A NOTE ON ADJOINT SEMIGROUPS ASSOCIATED WITH
SOME LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Set $E = R^n$ or C^n,

$$C_\gamma = \{\varphi : (-\infty, 0] \to E : \varphi \text{ is continuous and}$$
$$a_\gamma(\varphi) = \lim_{\theta \to -\infty} e^{\gamma\theta}\varphi(\theta) \text{ exists}\},$$

where $\gamma \in R$, and set

$$|\varphi|_\gamma = \sup\{e^{\gamma\theta}|\varphi(\theta)| : -\infty < \theta \leq 0\} \quad \varphi \in C_\gamma.$$

Then C_γ is a Banach space with respect to the norm $|\cdot|_\gamma$, and it is isomorphic to the Banach space $C = C([-1,0], E)$: an isometric isomorphism ι_γ from C_γ onto C is given by

$$(\iota_\gamma\varphi)(s) = \begin{cases} e^{\gamma s/1+s}\varphi(s/1+s) & -1 < s \leq 0 \\ a_\gamma(\varphi) & s = -1. \end{cases}$$

The t-segment x_t of a function $x:(-\infty, t] \to E$ is a function, on $(-\infty, 0]$ into E, defined by $x_t(\theta) = x(t+\theta)$ $\theta \leq 0$. Consider a linear functional differential equation with the phase space C_γ:

(1) $\dot{x} = L(x_t)$,

where L is a continuous linear operator on C_γ into E. It
is well known [3,4] that this equation has a unique solution $x(\varphi)$
satisfying the initial condition $x_0 = \varphi$ for any φ in C_γ,
and that the one parameter family of operators T(t), $t \geq 0$,
defined by

$$T(t)\varphi = x_t(\varphi) \quad \text{for} \quad \varphi \in C_\gamma$$

is a strongly continuous semigroup of bounded linear operators
on C_γ. This semigroup is called the solution semigroup of
Equation (1). We consider the representation of the adjoint
semigroup of T(t).

For linear functional differential equations with finite
delays on the phase space C, it is shown [2] that C is
sun-reflexive with respect to the solution semigroups. On the
other hand, for the case $\gamma = 0$, the space C_0 is not
sun-reflexive with respect to T(t), [5]. The space $C_0^{\odot\odot}$
defined below is isomorphic to the space E × BU, where BU is
the space of bounded, uniformly continuous functions on $(-\infty, 0]$
into E with the supremum norm; C_0 is imbedded onto the
subspace $\{(a_0(\varphi), \varphi) : \varphi \in C_0\} \subset E \times BU$. Diekmann and Greiner
suggested that, for any γ, the space C_γ would not be
sun-reflexive with respect to T(t) from the imformation about
the spectrum of the resolvent $(\lambda I - A)^{-1}$, where A is the

infinitesimal generator of $T(t)$. What is $C_\gamma{}^{\odot\odot}$? This note is an unaccomplised approach to this question.

Before proceeding, we give the definition of sun-reflexivity. For a moment, denote by $T(t)$ a strongly continuous semigroup of bounded linear operators on a Banach space X. Then the adjoins semigroup $T^*(t)$ is not necessarily a strongly continuous on X^*. The maximal closed subspace of X^* on which $T^*(t)$ becomes a strongly contious semigroup is the one defined by

$$X^\odot = \{x^\odot \in X^* : \lim_{t \to 0+} |T^*(t)x^\odot - x^\odot| = 0\}.$$

It is known that this space coincides with the closure of the $\mathcal{D}(A^*)$, the domain of the adjoint operator of the infinitesimal generator A of $T(t)$. Denote by $T(t)$ the restriction of $T^*(t)$ on $X^\odot$. Repeating the above process for $T^\odot(t)$, we have a strongly continuous semigroup $T^{\odot\odot}(t)$ which is the restriction of $T^{\odot*}(t)$ on the space $X^{\odot\odot} = \{x^{\odot\odot} \in X^{\odot*} : \lim_{t \to 0+} |T^{\odot*}(t)x^{\odot\odot} - x^{\odot\odot}|$
$= 0\}$. The original space X is isomorphically imbedded to a subspace of $X^{\odot\odot}$, and $T^{\odot\odot}(t)$ is an extension of $T(t)$. If X is isomorphic to the whole space $X^{\odot\odot}$, we say that X is sun-reflexive with respect to the semigroup $T(t)$. This occurs if and only if the resolvent $(\lambda I - A)^{-1}$ is weakly compact for any λ in the resolvent set $\rho(A)$, see [6]. Refer the book [1] for the theory of adjoint semegroups. The space $X^{\odot\odot}$ depends on $T(t)$; we may obtain another space as $X^{\odot\odot}$ from a different

semigroup. But for the solution semigroup of Equation (1), the spaces $C_\gamma^{\odot}$, $C_\gamma^{\odot\odot}$ are independent of the choice of the linear operator L. This fact holds for solution semigroups of linear functional differential equations with infinite delays on the very general phase space, [5].

Now we return to the solution semigroup $T(t)$ of Equation (1). It is easy to show the following result, cf [5].

Theorem 1. A function φ in C_γ is in $\mathscr{D}(A)$ if and only if it is continuously differentiable, $\dot\varphi$ is in C_γ and $\dot\varphi(0) = L(\varphi)$; and $A\varphi = \dot\varphi$ for φ in $\mathscr{D}(A)$.

To obtain the information about the space $C_\gamma^{\odot\odot}$, we take L = 0 in Equation (1); that is, we consider the equation

$$(2) \qquad \dot x = 0.$$

Denote by $S(t)$ the solution semigroup of this equation, and by B its infinitesimal generator. As claimed in the above, $\mathscr{D}(A)$ and $\mathscr{D}(B)$ have the same closure in C_γ^{*}, which we denote by $C_\gamma^{\odot}$. Since the case that $\gamma = 0$ is already treated in [5], we assume that $\gamma \neq 0$ hereafter.

If φ is in C_γ, the function u defined by $u(\theta) = e^{-\gamma\theta}\varphi(\theta)$ for θ in $(-\infty,0]$ is in C_0. Set $j_\gamma(\varphi) = u$. Then j_γ is an isometric isomorphism between C_γ and C_0. We characterize $\mathscr{D}(B)$ by the condition on u.

Theorem 2. Assume that $\gamma \neq 0$. Then φ in C_γ is in $\mathfrak{D}(B)$ if and only if $u = j_\gamma(\varphi)$ is represented as

$$(3) \qquad u(\theta) = \frac{1}{\gamma} v(0) - \int_\theta^0 v(s)\, ds \quad \text{for} \quad \theta \text{ in } (-\infty, 0]$$

and for some function v in C_0 such that

$$(4) \qquad \lim_{\theta \to -\infty} v(\theta) = 0 \quad \text{and} \quad \lim_{r \to -\infty} \int_r^0 v(s)\, ds \quad \text{converges.}$$

Proof. A function φ in C_γ is continuously differentiable if and only is u is so, and $\dot{\varphi}(\theta) = e^{-\gamma\theta}\{-\gamma u(\theta) + \dot{u}(\theta)\}$. Thus $a_\gamma(\dot{\varphi})$ exists if and lnly if $\lim_{\theta \to -\infty}\{-\gamma u(\theta) + \dot{u}(\theta)\}$ exists; and $\dot{\varphi}(0) = 0$ if and only if $-\gamma u(0) + \dot{u}(0) = 0$. Set $v(\theta) = \dot{u}(\theta)$. Then the last condition is equivalent that u is represented as in the above form (3). The condition that u is in C_0 is equivalent that v has the second dondition in (4). If $u(\theta)$ and $-\gamma u(\theta) + \dot{u}(\theta)$ converge as $\theta \to -\infty$, then $u(\dot{\theta}) \to 0$ as $\theta \to -\infty$. Thus we have the first dondition for v in (4).

If η is a function of bounded variation on $[-1, 0]$ and φ is continuous on $[-1, 0]$, we can write

$$\int_{-1}^0 d\eta(t)\varphi(t) = [\eta(-1+) - \eta(-1)]\varphi(-1) + \lim_{r \to -1+}\int_r^0 d\eta(t)\varphi(t),$$

where $\eta(-1+) = \lim_{t \to -1+}\eta(t)$. Since $\iota_0 : C_0 \to C$ is an

isomorphism, ι_0^* ; $C^* \to C_0^*$ is also an isomorphism. Hence, from Riesz's representation theorem, for any u^* in C_0^* there exist a vector a in E^*, and a E^*-valued function f, of bounded variation on $(-\infty, 0]$, such that

$$<u^*, u> = au(-\infty) + \int_{-\infty}^{0} df(\theta)u(\theta) \qquad \text{for } u \text{ in } C_0,$$

where $u(-\infty) = \lim_{\theta \to -\infty} u(\theta)$ and $\int_{-\infty}^{0} = \lim_{r \to -\infty} \int_{r}^{0}$. Of course, the norm of u^* is given by

$$|u^*| = |a| + \text{Var}(f, (-\infty, 0]) = |a| + \lim_{r \to -\infty} \text{Var}(f, [r, 0]).$$

If f is normalized in the sense that $f(0) = 0$ and f is left continuous on $(-\infty, 0)$, the pair (a, f) is uniquely determined by u^*. Denote by NBV the class of those normalized functions. We may identify the space C_0^* with $E^* \times \text{NBV}$, and regard the isomorphism j_γ^* as the one from $E^* \times \text{NBV}$ to C_γ^*: $<j_\gamma^*(a, f), \varphi> = <(a, f), u>$, where $u = j_\gamma(\varphi)$. Namely the space $E^* \times \text{NBV}$ is the coordinate space of C_γ^*.

Theorem 3. Assume that $\gamma \neq 0$. Then an element (a, f) in $E^* \times \text{NBV}$ is a coordinate of an element of $\mathcal{D}(B)$ if and only if a is arbitrary, $f(\theta)$ is absolutely continuous on $(-\infty, 0)$, and the equivalence class of $\dot{f}(\theta)$ in $L^1((-\infty, 0), E^*)$ contains a function which is in NBV and converges to 0 as $\theta \to -\infty$.

For such a (a,f) the coordinate (b,g) of $B^*(j_\gamma^*(a,f))$ is given by

$$(5) \qquad b = -\gamma a$$

$$(6) \qquad g(\theta) = \begin{cases} 0 & \text{for } \theta = 0 \\ -\gamma[f(\theta) - f(0-)] - \dot{f}(\theta) & \text{for } \theta < 0, \end{cases}$$

where we read that $\dot{f}$ stands for the function in NBV mentioned in the above.

Proof. Consider the condition

$$(7) \qquad j_\gamma^*(a,f) \text{ is in } \mathscr{D}(B^*) \text{ and } B^*(j_\gamma^*(a,f)) = j_\gamma^*(b,g),$$

for (a,f) and (b,g) in $E^* \times \text{NBV}$. By the definition of B^*, this means that, for every φ in $\mathscr{D}(B)$, $<j_\gamma^*(a,f),B\varphi> =$ $<j_\gamma^*(b,g),\varphi>$, or $<(a,f),-\gamma u + \dot{u}> = <(b,g),u>$, where $u = j_\gamma(\varphi)$. The condition $f, g \in \text{NBV}$ implies that $f(0) = g(0) = 0$; and the condition $\varphi \in \mathscr{D}(B)$ implies that $\dot{u}(-\infty) = 0$. From integration by parts we then have that

$$<(a,f),-\gamma u + \dot{u}> = -[a-f(-\infty)]\gamma u(-\infty) + \int_{-\infty}^{0} [\gamma f(\theta)d\theta + df(\theta)]\dot{u}(\theta)$$

$$<(b,g),u> = [b-g(-\infty)]u(-\infty) - \int_{-\infty}^{0} dg(\theta)\dot{u}(\theta).$$

Since u is represented as in (3), it holds that

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$$u(-\infty) = \frac{1}{\gamma}v(0) - \int_{-\infty}^{0} v(\theta)d\theta.$$

Hence we can write

$$\int_{-\infty}^{0} [df(\theta)+\gamma f(\theta)d\theta]v(\theta) = -c\{\frac{1}{\gamma}v(0) - \int_{-\infty}^{0} v(\theta)d\theta\} - \int_{-\infty}^{0} dg(\theta)v(\theta),$$

where

$$c = \gamma(f(-\infty) - a) + g(-\infty) - b.$$

Furthermore, if we define a function h in NBV by

$$h(\theta) = \begin{cases} 0 & \theta = 0 \\ \dfrac{c}{\gamma} & \theta < 0, \end{cases}$$

we have that

$$(8) \qquad \int_{-\infty}^{0} [df(\theta)+\gamma f(\theta)]d\theta]v(\theta) = \int_{-\infty}^{0} [dh(\theta)+cd\theta-dg(\theta)]v(\theta).$$

Consequently Condition (7) is equivalent that Relation (8) holds for every v in C_0 having Property (4). Since every function with compact support has this property for v, it follows that $df(\theta)+ \gamma f(\theta) = dh(\theta)+cd\theta-dg(\theta)$, or

(9) $\qquad f(\theta) - \gamma \int_{\theta}^{0} f(s)\, ds = h(\theta) + c\theta + \int_{\theta}^{0} g(s)\, ds \qquad \text{for} \quad \theta \leq 0.$

Notice that the functions in both sides are normalized.

Suppose that Equation (9) holds for f and g in NBV. Then f is locally absolutely continuous on $(-\infty, 0)$, which implies that

$$\text{Var}(f, [s,t]) = \int_{s}^{t} |\dot{f}(\theta)|\, d\theta \qquad \text{for} \quad -\infty < s < t < 0.$$

Since f is of bounded variation on $(-\infty, 0]$, it follows that $|\dot{f}(\theta)|$ is integrable on $(-\infty, 0)$: that is, f is absolutely continuous on $(-\infty, 0)$. Furthermore, from Equation (9) we have that

(10) $\qquad\qquad \dot{f}(\theta) + \gamma f(\theta) = c - g(\theta) \qquad \text{a.e. in} \quad (-\infty, 0).$

Since $f(0-) = h(0-) = \dfrac{c}{\gamma}$ from Equation (9), we obtain Relation (6); and the solution f of Equation (10) is

(11) $\qquad\qquad f(\theta) = \dfrac{c}{\gamma} + \int_{\theta}^{0} e^{-\gamma(\theta - s)} g(s)\, ds \qquad \text{for} \quad \theta < 0.$

Suppose that $\gamma > 0$. Sincce $f(-\infty)$ exists, we have that $\lim\limits_{\theta \to -\infty} e^{\gamma\theta} [f(\theta) - c/\gamma] = 0$, which implies that

$$\int_{-\infty}^{0} e^{\gamma s} g(s) \ ds = 0.$$

Hence f is rewritten as

$$f(\theta) = \frac{c}{\gamma} - \int_{-\infty}^{\theta} e^{-\gamma(\theta-s)} g(s) \ ds = \frac{c}{\gamma} - \int_{-\infty}^{0} e^{\gamma t} g(t+\theta) \ dt,$$

from which it follows that

(12) $$f(-\infty) = \frac{c}{\gamma} - \frac{g(-\infty)}{\gamma}.$$

Suppose $\gamma < 0$. Writing the integral in (11) as

$$\int_{\theta}^{0} e^{-\gamma(\theta-s)} g(s) \ ds = \left[\int_{\theta}^{N} + \int_{N}^{0} \right] e^{-\gamma(\theta-s)} g(s) \ ds \qquad \theta < N < 0,$$

we have directly Relation (12), or

$$c = \gamma f(-\infty) + g(-\infty).$$

This relation and the definition of c imply Relation (5); Equation (10) then becomes

$$\dot{f}(\theta) = -\gamma[f(\theta) - f(-\infty)] + g(\theta) - g(-\infty) \qquad \text{a.e. in} \quad (-\infty,0).$$

This means that the equivalence class of $\dot{f}$ in $L^1((-\infty,0),E^*)$ contain a function which is in NBV and converges to 0 as θ

$\rightarrow$ $-\infty$. Notice that such a function is unique for the equivalence class of $\dot{f}$ since it is left continuous on $(-\infty, 0)$.

Conversely, suppose f has this property, and let $k(\theta)$ be the function, in NBV $\cap$ $L^1((-\infty,0),\mathbb{E}^*)$, such that $\dot{f}(\theta) = k(\theta)$ a.e. in $(-\infty,0)$ and that $k(-\infty) = 0$. Define $g(\theta)$ by $g(0) = 0$ and $g(\theta) = -\gamma[f(\theta) - f(0-)] - k(\theta)$ for $\theta < 0$. Then g is in NBV and $\dot{f}(\theta) = -\gamma[f(\theta) - f(0-)] - g(\theta)$ a.e. in $(-\infty,0)$, which implies that

$$(13) \qquad f(\theta) - f(0-) = -\gamma\int_0^\theta f(s) \, ds + \gamma\theta f(0-) - \int_0^\theta g(s) \, ds$$

for $\theta < 0$. Since $g(-\infty) = -\gamma[f(-\infty) - f(0-)] - k(-\infty) = -\gamma[f(-\infty) - f(0-)]$, we have that $\gamma f(0-) = \gamma f(-\infty) + g(-\infty)$. If a and b in $\mathbb{E}^*$ satisfy Relation (5), it then follows that $\gamma f(0-) = \gamma[f(-\infty) - a] + g(-\infty) - b$. Therefore relation (13) becomes Relation (9), as required.

Theorem 3 says that $j_\gamma^*(a,f)$ is in $\mathscr{D}(B^*)$ if and only if f is represented as

$$(14) \qquad f(\theta) = d + \int_\theta^0 k(s) \, ds \quad \text{for} \quad \theta < 0,$$

and for some d in $\mathbb{E}^*$ and for some k in NBV $\cap$ $L^1((-\infty,0),\mathbb{E}^*)$ with $k(-\infty) = 0$. In this case the C_γ^* norm of $j_\gamma^*(a,f)$ is given by

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$$|j_\gamma^*(a,f)| = |a| + \text{Var}(f,(-\infty,0)) = |a| + |d| + \int_{-\infty}^0 |k(s)| \, ds.$$

Thus we have the following result, which is also valid in the case that $\gamma = 0$, [5].

Corollary4 4.

$$C_\gamma^\odot \simeq \mathbb{E}^* \times \mathbb{E}^* \times L^1((-\infty,0),\mathbb{E}^*).$$

REFERENCES

[1] P.L. Butzer and H. Berens, "Semi-Groups of Operators and Approximation", Springer-Verlag, Berlin, Heidelberg, New York 1967.

[2] O. Diekmann, Perturbed dual semigroups and delay equations, Centrum voor Wiskunde en Informatica, Report AM-R8604, 1986.

[3] T. Hagemann and T. Naito, Functional differential equations with infinite delay on the space C_γ, Lecture Notes in Math., 1017, 1983, 207-214.

[4] J. Hale and J. Kato, Phase space for retarded equations with infinite delay, Funkcialaj Ekvacioj, 21(1978), 11-41.

[5] Y. Hino, S. Murakami and T. Naito, "Functional Differential Equations with Infinite Delay", to appear.

[6] B. de Pagter, Characterizations of $\odot$-reflexivity, preprint.