$, $\mathbb{P} \in B(m)$.

<u>Def</u> Ω is called the <u>fund. 2-form</u> on $\mathbb{E}_m(\theta)$

§ 8. Monod. Preserving deform.

<u>Theorem</u> The monodromy preserving deformation is an Ω-invariant foliation on $\mathbb{E}_m(\theta)$.